Math 8200: Homework 9: due Friday 26 March

Please put your homework directly into Jim Stankewicz's mailbox by 5pm on Friday.

Reading

- We have been working on the section titled 'Exact Sequences and Excision' starting on page 113. You can skip Theorem 2.13 and its Corollaries for now, we'll get back to those shortly. Theorem 2.16 is the important one for this week since it gives us the long exact sequence for relative homology groups. Make sure you work through the proof of this step-by-step so that you believe this is an exact sequence. You should make sure you understand how to apply the long exact sequences in Examples 2.17 and 2.18 (note that these use the long exact sequence for *reduced* homology groups). You need to know the Excision Theorem (2.20) and how to use it. In particular, look at the Corollaries and Theorem on page 126.

Problems

Each problem is worth 5 points, except #6.

1. Let X be a space and A a subspaces of X. Prove that the long exact sequence

$$\cdots \to H_n(X) \xrightarrow{q} H_n(X,A) \xrightarrow{d_n} H_{n-1}(A) \to \dots$$

   is exact at $H_n(X,A)$, i.e. that the kernel of d_n is equal to the image of q. Do this directly from the definitions of q and d_n from class. Do not use Theorem 2.16.

2. (a) Suppose that we have abelian groups A and A', and a short exact sequence

$$0 \to A \to A' \to \mathbb{Z} \to 0.$$

   Prove that $A' \cong A \oplus \mathbb{Z}$. (Hint: construct a homomorphism from $A \oplus \mathbb{Z}$ to A'. Such a homomorphism consists of a homomorphism $A \to A'$ which you already have, and a homomorphism $\mathbb{Z} \to A'$ which you have to figure out how to get from the exact sequence.)

   (b) Find a short exact sequence of the form

$$0 \to \mathbb{Z} \to A' \to A \to 0$$

   such that A' is not isomorphic to $\mathbb{Z} \oplus A$.

3. We can consider S^{n-1} as the 'equator' in S^n. Find the homology groups $H_k(S^n, S^{n-1})$.

4. #15 on page 132
5. #16 on page 132
6. (10 points) #17 on page 132 (you may assume that simplicial and singular homology are isomorphic)
7. #18 on page 132
8. #27 on page 133
9. Recall that $\mathbb{R}P^n$ can be viewed as a disc D^n with antipodal points on the boundary identified. That boundary then becomes a copy of $\mathbb{R}P^{n-1}$. You may assume the following facts:
   - $H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ is isomorphic to $\mathbb{Z}$ with generator given by the relative n-cycle $\Delta^n \xrightarrow{\cong} D^n \xrightarrow{q} \mathbb{R}P^n$ where the first map is any homeomorphism from Δ^n to D^n, and q is the quotient map that identifies points on the boundary of D^n to get $\mathbb{R}P^n$;
   - $H_k(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ is zero for $k \neq n$;
   - for even n, the map $d_n : H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to H_{n-1}(\mathbb{R}P^{n-1})$ is given by multiplication by 2; (although I said you can assume this, try to figure out why this is true, at least intuitively)

   Use induction on n to calculate the homology groups of $\mathbb{R}P^n$, for all $n \geq 1$. Assuming that $H_k(\mathbb{R}P^\infty, \mathbb{R}P^n) = 0$ for $k \leq n$, also calculate the homology groups of $\mathbb{R}P^\infty$.

Harder Problems

You may substitute any of these problems for those above (though your total score cannot be more than 50 points).

10. (5 points) #19 on page 132
11. (5 points) #20 on page 132
12. (5 points) Calculate the homology groups of $\mathbb{C}P^\infty$ using only the tools we have developed in class so far.