Math 8200, Spring 2010
Homework 1 Comments

- Problems 1-3 were pretty standard. Please come and talk to me if you had any trouble with these.

- #4: A homotopy is given by

  $$h(x,t) = (1-t)f(x) + tg(x)$$

  where the multiplication and addition on the right-hand side use the standard vector space structure on $\mathbb{R}^n$.

- #5: There is only map $X \to \bullet$, so X is contractible if and only if this map is a homotopy equivalence. If you write down what this means, you see that X is contractible if and only if the identity on X is homotopic to *some* constant map c_x for *some* $x \in X$. The question asks you to show that X is contractible if and only if the identity is homotopic to c_x for *any* $x \in X$. So the question boils down to showing that if the identity is homotopic to c_x for one $x \in X$, then it is homotopic to c_x for all $x \in X$. One way you might do this is to show that c_x is homotopic to $c_{x'}$ for any two points $x, x' \in X$. What do you need to know for this to be true?

- #6: The easiest way is to give a formula in vector notation. A suitable deformation retraction is given by

  $$h(\mathbf{x}, t) = \frac{\mathbf{x}}{1 - t + t|\mathbf{x}|}.$$

- #7: The easiest way is to embed $S^1 \vee S^1$ directly into the plane with one of the circles containing $(1,0)$, and one containing $(-1,0)$. You can then draw a diagram showing a deformation retraction.

- #8: Again, you can embed $S^2 \vee S^2$ directly as a subspace of $\mathbb{R}^3 - \{(-1,0,0),(1,0,0)\}$ and give a deformation retraction.

- #9: This is the hardest problem on this problem set. The deformation retraction from X to the point x gives us a homotopy $h : X \times [0,1] \to X$ between the identity and the constant map c_x. So the identity map $X \to X$ is nullhomotopic. This means that if $U = X$, then you can also take $V = X$. You might initially think that this should also mean that the identity $U \to U$ is nullhomotopic, but here is the problem with that. We can restrict the homotopy h to $U \times [0,1]$, but the image might not be in U, so in this case h would *not* give a homotopy between the identity on U and a constant map. Put more informally, the paths along which points in U travel to get to x might go outside of U on their way.

  So the problem is essentially to show that by taking a V even smaller than U, the homotopy h does restrict to a map $V \times [0,1] \to U$, which then is a homotopy between the inclusion $V \to U$ and the constant map. Here we can use continuity of h. Since h

is continuous and U is open, $h^{-1}(U)$ is open in $X \times [0,1]$. We now want to show that $h^{-1}(U)$ contains $V \times [0,1]$ for some neighbourhood V of x. We know that $h^{-1}(U)$ is open and contains all the points of the form (x_0, t) for all $t \in [0,1]$. (Why?) For each such t, it therefore contains a subset of the form $V_t \times (t - \epsilon, t + \epsilon)$ where $V_t \subset X$ is open. (This uses the definition of the product topology on $X \times [0,1]$.)

Now we use the fact that $[0,1]$ is compact! This tells us that there are finitely many $t_0, \ldots, t_n$ such that the intervals $(t_i - \epsilon_i, t_i + \epsilon_i)$ cover $[0,1]$. Let V be the intersection of $V_{t_0}, \ldots, V_{t_n}$. This is the intersection of finitely many open sets, so V is open. (This is why compactness of $[0,1]$ was necessary, we could not take the intersection of *all* the sets V_t, so we needed to reduce to finitely many.) Putting this all together you see that $V \times [0,1]$ is contained in $h^{-1}(U)$, which means that h restricts to a map $V \times [0,1] \to U$ as required.

- #10: We can give a deformation retraction to the a point on the horizontal line by getting every point to travel down along the vertical lines, and then along the bottom horizontal line as necessary. This can be done continuously, with the relevant point fixed all the time. Every other point $x \in X$ (i.e. not on the horizontal line) has the property that every sufficiently small neighbourhood of X is disconnected. However closely you look at a point on one of the vertical lines, an open set will contain other vertical lines as well. This contradicts the conclusion of the previous problem which tells us that X cannot be deformation retracted to a point not on the horizontal line. (This is a bit counterintuitive: it can be deformation retracted onto some points, but not others. The key thing to remember is that during the deformation retraction, the target point should remain stationary. If we dropped this condition and allowed the target point to move around, then you would be able to deform X down to any point.)

- #11: The key here is to collapse the contractible subspace consisting of the disc where the Klein bottle intersects itself. First you have to remember that that disc is actually filled in. It's easy to imagine that this disc has been 'punched out' by the tube passing through it, but that's not the case, the disc is still there. This means you can collapse it and get something homotopy equivalent. (You could not collapse it if it was a circle, as opposed to a disc, since a circle is not contractible.)

  After doing this collapse, you have two tubes that taper off and meet the surface of the bottle again at a single point. We can stretch these away, leaving a line connecting the endpoints of the tubes to the surface. (What you are really doing here is 'uncollapsing' a contractible subspace.) Now you have a sphere with two arcs attached, one outside and one inside. Finally, collapsing some more arcs on the surface of the sphere, you get a sphere with two circles attached at a point, or $S^2 \vee S^1 \vee S^1$. Note that it doesn't matter whether the circles are attached outside or inside the sphere - the spaces you get are homeomorphic.

If you have any questions about the harder problems, please come and talk to me about them.