Math 8200: Homework 4: due Thursday 11 February

Reading

- Read the remainder of the section on Van Kampen's Theorem. This is page 43 to the top of page 50. The book states the most general version of the Theorem in which the space X is the union of arbitrarily many open sets. In class, we only considered the case of two spaces. You only need to know the version we did in class, but it would be instructive to work on understanding the statement and proof of the general version, at least as far as they relate to what we went over.

  The applications covered in the book are more complex than those we did in class. I recommend you make sure you understand the ones we did before trying to read the others. Examples 1.23 and 1.24 both concern finding the fundamental group of the complement of a knot, or link, in $\mathbb{R}^3$. This is a standard way to try to tell apart two knots, for example, to show that one is knotted and the other is knot. (In class we covered the case of the complement of an 'unknot', i.e. the standard circle in $\mathbb{R}^3$.)

Problems

1. (10 points) Suppose that $X = U \cup V$ where U, V are open subsets of X, and suppose that $x_0 \in U \cap V$. Suppose also that $U \cap V$ is path-connected. Prove that the homomorphism

$$(i_U)_* * (i_V)_* : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0),$$

   determined by the inclusions $i_U : U \to X$ and $i_V : V \to X$, is surjective. The outline of this argument was given in class. I want you to fill in the details and prove this as carefully as possible. (You do not need to prove that there is a homomorphisms like this, just that it is surjective. You do not need to explain anything about free groups. Assume the reader is an expert on group theory and concentrate on the topology.)

2. (5 points) Use the Van Kampen Theorem to calculate the fundamental group of the Klein bottle K. You may give your answer in terms of generators and relations, but explain what loops in K give your generators.

3. (5 points) Find the fundamental group of the surface M_2 (the orientable surface of genus 2). Give your answer in terms of generators and relations, but explain what loops in M_2 give your generators.

4. (5 points) Let X be the space obtained from a Möbius band by attaching a disc D^2 along the boundary circle. Find the fundamental group of X.

5. (5 points) Find the fundamental group of the surface given by the connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2$. You may give your answer in terms of generators and relations.

6. (10 points) Consider the homomorphism

$$i_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^3)$$

induced by the standard inclusion $\mathbb{R}P^2 \to \mathbb{R}P^3$. Prove that i_* is an isomorphism and hence deduce that $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}/2$. (Hint: view $\mathbb{R}P^3$ as a solid ball D^3 with antipodal points on the boundary identified. The subspace $\mathbb{R}P^2$ then consists of those boundary points after identification. Show separately that i_* is injective and surjective.)

7. (5 points) #8 on page 53.

8. (5 points) #10 on page 53.

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points.

- (5 points) #11 on page 53
- (5 points) #16 on page 54
- (5 points) #20 on page 55
- (5 points) Show that each of the surfaces S^2, M_g (for $g \geq 1$) and $\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ (k factors) has a different fundamental group, so that no two of these surfaces are homotopy equivalent.