**Math 8200: Homework 8: due Thursday 18 March**

# Reading

- Read from page 108 to the top of page 113. The opening remarks are interesting, particularly the paragraph beginning "In particular, elements of $H_1(X)$ ..." on page 109. We did Proposition 2.6 in class, and Proposition 2.7 is very similar to one of the exercises from Homework 7, except with singular homology instead of simplicial homology.

  The *reduced homology groups* on page 110 can be very useful for stating certain results. The only difference between them and the ordinary homology groups is that $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$. For all $n > 0$, we have $\tilde{H}_n(X) = H_n(X)$. We will use reduced homology occasionally so make sure you read this. Also, make sure you use the definition of $\tilde{H}_n(X)$ from the book when you are trying to prove things about them.

  The section titled 'Homotopy Invariance' is really important. You can skim the details of the proof of 2.10, though I will expect you to know what a chain homotopy is, and to be able to show that a homotopy between maps of spaces determines a chain homotopy $P$ between the corresponding chain maps. You also need to be able to prove Proposition 2.12.

# Problems

1. (5 points) Prove that there is no retract of $D^3$ onto its boundary $S^2$.

2. (10 points) Let $C_\bullet$ and $D_\bullet$ be chain complexes. Show that chain homotopy is an equivalence relation on the set of chain maps $C_\bullet \to D_\bullet$.

3. (5 points) Show that the spaces $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic simplicial homology groups, but are not homeomorphic.

4. (5 points) Let $f : X \to Y$ be a continuous map. Show that $f$ induces a homomorphism $f_* : \tilde{H}_n(X) \to \tilde{H}_n(Y)$ between the *reduced* homology groups of $X$ and $Y$ in such a way that $(f \circ g)_* = f_* \circ g_*$ and $(1_X)_* = 1_{\tilde{H}_n(X)}$.

5. (5 points) Consider the following two chain complexes:

$$C_\bullet : \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$$

   where the superscript 2 tells us that this map is multiplication by 2, and

$$D_\bullet : \quad 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

   (a) Prove that $C_\bullet$ and $D_\bullet$ have isomorphic homology groups.

   (b) Show that $C_\bullet$ and $D_\bullet$ are *not* chain homotopy-equivalent.

6. (5 points) Calculate the homology groups of the space $X$ that consists of two points $0, 1$ with open sets $\emptyset, \{0\}, X$.

7. (10 points) Let $X$ be a topological space and take $x_0 \in X$. Define a group homomorphism

$$h : \pi_1(X, x_0) \to H_1(X)$$

by

$$h([\gamma]) = [\gamma]$$

where, on the left hand side, $[\gamma]$ denotes the homotopy class of a loop $\gamma$ in $X$ based at $x_0$, and on the right hand side, $[\gamma]$ denotes the homology class of the 1-cycle in $X$ determined by the 1-simplex $\gamma$.

   (a) Show that $h$ is well-defined.

   (b) Show that $h$ is surjective if $X$ is path-connected.

   The homomorphism $h$ is called the *Hurewicz homomorphism* and its kernel is the commutator subgroup of $\pi_1(X, x_0)$. Thus if $X$ is path-connected, then $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$.

8. (5 points) Let $M$ and $N$ be compact surfaces. Prove that the Euler characteristic of the connected sum $M \# N$ is given by

$$\chi(M \# N) = \chi(M) + \chi(N) - 2.$$

   (You may assume that homotopy equivalent $\Delta$-complexes have equal Euler characteristics.)

   Use this to calculate the Euler characteristic of each of the following surfaces:

$$S^2, \quad \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{k \text{ times}}, \quad \underbrace{T \# \ldots \# T}_{g \text{ times}}$$

   where $k, g \geq 1$ and $T = S^1 \times S^1$ is the torus.

## Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points).

9. (5 points) Find the homology groups of the space $X$ consisting of two points with the indiscrete topology.

10. (10 points) Let $f : C_\bullet \to D_\bullet$ be a chain map such that the induced map $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism for all $n$. Suppose that each of the terms $D_n$ in the chain complex $D_\bullet$ is a *free* abelian group. Prove that $f$ is a chain homotopy equivalence.