Math 8200: Homework 6: due Thursday 25 February

Reading

- Read the remainder of the covering spaces section. Pages 63-67 cover most of the things we did on the relationship between subgroups of the fundamental group of X and covering spaces of X. You can skip the section titled 'Representing Covering Spaces by Permutations' although that is interesting and you should read it if you have the chance. We talked about deck transformations (page 70) and essentially did Proposition 1.39.

  You should read the rest of this section (up to just before Example 1.41) on your own. This covers a way to think about getting covering maps from the action of the group of deck transformations on the total space Y. Then X is the quotient space (or **orbit space**) of the group action. This is slightly different than the way we have mainly been thinking about covering spaces. Here you start with Y and construct X rather than the other way around. Of the examples, you should definitely look at 1.42 and 1.43. The others are interesting too, but more complicated. The section on 'Cayley Complexes' is an aside but, again, look at it if you have time.

  Please feel free to ask me questions about any of this that doesn't make sense, especially the group actions part which you do need to know, but we will not cover in class.

Problems

1. (5 points)

   (a) Let $p : Y \to X$ be a covering map with Y path-connected, and take $y_0 \in Y$ with $p(y_0) = x_0$. Last week you showed that if $y_1 \in p^{-1}(x_0)$, then $p_*\pi_1(Y, y_0)$ and $p_*\pi_1(Y, y_1)$ are conjugate subgroups of $\pi_1(X, x_0)$. Now let H be any other conjugate of $p_*\pi_1(Y, y_0)$. Prove that there is a $y_1 \in p^{-1}(x_0)$ for which $H = p_*\pi_1(Y, y_1)$.

   (b) Now suppose that p is a normal covering map. Use your answer to (a) to explain why $p_*\pi_1(Y, y_0)$ must be a normal subgroup of $\pi_1(X, x_0)$.

2. (5 points) Let $p : Y \to X$ be a covering map with X, Y path-connected and locally path-connected. Say that $x \in X$ has the property (*) if

   $$\text{for any } y_1, y_2 \in p^{-1}(x), \text{ there is a deck transformation} \\ h : Y \to Y \text{ such that } h(y_1) = y_2.$$

   Prove that if one point in X has property (*) then every point in X has this property. (This fact is part of the proof of the result that if $p_*\pi_1(Y, y_0)$ is a normal subgroup for some $y_0 \in Y$, then p is a normal covering space. You therefore may not just quote that result.)

3. (10 points)

(a) Let X be a path-connected, locally path-connected, semilocally simply-connected space with universal cover $q : \tilde{X} \to X$. Prove that if $p : Y \to X$ is another covering space with Y path-connected, then there is a map $r : \tilde{X} \to Y$ such that $p \circ r = q$. (This explains why we call $\tilde{X}$ the *universal* cover of X.)

(b) How many different such maps r are there?

(c) Show that r is a covering map (and hence that $\tilde{X}$ is also the universal cover of Y).

4. (5 points) #4 from page 79 (only do the first half)
5. (5 points) #8 from page 79 (note: in my version of the book, it says to use Exercise 10 from Chapter 0 - that is a misprint, you might want to use Exercise **11** from Chapter 0)
6. (5 points) #10 from page 79 (you should explain how you know you got them all)
7. (5 points) #20 from page 81
8. (5 points) #21 from page 81 (only do the space X)
9. (5 points) #24 from page 81 (you will have to read the section on group actions and how they give rise to covering maps to be able to do this)

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points). Each is worth 5 points.

10. For which connected spaces Y is there a covering map $Y \to K$ where K is the Klein bottle? Describe all these covering maps.
11. #18 in page 80
12. #25 on page 81
13. #28 on page 82