Math 8200: Homework 5: due Thursday 18 February

Reading

- Start looking at section 1.3 on covering spaces. Read this up to the top of page 63. There is an interesting discussion of covering spaces of $S^1 \vee S^1$ which should give you an idea of how varied these can be. The results in the section on 'Lifting Properties' are the most important. The 'homotopy lifting property' (1.30) is stated more generally here than we did in class. In particular, it applies not only to paths and homotopies between paths. You will need this more general version for some of the problems. There is also the 'Lifting Criterion' (1.33) which we did not cover in class but is important. It gives a more general answer to the question of when you can find a lift of a map into a covering space.

Problems

1. (5 points) Prove that if $p : Y \to X$ is a covering map with X connected then $|p^{-1}(x)|$ is the same for all $x \in X$.

2. (5 points)

   (a) Draw a picture of a finite-sheeted connected covering space $p : Y \to S^1 \vee S^1$ not the same as any of those on page 58. Describe the subgroup $p_*\pi_1(Y) \subset \mathbb{Z} * \mathbb{Z}$.

   (b) Draw a picture of an infinite-sheeted connected covering space $p : Y \to S^1 \vee S^1$ not the same as any of those on pages 58-59, and such that the subgroup $p_*\pi_1(Y) \subset \mathbb{Z} * \mathbb{Z}$ is finitely-generated. Describe that subgroup.

3. (5 points) Let H be the subgroup of $\mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta \rangle$ generated by the elements α^2 and β^3. Draw a picture of a connected covering space $p : Y \to S^1 \vee S^1$ such that $p_*\pi_1(Y) = H$. Explain how you know your covering space works. (Hint: number (14) on page 58.)

4. (5 points) Let $p : Y \to X$ be a covering map with Y path-connected and let y_0, y_1 be two points in Y with $p(y_0) = p(y_1) = x_0$. Prove that the subgroups

$$H_0 = p_*\pi_1(Y, y_0), \quad H_1 = p_*\pi_1(Y, y_1)$$

   are conjugate subgroups of $\pi_1(X, x_0)$, (i.e. there is $g \in \pi_1(X, x_0)$ such that $gH_0g^{-1} = H_1$).

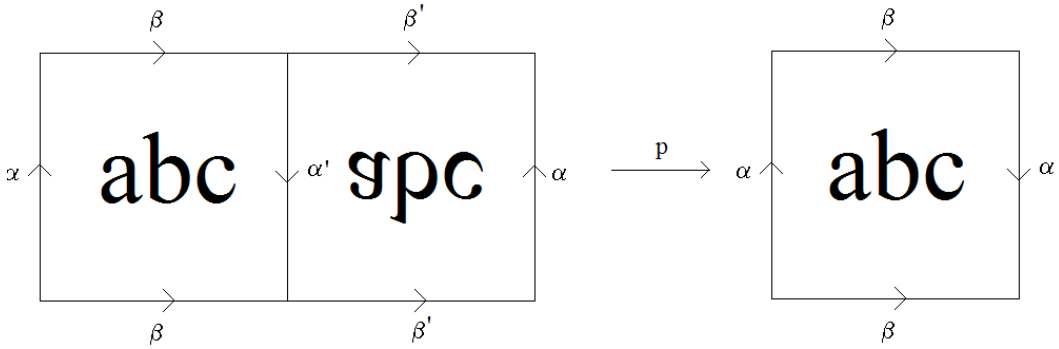
5. (5 points) True or false? (Explain your answers.)

   (a) if $Y \to X$ and $Y' \to X'$ are covering maps, then $Y \vee Y' \to X \vee X'$ is a covering map;

   (b) if $Y \to X$ and $Y' \to X'$ are covering maps, then $Y \times Y' \to X \times X'$ is a covering map.

6. (10 points) Let $T = S^1 \times S^1$ be the standard torus, and K the Klein bottle.

   (a) The following picture is meant to illustrate a covering map $p : T \to K$. (The writing 'abc' is there to help you visualize how the points in the torus on the left-hand side are mapped to points in the Klein bottle on the right-hand side. The



   If we write $\pi_1(K)$ as the group $\langle \alpha, \beta | \alpha\beta = \beta^{-1}\alpha \rangle$, then describe the subgroup of $\pi_1(K)$ given by the image of p_*. What is the index of this subgroup?

   (b) Prove that there is no covering map $K \to T$.

7. (5 points) Describe a finite-sheeted connected nontrivial covering space of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

8. (5 points) From the book: #3 on page 79.

9. (5 points) From the book: #9 on page 79.

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points). Each is worth 5 points.

- #6 on page 79
- #11 on page 80
- #13 on page 80
- #16 on page 80 (only the first statement)