Math 8200: Homework 12: due Friday 16 April

Please put your homework directly into Jim Stankewicz's mailbox by 5pm on Friday.

Reading

- We focused this week on cellular homology, so read the section on this (page 137-149). You should definitely know Lemma 2.34 and how to prove it. For part (c), the book mentions two proofs: the first (second paragraph of page 138) relies on some a fact about the topology of CW complexes from the Appendix (that a compact subset is contained in a finite skeleton). You should understand this proof, *including* the proof of the Proposition from the Appendix. The second proof (third paragraph of page 138 and onwards) is optional.

  You should know how to prove Theorem 2.35, and be able to apply the Cellular Boundary Formula as we did in class on Thursday. I don't expect you to prove this formula, although it is not particularly hard (top of page 141). We'll work through Examples 2.36 and 2.37 in class, so you should look at those at some point. Also work through Example 2.38 since this shows that a space with the same homology groups as a point need not be contractible. Example 2.39 gives you good practice at calculating a higher-dimensional cellular boundary map, so look at this too. There is a homework exercise this week filling in some of the details of Example 2.40, 2.42 we did in class, and 2.43 is optional but more good practice.

  The Euler characteristic is very important - make sure you can prove 2.44 (you already did a homework exercise similar to this. The rest of the section is not as important: split exact sequences are a useful tool that you have already seen in a homework exercise, and homology of groups provides an interesting connection to algebra - it is essentially using topology to study algebraic objects (kind of the opposite to how algebraic topology normally works).

Problems

Each question is worth 5 points except number 7.

1. Consider the n-sphere with the CW complex structure described in the last paragraph of page 7, with two r-cells for each $0 \leq r \leq n$. Find the cellular chain complex for this CW complex and use it to give another calculation of the homology of S^n. (Include the case $n = \infty$.)

2. Describe a CW complex that is homeomorphic to the Möbius band and calculate its cellular homology.

3. Calculate the homology of the space obtained by adding a 2-cell to the Möbius band M along a degree three map $\phi : S^1 \to \partial M$.

4. Let $f : S^n \to X$ be any continuous map. The *space obtained by attaching an $n+1$-cell to X along f* is the space obtained from the disjoint union $D^{n+1} \amalg X$ by identifying the point $x \in \partial D^{n+1}$ to the point $f(x) \in X$.

   (a) Show that the space obtained by attaching an $n+1$-cell to X along f is homeomorphic to the mapping cone C_f.

   (b) In the case that f is a degree d map $f : S^n \to S^n$, show that C_f has a CW structure consisting of one 0-cell, one n-cell and one $n+1$-cell, and use this to calculate the homology of C_f.

5. Describe a topological space X whose only nonzero homology groups are

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = \mathbb{Z}/2, \quad H_2(X) = \mathbb{Z}/3.$$

   (Hints: find a chain complex $C_\bullet$ of free abelian groups whose homology is as required, then find a CW complex whose cellular chain complex is $C_\bullet$.)

6. (a) Let $f : \Delta^n \to D^n$ be a homeomorphism. Show that $\partial_n f \in C_{n-1}(S^{n-1})$ and that $[\partial_n f]$ is a generator for $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. (You may use the isomorphism between simplicial and singular homology.)

   (b) Let M be an n-manifold and take $x \in M$. Show that $H_n(M, M-\{x\}) \cong \mathbb{Z}$ and give an explicit generator for this group.

7. (10 points) Let $f : X \to Y$ be any continuous map between topological spaces. Let C_f denote the mapping cone of f (see page 13). Notice that there is an inclusion map

$$i(f) : Y \to C_f.$$

   (a) Show that the sequence

$$\tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(X) \xrightarrow{i(f)_*} \tilde{H}_n(C_f)$$

   is exact for all $n \geq 0$.

   (b) The *Puppe sequence* for f is the sequence of maps obtained by iterating the mapping cone construction:

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C_f \xrightarrow{i(i(f))} C_{i(f)} \xrightarrow{i(i(i(f)))} C_{i(i(f))} \longrightarrow \dots$$

   Show that $C_{i(f)}$ is homotopy equivalent to SX, the suspension of X.

(c) By taking homology of the Puppe sequence, deduce that there is a long exact sequence

$$\cdots \to \tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(Y) \xrightarrow{i_*} \tilde{H}_n(C_f) \to \tilde{H}_{n-1}(X) \xrightarrow{f_*} \tilde{H}_{n-1}(Y) \xrightarrow{i_*} \dots$$

This is the *long exact sequence of the map* f.

(d) Referring to question 4, let $f : S^n \to Y$ be a continuous map, and let $Y \cup_f e^{n+1}$ be the space obtained by attaching an $n+1$-cell to Y along f. Show that the homomorphism

$$i_* : H_k(Y) \to H_k(Y \cup_f e^{n+1})$$

is

i. an isomorphism if $k \neq n, n+1$;
ii. an injection if $k = n+1$;
iii. a surjection if $k = n$;

8. #29 on page 158

9. #33 on page 158

Harder Problems

You may substitute each of the following problems for one of the regular problems. Each is worth 5 points.

10. Let $A \in GL_2(\mathbb{Z})$ (that is, A is a 2×2-matrix of integers whose inverse also has integer entries).

(a) Define $f_A : S^1 \times S^1 \to S^1 \times S^1$ by

$$f_A(z_1, z_2) = (z_1^{A_{11}} z_2^{A_{12}}, z_1^{A_{21}} z_2^{A_{22}}).$$

Show that f_A is a homeomorphism from the torus to itself.

(b) As in #29 on page 158, glue together two solid tori using f_A to identify points on their boundaries. The result is a compact 3-manifold X without boundary. Calculate the homology groups of X.

11. #24 on page 157