Math 8200, Spring 2010: Final Exam Study Guide

The final exam will take place on Thursday May 6 at 12:00pm in Boyd 326 (the usual classroom). It will be 3 hours long. You will not be allowed to use notes or books of any kind. The exam will cover all the material of the semester.

Format of the exam

The exam will have ten questions, each will be worth 10 points. You should do as many of the questions as fully as possible. The questions will be similar in style to the homework problems and will vary in difficulty. The first five questions will be basic calculations and applications. The later questions will be more involved, requiring proofs and generalizations of the theory from class. At least the last one or two questions will be more challenging.

Your emphasis in working through the exam should be in providing complete clear precise answers, justifying as much as possible. I would much prefer to see you spend extra time on a few questions justifying your reasoning than rushing through them all giving only a rough outline of the solution. Points will be lost for statements that are wrong or imprecise even if I know what you mean.

General advice for review

For the most part, the best way to review older material is to look back at the homeworks. Try working through these again to see how much you can recall. If there are things you cannot figure out, look at the comments/solutions I posted on the website. If these don't make sense, please come and see me and we can go through the problems together. Below I have made a list of the homework exercises that are most important to master. If you don't know where to start or are worried about some of the more difficult homework problems, look at these first.

You should also work through the syllabus below, in conjunction with notes from class, making sure you understand what everything means and are able to carry out the skills. One thing you could try is making up problems of your own that capture each item. Highlighted in bold below are the most important skills. If the whole list is overwhelming, you should concentrate on these first.

Syllabus

Below I've tried to give a list of the main skills and facts that you need to know for the final. A list like this cannot cover the general intuition that you have built up over the course of the semester but this should give you something to work from. In essence, the syllabus consists of everything I mentioned in class, a few extra things that I told you to read for the homeworks, minus a few things I talked about in class that I don't expect you to know. For each of these concepts, you need to be able to apply it and recognize it in examples. When notes from class do not contain all the details, you should look in the textbook.

- homotopy and homotopy equivalences
  - **definition of a homotopy between two continuous maps** $f, g : X \to Y$
  - **homotopy equivalences and homotopy inverses**
  - **difference between 'homotopic' and 'homotopy equivalent', 'contractible' and 'nullhomotopic'**
  - **homeomorphic implies homotopy equivalent**
  - deformation retractions, homotopy equivalences by collapsing contractible subspaces
- fundamental group
  - **definition of fundamental group, including homotopies relative to a basepoint**
  - **well-defined-ness of fundamental group structure, group axioms**
  - isomorphism between fundamental groups at basepoints in same path component, choice of isomorphism depends on choice of path
  - **homomorphism of fundamental groups induced by a continuous map** $f : X \to Y$**, properties**
  - **homotopy equivalence determines isomorphism of fundamental groups**, and proof
  - **fundamental group for** S^1 **and idea as counting number of times a loop wraps around the circle**, proof via lifting loops to $\mathbb{R}$
  - **fundamental group of a product of two spaces, torus**
  - **fundamental group of** S^2, idea of proof
  - **definition of simply-connected**
  - **applications of** $\pi_1(S^1)$**:** fundamental theorem of algebra, **no retraction from** D^2 **to** S^1 **and proof**, Brouwer Fixed Point Theorem for D^2
- Van Kampen Theorem
  - **definition of free product of groups**
  - **statement of Van Kampen Theorem for** $X = U \cup V$ **where** $U \cap V$ **is path-connected**
  - sketch proof as given in class

- **applications: fundamental group of** $S^1 \vee S^1$, S^2 **(again), torus (again), Klein bottle**

- real projective spaces
  - **three definitions of** $\mathbb{R}P^n$**: lines through origin in** $\mathbb{R}^{n+1}$, S^n **with antipodal points identified,** D^n **with antipodal points on boundary identified**
  - identifying $\mathbb{R}P^{n-1}$ with a subspace of $\mathbb{R}P^n$
  - $\pi_1(\mathbb{R}P^2)$ **using Van Kampen Theorem**

- covering spaces
  - **definition of covering maps**
  - **examples:** $\mathbb{R} \to S^1$, $S^1 \to S^1$, $S^n \to \mathbb{R}P^n$, **coverings involving Klein bottles and tori**
  - covering maps $X \to S^1 \vee S^1$
  - **unique path lifting for covering maps**
  - **covering map** $Y \to X$ **makes** $\pi_1(Y)$ **a subgroup of** $\pi_1(X)$, the subgroup of loops in X that lift to loops in Y
  - bijection between the cosets of the subgroup $\pi_1(Y)$ in $\pi_1(X, x)$ and the elements of $p^{-1}(x)$
  - **fundamental group of** $\pi_1(\mathbb{R}P^n)$
  - finding the subgroup corresponding to a given covering space of $S^1 \vee S^1$
  - **theorem stating the existence of a covering space of** X **for each subgroup of** $\pi_1(X)$ and sketch proof
  - **definition of universal cover and examples**
  - **isomorphisms between covering spaces**, condition for two covering spaces to be isomorphic in terms of the subgroups $p_*\pi_1(Y, y)$ and proof
  - **uniqueness of the universal cover**
  - deck transformations, normal covering maps, connection with normal subgroups
  - covering spaces coming from actions of groups (see pages 71-73 in book)

- simplicial homology
  - Δ**-complexes and examples**
  - **the simplicial chain complex of a** Δ**-complex**, proof that $\partial_{n-1} \circ \partial_n = 0$
  - **chain complexes in general and their homology groups, simplicial homology groups of a** Δ**-complex**
  - **examples of simplicial homology: point,** S^1**, Klein bottle,** $\mathbb{R}P^2$

- singular homology

  - **singular n-simplexes, n-chains, the singular chain complex**
  - **n-cycles, n-boundaries, the singular homology groups of a space**
  - **singular homology of a point, singular homology of a disjoint union**
  - **relationship between $H_0(X)$ and path components of X**, and proof
  - chain maps, a continuous map induces a chain map on singular chain complexes, a chain map induces a homomorphism on homology groups, hence **a continuous map induces homomorphisms on homology groups**
  - **homology groups are functors, homeomorphic spaces have isomorphic homology groups**
  - **homotopic maps induce equal homomorphisms on homology groups**, sketch proof using chain homotopies
  - **homotopy equivalent spaces have isomorphic homology groups**
  - reduced homology

- relative homology groups
  - **pairs of spaces and maps of pairs of spaces**
  - **singular chain complex of a pair (X, A), relative homology groups $H_n(X, A)$, $H_n(X) - H_n(X, \emptyset), H_n(X, X) = 0$**
  - relative simplicial homology
  - **exact sequences**
  - **long exact sequence for relative homology of a pair, including definition of the boundary map $d_n : H_n(X, A) \to H_{n-1}(A)$,**
  - proof of long exact sequence using theorem that a short exact sequence of chain complexes determines a long exact sequence of homology groups, proof of this theorem using diagram chasing
  - long exact sequence for reduced homology groups and for simplicial homology

- excision and Mayer-Vietoris
  - **statement of excision theorem** (not the proof)
  - finding homology of $S^1 \vee S^1$ using excision and long exact sequences for pairs
  - **Mayer-Vietoris sequence including definition of maps involved in the sequence**, proof
  - **calculating homology groups using Mayer-Vietoris, S^n, $H_k(D^n, S^{n-1})$, torus, Klein bottle $\mathbb{R}P^n$ (by induction on n)**
  - **isomorphism between simplicial and singular homology groups**, proof by induction on skeleta using long exact sequence in homology, the five lemma, fact that a compact subset of a Δ-complex is contained in some finite skeleton

- degree for maps of spheres

  - **the degree of a map $S^n \to S^n$ and properties (from book)**
  - **Hairy Ball Theorem, degree of antipodal map**
  - local degrees, especially when map is a local homeomorphism

- CW complexes and cellular homology
  - **definition of CW complexes, cells**
  - **examples of CW complexes: S^n, $\mathbb{R}P^n$**
  - **cellular chain complex including abstract definition as $H_n(X^n, X^{n-1})$ and boundary maps**
  - identifying $H_n(X^n, X^{n-1})$ with the free abelian group on the set of n-cells of X
  - $H_n(X^k)$ for $k < n$, and the maps $H_n(X^{n+1}) \to H_n(X^k) \to H_n(X)$
  - **cellular homology**, isomorphism between cellular and singular homology
  - **calculating the boundary maps ∂_1 and ∂_2 in the cellular chain complex (these are easier) and hence the cellular homology of 2-dimensional CW complexes**
  - the cellular boundary formula and using degree to calculate general boundary maps ∂_n, homology of $\mathbb{R}P^n$
  - **Euler characteristic for finite CW complexes, relationship to ranks of the homology groups**

- classification of surfaces
  - **definition of n-manifold, surface, connected sum of manifolds**
  - **statement of classification of compact connected surfaces with or without boundary, as determined by orientability, Euler characteristic and number of boundary components**
  - **homology groups of all the compact connected surfaces**, how to calculate these using cellular homology
  - **classifying surfaces given by identifying edges on a polygon**

- orientability
  - orientation of an n-manifolds M at $x \in M$, orientations for M
  - connection between orientations and $H_n(M, \partial M)$ (not the proof)
  - orientability of surfaces
  - degree of a map between oriented manifolds

Homework exercises

To help prioritize your review, here is a list of the most basic and important homework exercises from the semester. You should make sure you understand these as many of the exam questions will be like them. Please look at my comments and solutions on the website and let me know if you have any questions. The exercises not listed here are still relevant but you should focus on these first.

- (HW1) #1, 2, 3, 4, 6, 7, 8, 11
- (HW2) #2, 3, 4, 8, 9
- (HW3) #2, 3, 4, 5
- (HW4) #2, 3, 4, 5, 7
- (HW5) #1, 3, 5, 6, 7
- (HW6) #1, 2, 4
- (HW7) #1, 2, 3, 6, 8, 10
- (HW8) #1, 3, 4, 8
- (HW9) #1, 2, 4, 5, 6, 7
- (HW10) #3, 4, 5, 7, 8, 9
- (HW11) #1, 2, 3, 19, 20, 21
- (HW12) #2, 3, 5
- (HW13) #1, 2, 3, 4, 6, 8