

## Math 8200: Homework 7: due Thursday 4 March

### Reading

- Read Chapter 2 from the start up to and including page 107. There's some terminology for chain complexes at the top of page 106 that I didn't mention in class: in particular the *cycles* and *boundaries* in a chain complex, and what it means to say that two cycles are *homologous*. I strongly recommend you work through Examples 2.3 and 2.4 as well as through the example of the Klein bottle from class. Ask me if you have any questions.

### Problems

Each problem is worth 5 points.

1. Let  $X$  be a  $\Delta$ -complex. Prove that  $H_0(X)$  is isomorphic to the free abelian group generated by the set of path components of  $X$ . (The free abelian group generated by a set  $A$  is sometimes written  $\mathbb{Z}[A]$ , and the set of path components of  $X$  is usually written  $\pi_0(X)$ . Therefore this question is asking you to prove that  $H_0(X) \cong \mathbb{Z}[\pi_0(X)]$ . Hint: define a homomorphism from  $\Delta_0(X)$  to  $\mathbb{Z}[\pi_0(X)]$  whose kernel is equal to the image of  $\partial_1$ .)

2. Fill in the gaps from the proof in class that  $H_1(K) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$ , where  $K$  is the Klein bottle. You may assume that

$$H_1(K) \cong \mathbb{Z}^3 / \langle \alpha - \beta + \gamma, \alpha + \beta - \gamma \rangle$$

where  $\alpha, \beta, \gamma$  are the generators of  $\mathbb{Z}^3$ .

3. Give a  $\Delta$ -complex structure for the real line  $\mathbb{R}$  and use it to calculate  $H_n(\mathbb{R})$  for all  $n$ .
4. Show that *any*  $\Delta$ -complex structure for  $S^1$  yields the same homology groups:  $H_1(S^1) = H_0(S^1) = \mathbb{Z}$ ,  $H_n(S^1) = 0$  otherwise. (You may assume that every  $\Delta$ -complex structure for  $S^1$  consists of finitely many copies of  $\Delta^1$  with endpoints identified appropriately.)
5. Let  $n$  be an even positive integer. Suppose that the  $\Delta$ -complex  $X$  has one  $n$ -simplex and no simplexes of dimension higher than  $n$ . Prove that  $H_n(X) = 0$ .
6. Consider the  $\Delta$ -complex structure for  $S^2$  given by identifying corresponding edges of two 2-simplexes. Use this to calculate the homology groups of  $S^2$ .
7. #1 on page 131 (and find the homology of this space)

8. #4 on page 131
9. #7 on page 131
10. Recall that the *rank* of a finitely generated abelian group  $A$ , written  $r(A)$  is the number of copies of  $\mathbb{Z}$  in the canonical presentation of  $A$  given by the classification theorem. (Equivalently, we can say the rank of  $A$  is equal to the largest possible size of a linear independent set of elements in  $A$ .) A key result in linear algebra says that:

if  $f : A \rightarrow A'$  is a homomorphism of finitely generated abelian groups, then

$$r(\ker f) + r(\operatorname{im} f) = r(A).$$

Use this result to prove the following claim:

- Let  $X$  be a 2-dimensional  $\Delta$ -complex with finitely many simplexes. (So there are finitely many simplexes in total, and none of dimension higher than 2.) Prove that the Euler characteristic of  $X$  satisfies

$$\chi(X) = r(H_0(X)) - r(H_1(X)) + r(H_2(X)).$$

## Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points). Each is worth 5 points.

11. Find the homology groups of  $\Delta^n$  (the  $\Delta$ -complex with a single  $n$ -simplex and nothing glued).
12. Describe a  $\Delta$ -complex structure for  $\mathbb{R}P^n$  and use it to calculate the homology groups. Include the case  $n = \infty$ .
13. Let  $X$  be a connected  $n$ -dimensional  $\Delta$ -complex with finitely many simplexes. Suppose that  $X$  is a manifold. Prove that  $H_n(X)$  is isomorphic either to  $\mathbb{Z}$  or 0.