Math 8200: Homework 13: due Wednesday 28 April by 5pm

Please put your homework directly in Jim Stankewicz's mailbox by 5pm.

Reading

- The classification of surfaces is not described comprehensively in Hatcher, but it *is* an important part of the class. I expect you to know what we covered in class, so use your notes from there as a guide. Examples 2.36 and 2.37 in Hatcher contain the calculations of the cellular homology groups of the surfaces, so look at hose as well. I don't know of a good reference for all of the theory, including the surfaces with boundary. In the qualifying exam syllabus the reference listed is:

  - W. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, 1991.

  I haven't looked at this. I have looked at:

  - J. Munkres, *Topology*, Prentice Hall, 2000,

  which I think gives full detail on the cutting up and gluing procedures for polygons. There is also the Wikipedia page on 'Surfaces' for a quick reference. As with everything, let me know if you have any questions on this.

  The material on orientations is on pages 233-237 of Hatcher. He uses a more general version based on homology 'with coefficients' in which instead of abelian groups, we obtain R-modules for some fixed commutative ring R. The case $R = \mathbb{Z}$ is 'ordinary' singular homology as we covered in the class. I expect you to know the definitions of orientation at a point and orientation for a manifold, and the statement of the theorem about $H_n(M, \partial M)$.

  For questions 7 and 8 you need to read the section on the Lefschetz Fixed Point Theorem (page 179).

Problems

Each question is worth 5 points except number 4.

1. (a) Let X_n be the surface obtained by twisting a rectangular strip n times before gluing the ends (so that X_0 is a cylinder (without ends) and X_1 is a Möbius band). Which surface is X_n? Explain.

   (b) Find all compact surfaces homotopy equivalent to S^1. Justify your answer.

2. Let K_1 and K_2 be two copies of the Klein bottle, each with n discs removed. Glue each boundary circle of K_1 to a unique boundary circle of K_2. Prove that the resulting surface Σ does not depend on how the circles are attached and describe where Σ fits in the classification of compact connected surfaces.

3. Which compact connected surfaces are obtained from a polygon by attaching edges according to the words:

   (a) $abcc^{-1}db^{-1}ea^{-1}e^{-1}$ (surface with boundary) or $abcc^{-1}db^{-1}ea^{-1}e^{-1}d^{-1}$ (surface without boundary)

   (b) $a_1 \ldots a_n a_1 \ldots a_n$

   (c) $a_1 \ldots a_n a_n \ldots a_1$

4. (10 points) Calculate the homology groups of all the compact connected surfaces with boundary.

5. Let $M = S^1 \times D^2$ be a solid torus. Show that M is orientable.

6. Find all path-connected spaces X such that there exists a finite-sheeted covering map $X \to K$ where K is the Klein bottle. Justify your answer fully. (You may use the result of #22 on page 157 without proving it.)

7. Let $A = \{a_{ij}\}$ be a 2×2-matrix whose entries are integers and define a map $f : S^1 \times S^1 \to S^1 \times S^1$ by

$$f(z, w) = (z^{a_{11}} w^{a_{12}}, z^{a_{21}} w^{a_{22}})$$

   Calculate the Lefschetz number of f.

8. Recall that $\mathbb{R}P^{2n-1}$ is orientable and fix an orientation. This means that the degree of a map $f : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ is well-defined. For which degrees is such a map guaranteed to have a fixed point?

9. Let $p : M \to N$ be a covering map between connected compact oriented n-manifolds without boundary. Prove that the degree of p is equal to $\pm d$ where d is the number of sheets in the cover. (The sign is ambiguous because there is a choice of orientation on each of the manifolds and changing that choice changes the sign of the degree.)

Harder Problems

There are no harder problems this week - some of the regular problems are fairly tough I think.