Math 8200: Homework 3: due Thursday 4 February

Reading

- You should read the rest of section 1.1 (pages 29-34) on the fundamental group of the circle, and applications. The book does give a slightly different presentation of the calculation and I recommend that you read this and try to understand how each part is related to what we did in class. The book also proves a slightly more general result (labelled (c) on page 30) that goes beyond what was needed for this proof, but will be useful later on.

  In section 1.2, you should read pages 40-42 for now. We'll get to the full statement of Van Kampen's Theorem, and applications, next time.

Problems

1. (10 points) Let $p : X \to Y$ be a continuous map and suppose $U \subset Y$ has the property that
$$p^{-1}(U) \equiv \coprod_{\alpha \in A} U_\alpha$$
where p restricts to a homeomorphism
$$p : U_\alpha \to U$$
for each $\alpha \in A$. (So the inverse image of U under p is homeomorphic to a disjoint union of copies of U, and each of those copies is mapped by p to U via a homeomorphism.)

   Let $\gamma : [0, 1] \to U$ be a path and choose $x \in X$ with $p(x) = \gamma(0)$. Prove that there is a unique map
$$\tilde{\gamma} : [0, 1] \to X$$
such that

   - $\tilde{\gamma}(0) = x$; and
   - $p \circ \tilde{\gamma} = \gamma$.

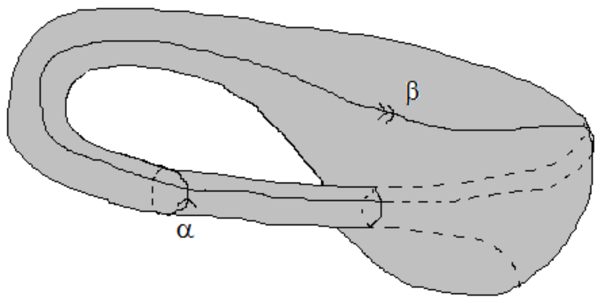
   Give as clear and rigorous a proof of this as you can. (I recommend that you write a draft before writing up your final answer.)

2. (5 points) Define a function
$$\Phi : \pi_1(S^1) \to \mathbb{Z}$$
by
$$\Phi([\gamma]) = \tilde{\gamma}(1)$$
where $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ is the unique lift of γ such that $\tilde{\gamma}(0) = 0$, as described in class. Prove carefully that Φ is a group homomorphism. (You may assume that Φ is well-defined.)

3. (5 points) Let $f : S^1 \to S^1$ be a map with $f(1) = 1$. The *degree* of f is $\Phi([f])$ where Φ is as in question 2. If g is another such map, find and prove a formula for the degree of $g \circ f$ in terms of the degrees of g and f.

4. (5 points) Let K be the Klein bottle, and let α and β be the two loops in the picture below. Prove that

$$\beta \cdot \alpha \simeq \alpha^{-1} \cdot \beta.$$



5. (3 points for each part) #16 on page 39 (all parts except (e)).

6. (5 points) #1 on page 52.

7. (5 points) #4 on page 53.

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points.

- (5 points) #8 on page 38.
- (5 points) #9 on page 38.
- (10 points) #17 on page 39 (but not if you did it on the previous homework!)
- (10 points) #12 on page 53.