

Math 8200: Homework 2: due Thursday 28 Jan
Homotopies between paths and loops

Reading

- All the reading here is from section 1.1. You should read pages 25-28, and from ‘Induced Homomorphisms’ on page 34 to the end, page 37. You should also look over notes from class. The book does use slightly different notation than I used in class, so make sure you can reconcile these.

Problems

1. (5 points) Let $\gamma : [0, 1] \rightarrow X$ be a loop in X based at x_0 , and let $\bar{\gamma}$ be its reverse, given by $\bar{\gamma}(t) = \gamma(1 - t)$. Show that $\gamma \cdot \bar{\gamma}$ is homotopic relative to the basepoint to the constant loop e_{x_0} based at x_0 . (You should give an *explicit* formula for a homotopy.)
2. (5 points for each part) Let α be a path in X from x_0 to x_1 . We defined a map

$$c_\alpha : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$$

by

$$c_\alpha : [\gamma] \mapsto [\bar{\alpha} \cdot \gamma \cdot \alpha].$$

- (a) Show that c_α is well-defined.
- (b) Show that c_α is a group homomorphism.
- (c) Show that $c_{\bar{\alpha}} \circ c_\alpha$ is equal to the identity homomorphism on $\pi_1(X, x_0)$

(You do not need to give explicit formulas for homotopies in this question. You can use diagrams and other ways to describe any homotopies you need. But do try to describe the homotopies carefully so it is clear what you mean.)

3. (5 points) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps. Prove that

$$(g \circ f)_* = g_* \circ f_*$$

as homomorphisms $\pi_1(X, x_0) \rightarrow \pi_1(Z, g(f(x_0)))$.

4. (10 points) Let α and β be two paths in X from x_0 to x_1 . Suppose that h is a homotopy relative to the endpoints from α to β . This means that $h(0, t) = x_0$ and $h(1, t) = x_1$ for all $t \in [0, 1]$. Prove that $c_\alpha = c_\beta$ (see the definition in #2).
5. (5 points) Prove the following lemma from class. Let $f, g : X \rightarrow Y$ be maps and let h be a homotopy from f to g . Let α be the path from $f(x_0)$ to $g(x_0)$ given by $\alpha(t) = h(x_0, t)$. Prove that

$$c_\alpha \circ f_* = g_*$$

6. (10 points) #1 on page 38.

7. (10 points) #4 on page 38.

8. (5 points) #11 on page 39.

9. (5 points) #12 on page 39.

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 70 points).

- (5 points) #3 on page 38.

- (10 points) #6 on page 38.

- (5 points) #15 on page 39.

- (10 points) #17 on page 39.

- (5 points) #20 on page 40.