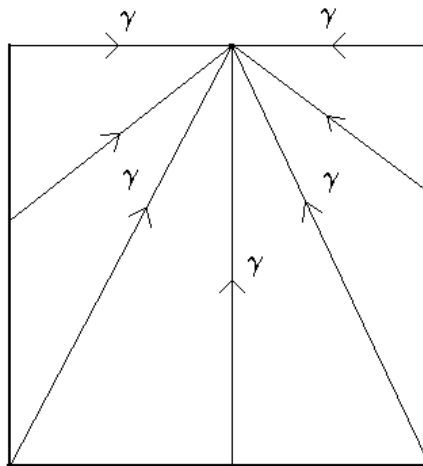


**Math 8200, Spring 2010**  
**Homework 2 Comments**

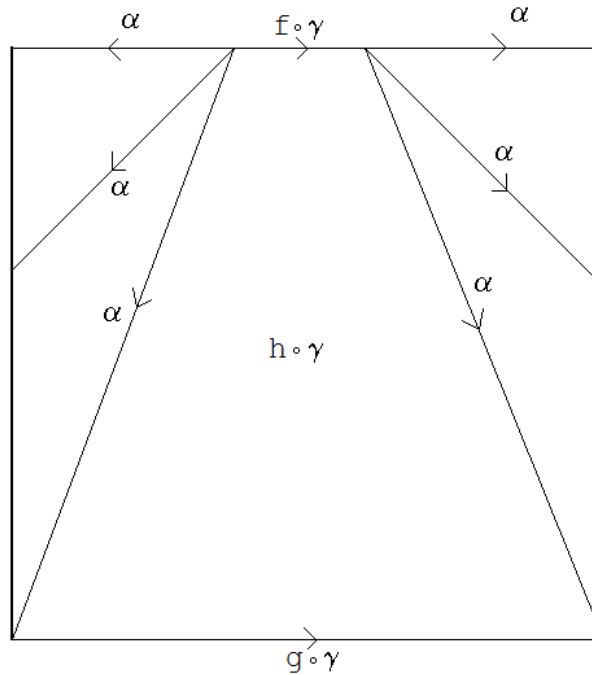
- #1: An explicit homotopy is given by

$$h(s, t) := \begin{cases} \gamma(2t) & \text{if } 2t \leq s; \\ \gamma(s) & \text{if } s \leq 2t \leq 2 - s; \\ \gamma(2 - 2t) & \text{if } 2 - s \leq 2t. \end{cases}$$

A picture of this homotopy looks like



- #2: Part (a) is similar to how we show that the group operation in the fundamental group is well-defined. Parts (b) and (c) both rely on the fact you proved in #1.
- #3: Just work through the definitions.
- #4: You need to show that  $\bar{\alpha} \cdot \gamma \cdot \alpha$  is homotopic to  $\bar{\beta} \cdot \gamma \cdot \beta$ . You can piece together the given homotopy  $h$  in the right way, you get the required homotopy.
- #5: You need to show that  $\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$  is homotopic to  $g \circ \gamma$ . The following picture shows how to use  $h$  to get that homotopy. Here  $h \circ \gamma$  stands for the map given by  $h(\gamma(-), -)$ .



Let me know if you have trouble interpreting this picture.

- #6 (#1 from book): The idea comes from group theory where we can ‘cancel’ by multiplying by the inverse:

$$f_0 \simeq f_0 \cdot e_{x_0} \simeq f_0 \cdot g_0 \cdot \bar{g}_0 \simeq f_1 \cdot g_1 \cdot \bar{g}_1 \simeq f_1 \cdot e_{x_0} \cdot f_1.$$

This sequence of homotopies uses several different properties of homotopies of paths:

- the fact that the constant path is a kind of identity (up to homotopy) for concatenation of paths
  - the fact that the reverse path is an inverse up to homotopy
  - the ‘associativity’ of paths up to homotopy:  $f_0 \cdot (g_0 \cdot \bar{g}_0) \simeq (f_0 \cdot g_0) \cdot \bar{g}_0$
  - the fact that a homotopy between  $g_0$  and  $g_1$  determines a homotopy between  $\bar{g}_0$  and  $\bar{g}_1$
  - the fact that concatenation respects homotopies: if  $f_0 \cdot g_0$  is homotopic to  $f_1 \cdot g_1$ , and  $\bar{g}_0$  is homotopic to  $\bar{g}_1$ , then  $(f_0 \cdot g_0) \cdot \bar{g}_0$  is homotopic to  $(f_1 \cdot g_1) \cdot \bar{g}_1$ .
- #7 (#4 from book): Use compactness of  $[0, 1]$  to obtain a finite decomposition of  $[0, 1]$  into intervals  $[t_i, t_{i+1}]$  such that  $\gamma([t_i, t_{i+1}])$  is contained in a star-shaped subset of  $X$ . Then show there is a homotopy between each of these parts of  $\gamma$  and the union of two line segments (one from  $\gamma(t_i)$  to the ‘center’ of the star, and one from the center to  $\gamma(t_{i+1})$ ).

- #8 (#11 from book): We get an inverse to this homomorphism by observing that every loop in  $X$  based at  $x_0$  is actually a loop in  $X_0$ , and that every homotopy between loops in  $X$  based at  $x_0$  is actually a homotopy between loops in  $X_0$ . (You need both these parts to get a well-defined inverse.)
- #9 (#12 from book): Since  $\pi_1(S^1) \cong \mathbb{Z}$ , the homomorphisms  $\pi_1(S^1) \rightarrow \pi_1(S^1)$  are in 1-1 correspondence with the homomorphisms  $\mathbb{Z} \rightarrow \mathbb{Z}$ . There is exactly one of these for each  $n \in \mathbb{Z}$ , given by  $x \mapsto nx$ . You can check that the map  $f : S^1 \rightarrow S^1$  given by  $f(z) = z^n$  has  $f_*$  corresponding to this homomorphism.

If you have any questions about the harder problems, please come and talk to me about them.