A NOTE ON THE COMPOSITION PRODUCT OF SYMMETRIC SEQUENCES

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Abstract. We consider the composition product of symmetric sequences in the case where the underlying symmetric monoidal structure does not commute with coproducts. Even though this composition product is not a monoidal structure on symmetric sequences, it has enough structure, namely that of an 'oplax' monoidal product, to be able to define monoids (which are then operads on the underlying category) and make a bar construction. The main benefit of this work is in the dual setting, where it allows us to define a cobar construction for cooperads.

Introduction

The category of symmetric sequences, in a closed symmetric monoidal category $(\mathcal{C}, \wedge, S)$, is well known to have a monoidal product (called the composition product) whose monoids are precisely the operads in $\mathcal{C}$. This perspective allows for standard techniques in the theory of monoids, in particular bar constructions, to be applied to operads. This result depends, however, on the assumption that the monoidal structure $\wedge$ on $\mathcal{C}$ is *closed*, or more specifically, on the fact that $\wedge$ commutes with colimits. This condition is necessary in order that the composition product be associative.

On the other hand, operads can be defined in any symmetric monoidal category without the closed condition. It is therefore natural to wonder to what extent operads can still be treated as monoids when the composition product is not strictly associative. In particular, can we form a bar construction for operads in this wider context? In this paper, we show that the composition product is an 'oplax monoidal product' in the sense of Day-Street [2] and that this slightly weaker structure allows, nonetheless, for sensible definitions of monoids and of bar constructions over them. In particular, it follows that operads in any symmetric monoidal category have a natural bar construction.

The main motivation of this paper is, in fact, *co*operads. A cooperad in $\mathcal{C}$ can be considered as an operad in $\mathcal{C}^{op}$ (with the canonical symmetric monoidal structure on the opposite category). Although most interesting symmetric monoidal categories are closed symmetric monoidal (for example, compactly-generated topological spaces, or any of the standard models for stable homotopy theory), their opposite categories are less often so. This is reflected in the fact that the monoidal product does not generally commute with finite *products* in these topological examples. Our results therefore are necessary to define cobar constructions for cooperads in these categories.

Such cobar constructions were described previously by the author in [1], in which the theory presented here was outlined vaguely. The present paper is intended to fill in the gaps in that presentation and to demonstrate the usefulness of considering oplax monoidal structures, and their monoids, in homotopy theory.

Here is an outline of the paper. In §1 we describe what is meant by an oplax monoidal structure. In §2 we show that the composition product forms part of an oplax monoidal structure on the category of symmetric sequences. In §3 we describe monoids for oplax monoidal structures, and show that an operad is precisely such an monoid for the composition product. Finally, in §4 we describe the simplicial bar construction in this setting.

Acknowledgements

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1. Oplax monoidal structures

In this section we define the notion of an oplax monoidal structure on a category. This structure appears in [2] as an example of a lax monoid (in the 2-cell dual of the 2-category of categories). It is a weakening of the notion of monoidal structure in the sense that the unitivity and associativity morphisms are not required to be isomorphisms. To deal with this change, we need explicit functors to stand for the higher iterates of the usual monoidal structure. These functors are then related by a set of associativity morphisms that satisfy an appropriate set of conditions analogous to the usual axioms for a monoidal structure.

Definition 1.1. Let $\mathcal{E}$ be a category. An *oplax monoidal structure* on $\mathcal{E}$ consists of the following data:

- a collection of *products* $\mu_n : \mathcal{E}^n \to \mathcal{E}$ for $n \geq 0$;
- natural *associativity* morphisms

  $$\alpha_{n,l,r} : \mu_n(X_1, \dots, X_n) \to \mu_{n-r+1}(X_1, \dots, X_l, \mu_r(X_{l+1}, \dots, X_{l+r}), X_{l+r+1}, \dots, X_n)$$

  for $0 \leq l \leq l + r \leq n$, that collect the r terms starting after the l^{th} into a separate product;

satisfying the following conditions:

(1) for $l + s \leq k$ we have

$$\begin{array}{ccc}
\mu_n(X_1,\dots,X_n) & \xrightarrow{\alpha_{n,l,s}} & \mu_{n-s+1}(X_1,\dots,\mu_s(X_{l+1},\dots,X_{l+s}),\dots,X_n) \\
\downarrow{\scriptstyle \alpha_{n,k,r}} & & \downarrow{\scriptstyle \alpha_{n-s+1,k-s+1,r}} \\
\mu_{n-r+1}(X_1,\dots,\mu_r(X_{k+1},\dots,X_{k+r}),\dots,X_n) & \xrightarrow[\alpha_{n-r+1,l,s}]{} & \mu_{n-r-s+2}(X_1,\dots,\mu_s(X_{l+1},\dots,X_{l+s}),\dots,\mu_r(X_{k+1},\dots,X_{k+r}),\dots,X_n)
\end{array}$$

(2) for $l \leq k < k + r \leq l + s$ we have

$$\begin{array}{ccc}
\mu_n(X_1,\dots,X_n) & \xrightarrow{\alpha_{n,l,s}} & \mu_{n-s+1}(X_1,\dots,\mu_s(X_{l+1},\dots,X_{l+s}),\dots,X_n) \\
\downarrow{\scriptstyle \alpha_{n,k,r}} & & \downarrow{\scriptstyle \mu_{n-s+1}(X_1,\dots,X_l,\alpha_{s,k-l,r},X_{l+s+1},\dots,X_n)} \\
\mu_{n-r+1}(X_1,\dots,\mu_r(X_{k+1},\dots,X_{k+r}),\dots,X_n) & \xrightarrow[\alpha_{n-r+1,l,s-r+1}]{} & \mu_{n-s+1}(X_1,\dots,\mu_{s-r+1}(X_{l+1},\dots,\mu_r(X_{k+1},\dots,X_{k+r}),\dots,X_{l+s}),\dots,X_n)
\end{array}$$

Note that μ_0 consists of a single object in $\mathcal{E}$ which we denote I and plays the role of the unit. We say that an oplax monoidal structure is *normal* if $\mu_1 : \mathcal{E} \to \mathcal{E}$ is the identity functor.

Example 1.2. Let $\mathcal{E}$ be a category with a monoidal structure given by the bifunctor $\otimes : \mathcal{E}^2 \to \mathcal{E}$ with unit object I. Set

$$\mu_n(X_1, \dots, X_n) := (\dots(X_1 \otimes X_2) \otimes \dots) \otimes X_n$$

where μ_0 taking the value I. The unit isomorphism for the monoidal structure determines morphisms $\alpha_{n,l,0}$ and the associativity isomorphisms determine $\alpha_{n,l,r}$ for $r \geq 1$, as in Definition 1.1. The necessary conditions follow from the axioms for a monoidal structure. Thus a monoidal structure determines a normal oplax monoidal structure.

Conversely, suppose we have a normal oplax monoidal structure on $\mathcal{E}$ in which all the associativity morphisms $\alpha_{n,r,k}$ are isomorphisms. Then μ_2 is a monoidal structure on $\mathcal{E}$ with associativity isomorphism

$$\alpha_{3,1,2} \circ \alpha_{3,0,2}^{-1} : \mu_2(\mu_2(X,Y),Z) \to \mu_3(X,Y,Z) \to \mu_2(X,\mu_2(Y,Z))$$

and unit isomorphisms

$$\alpha_{1,0,0} : X \to \mu_2(I,X) \text{ and } \alpha_{1,1,0} : X \to \mu_2(X,I).$$

Remark 1.3. There is a dual notion of 'lax monoidal structure' on a category $\mathcal{E}$. The only difference is that the directions of the morphisms $\alpha_{n,l,r}$ are reversed, that is, a lax monoidal structure on $\mathcal{E}$ is the same as an oplax monoidal structure on $\mathcal{E}^{op}$.

Remark 1.4. From a category $\mathcal{E}$ with normal oplax monoidal structure we can construct, in a natural way, a 'multicategory' (see, for example, [3, I.2]) with the same class of objects as $\mathcal{E}$. For $X_1, \dots, X_n, Y \in \mathcal{E}$, we define the set of 'multi-maps' from $(X_1, \dots, X_n)$ to Y by

$$\operatorname*{Hom}_{\mathcal{E}}(X_1, \dots, X_n; Y) := \operatorname*{Hom}_{\mathcal{E}}(\mu_n(X_1, \dots, X_n), Y).$$

In this way, we can view a normal oplax monoidal structure as a special kind of multicategory: namely, one in which the sets of multimaps are represented, in a natural way, by single objects of $\mathcal{E}$.

2. Composition of symmetric sequences

Our main example of a normal oplax monoidal structure is the composition product on symmetric sequences.

Definition 2.1. A *symmetric sequence* in a category $\mathcal{C}$ is a functor $F : \mathsf{FinSet} \to \mathcal{C}$ from the category FinSet, whose objects are finite sets and whose morphisms are bijections, to $\mathcal{C}$. Denote the category of all symmetric sequences in $\mathcal{C}$ by $\mathcal{C}^\Sigma$ (in which morphisms are natural transformations). We write $\underline{r} := \{1, \dots, r\}$.

Remark 2.2. From now on, we fix a symmetric monoidal structure $\wedge$ with unit object S on the category $\mathcal{C}$. The symmetry and associativity isomorphisms for the monoidal product $\wedge$ allow us to write expressions such as

$$X \wedge Y \wedge Z \quad \text{and} \quad \bigwedge_{\alpha \in A} X_\alpha$$

without caring about parentheses or ordering of the factors. These expressions contain an implicit choice of ordering and bracketing with different choices related by the appropriate isomorphisms.

We assume throughout this paper that the underlying category $\mathcal{C}$ has all colimits (and, when we talk about the dual case of cooperads, all limits). In particular $\mathcal{C}$ has an initial object which we denote by 0.

Definition 2.3. For a finite set A, denote by A/FinSet the category whose objects are all functions $f : A \to I$ for some finite set I, and whose morphisms from $f : A \to I$ to $f' : A \to I'$ are the *bijections* $\sigma : I \to I'$ such that $f' = \sigma \circ f$.

Now let F and G be two symmetric sequences in $\mathcal{C}$. For each finite set A, we define a functor

$$(F,G) : A/\mathsf{FinSet} \to \mathcal{C}$$

on objects by

$$(F,G)(f : A \to I) := F(I) \wedge \bigwedge_{i \in I} G(f^{-1}(i)).$$

For a morphism $\sigma : I \to I'$ in A/FinSet we define

$$(F,G)(\sigma) := F(I) \wedge \bigwedge_{i \in I} G(f^{-1}(i)) \to F(I') \wedge \bigwedge_{i' \in I'} G(f^{-1}(\sigma^{-1}(i')))$$

by combining map $F(\sigma)$ with the permutation of the smash product identifying the term corresponding to $i \in I$ with the term corresponding to $\sigma(i) \in I'$.

We then define

$$(F \circ G)(A) := \operatorname*{colim}_{f \in A/\mathsf{FinSet}} (F,G)(f).$$

A bijection $\alpha : A \to A'$ determines a map $(F,G)(f) \to (F,G)(f \circ \alpha^{-1})$ that is the map

$$F(I) \wedge \bigwedge_{i \in I} G(f^{-1}(i)) \to F(I) \wedge \bigwedge_{i \in I} G(\alpha(f^{-1}(i)))$$

via the identity on $F(I)$ and the action of the symmetric sequence G on the bijections $f^{-1}(i) \cong \alpha(f^{-1}(i))$ given by restricting α. We thus obtain induced maps

$$(F \circ G)(\alpha) := (F \circ G)(A) \to (F \circ G)(A')$$

that make $F \circ G$ into a symmetric sequence in $\mathcal{C}$.

Remark 2.4. The definition of the composition product in 2.3 is isomorphic to more familiar forms involving the coproduct over partitions of A. Observe that the groupoid A/FinSet has disconnected pieces corresponding to different unordered partitions of A. Each function $f : A \to I$ determines such a partition, indexed by I, and two functions are in the same connected component of A/FinSet if and only if they determine the same partition. Thus $(F \circ G)(A)$ can be written instead as a coproduct over unordered partitions of A. The automorphism group of some $f : A \to I$ in A/FinSet is the symmetric group on the complement of the image of f. We therefore have

$$(F \circ G)(A) \cong \coprod_{A := \coprod_{i \in I} A_i} \left[F(I) \wedge \bigwedge_{i \in I} G(A_i) \right]_{\Sigma_{\{i | A_i = \emptyset\}}}.$$

Definition 2.5. The *unit symmetric sequence* 1 in the symmetric monoidal category $\mathcal{C}$ is given by

$$1(A) = \begin{cases} S & \text{if } |A| = 1; \\ 0 & \text{otherwise;} \end{cases}$$

where 0 is a fixed initial object in $\mathcal{C}$. The map $1(A) \to 1(A')$ induced by a bijection $A \to A'$ is the identity morphism on S or 0 as appropriate.

Proposition 2.6. *Let $\mathcal{C}$ be a symmetric monoidal category in which the monoidal structure $\wedge$ commutes with finite coproducts. Then the composition product determines a monoidal structure on the category $\mathcal{C}^{\Sigma}$ of symmetric sequences in $\mathcal{C}$ with unit object the symmetric sequence I of Definition 2.5.*

Proof. See §I.1.8 of [4]. □

We now drop the assumption that $\wedge$ commutes with finite coproducts. Our goal then is to show that the composition product is now part of a normal oplax monoidal structure on $\mathcal{C}^{\Sigma}$. To do this we need to define higher composition products for more than two terms.

Definition 2.7. For a finite set A and positive integer n, we denote by $A/\mathsf{FinSet}[n]$ the category whose objects are sequences of functions of finite sets

$$A \stackrel{f_{n-1}}{\longrightarrow} I_{n-1} \stackrel{f_{n-2}}{\longrightarrow} \dots \stackrel{f_1}{\longrightarrow} I_1.$$

and whose morphisms are commutative diagrams

$$\begin{array}{ccccccc} & & I_{n-1} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_1} & I_1 \\ & \nearrow^{f_{n-1}} & & & & & \\ A & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ & \searrow_{f'_{n-1}} & & & & & \\ & & I'_{n-1} & \xrightarrow{f'_{n-2}} & \cdots & \xrightarrow{f'_1} & I'_1 \end{array}$$

whose vertical maps are bijections. Notice that $A/\mathsf{FinSet}[2]$ is the category A/FinSet of Definition 2.3.

Now let $F_1, \dots, F_n$ be symmetric sequences in the symmetric monoidal category $\mathcal{C}$. We define a functor

$$(F_1, \dots, F_n) : A/\mathsf{FinSet}[n] \to \mathcal{C}$$

as follows. For the sequence

$$f : A \stackrel{f_{n-1}}{\longrightarrow} I_{n-1} \stackrel{f_{n-2}}{\longrightarrow} \dots \stackrel{f_1}{\longrightarrow} I_1$$

we set

$$(F_1, \dots, F_n)(f) := F_1(I_1) \wedge \bigwedge_{i \in I_2} F_2(f_1^{-1}(i)) \wedge \dots \wedge \bigwedge_{i \in I_{n-1}} F_n(f_{n-1}^{-1}(i)).$$

For a morphism $\sigma : f \to f'$ in $A/\mathsf{FinSet}[n]$, the bijections σ_k determine an isomorphism

$$(F_1, \dots, F_n)(f) \to (F_1, \dots, F_n)(f').$$

The *higher composition product* of $F_1, \dots, F_n$ is the symmetric sequence $(F_1 \circ \dots \circ F_n)$ defined by

$$(F_1 \circ \dots \circ F_n)(A) := \operatorname*{colim}_{f \in A/\mathsf{FinSet}[n]} (F_1, \dots, F_n)(f).$$

When $n = 2$ this reduces to the ordinary composition product of Definition 2.3.

Remark 2.8. As for the binary composition product, there is a formulation of the higher products in terms of partitions. The connected components of the groupoid $A/\mathsf{FinSet}[n]$ are the sequences of nested unordered partitions of A of length $n-1$. (For example, if $n=2$ we get a single partition, and if $n=3$ we get two partitions, one a refinement of the other, and so on.) We therefore have

$$(F_1 \circ \dots \circ F_n)(A) \cong \coprod_{f:\lambda_1 \leq \dots \leq \lambda_{n-1}} [(F_1, \dots, F_n)(f)]_{\Sigma_f}$$

where f denotes a sequence of partitions of A, and Σ_f denotes the automorphism group of the sequence f.

Remark 2.9. If $\wedge$ commutes with finite colimits in $\mathcal{C}$, the higher composition product $(F_1 \circ \dots \circ F_n)$ is isomorphic to any of the possible ways of iterating the binary product. For example,

$$(F_1 \circ F_2 \circ F_3) \cong (F_1 \circ F_2) \circ F_3 \cong F_1 \circ (F_2 \circ F_3).$$

Without this assumption, there are still maps from $(F_1 \circ F_2 \circ F_3)$ to each of these iterated composition products. We construct these next and show that they form a normal oplax monoidal structure on $\mathcal{C}^\Sigma$.

Definition 2.10. Let $F_1, \dots, F_n$ be symmetric sequences in the symmetric monoidal category $\mathcal{C}$. Our aim is to construct natural maps of symmetric sequences:

$$\alpha_{n,l,r} : (F_1 \circ \dots \circ F_n) \to (F_1 \circ \dots \circ F_l \circ (F_{l+1} \circ \dots \circ F_{l+r}) \circ F_{l+r+1} \circ \dots \circ F_n).$$

Fix a finite set A. At A, the required $\alpha_{n,l,r}$ maps out of a colimit over $A/\mathsf{FinSet}[n]$. It is therefore sufficient to define $\alpha_{n,l,r}$ for each $f \in A/\mathsf{FinSet}[n]$ in a way consistent with the maps. So fix a sequence

$$f : A \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_1} I_1.$$

At the finite set A, the target of $\alpha_{n,l,r}$ is a coproduct over $A/\mathsf{FinSet}[n-r+1]$. We choose to map $(F_1, \dots, F_n)(f)$ into the term in the target corresponding to the sequence

$$f' : A \xrightarrow{f_{n-1}} \dots \xrightarrow{f_{l+r}} I_{l+r} \xrightarrow{f_{l+r-1} \cdots f_l} I_l \xrightarrow{f_{l-1}} \dots \xrightarrow{f_1} I_1.$$

To do this we have to construct a map

$$(F_1, \dots, F_n)(f) \to (F_1, \dots, F_l, (F_{l+1} \circ \dots \circ F_{l+r}), F_{l+r+1}, \dots, F_n)(f').$$

Comparing these objects, we see it is sufficient to produce a map

$$\left[\left(\bigwedge_{i \in I_l} F_{l+1}(f_l^{-1}(i)) \right) \wedge \dots \wedge \left(\bigwedge_{i \in I_{l+r-1}} F_{l+r}(f_{l+r-1}^{-1}(i)) \right) \right] \to \bigwedge_{i \in I_l} (F_{l+1} \circ \dots \circ F_{l+r})((f_{l+r-1} \dots f_l)^{-1}(i)).$$

But the left-hand side here is precisely the smash product over $i \in I_l$ of the terms

$$(F_{l+1}, \dots, F_{l+r})(f[i])$$

where $f[i]$ is the sequence

$$(f_{l+r-1} \dots f_l)^{-1}(i) \xrightarrow{f_{l+r-1}} (f_{l+r-2} \dots f_l)^{-1}(i) \xrightarrow{f_{l+r-2}} \dots \xrightarrow{f_{l+1}} (f_l)^{-1}(i)$$

so the necessary map is formed by smashing together the inclusions of these terms in $(F_{l+1} \circ \dots \circ F_{l+r})(f^{-1}(i))$.

Proposition 2.11. *Let $\mathcal{C}$ be a symmetric monoidal category with all colimits. Taking*

$$\mu_n(F_1, \dots, F_n) := (F_1 \circ \cdots \circ F_n),$$

with $\mu_1(F) = F$ and $\mu_0 = 1$, the unit symmetric sequence, the maps $\alpha_{n,l,r}$ of Definition 2.10 form a normal oplax monoidal structure on the category $\mathcal{C}^\Sigma$ of symmetric sequences in $\mathcal{C}$.

Proof. The interested reader may do the necessary diagram-chasing. □

3. Monoids in oplax monoidal categories

We now return to the general context and say what is meant by a monoid in an arbitrary normal oplax monoidal category. It turns out that the relevant notion only depends on the first three levels of the oplax monoidal structure.

Definition 3.1. Let $(\mu_n, \alpha_{n,l,r})$ be a normal oplax monoidal structure on the category $\mathcal{E}$. A *monoid* with respect to this structure consists of an object $M \in \mathcal{E}$, a *multiplication map*

$$m_2 : \mu_2(M, M) \to M,$$

a *unit map*

$$m_0 : I \to M$$

such that the following diagrams commute:

(1)

$$\begin{array}{ccccccc} & & \mu_2(\mu_2(M,M),M) & \xrightarrow{\mu_2(m_2,M)} & \mu_2(M,M) & & \\ & \nearrow^{\alpha_{3,0,2}} & & & & \searrow^{m_2} & \\ \mu_3(M,M,M) & & & & & & M \\ & \searrow_{\alpha_{3,1,2}} & & & & \nearrow_{m_2} & \\ & & \mu_2(M,\mu_2(M,M)) & \xrightarrow{\mu_2(M,m_2)} & \mu_2(M,M) & & \end{array}$$

(2)

$$\begin{array}{ccccc} M & \xrightarrow{\alpha_{1,0,0}} & \mu_2(I,M) & \xrightarrow{\mu_2(m_0,M)} & \mu_2(M,M) \\ & \searrow^{1_M} & & & \downarrow{m_2} \\ & & & & M \end{array}$$

(3)

$$\begin{array}{ccccc} M & \xrightarrow{\alpha_{1,1,0}} & \mu_2(M,I) & \xrightarrow{\mu_2(M,m_0)} & \mu_2(M,M) \\ & \searrow^{1_M} & & & \downarrow{m_2} \\ & & & & M \end{array}$$

Example 3.2. If the oplax monoidal structure on $\mathcal{E}$ comes from an actual monoidal structure, then a monoid in the sense of Definition 3.1 is the same as a monoid in the usual sense.

Remark 3.3. It might seem strange that the definition of a monoid only relies on a small part of the oplax monoidal structure, namely that part up to and including μ_3. The rest of the oplax monoidal structure is necessary, however, for defining the simplicial bar construction on a monoid, as we do in the next section.

Definition 3.4. Let M be a monoid in a normal oplax monoidal category $\mathcal{E}$. We define maps

$$m_n : \mu_n(M, \dots, M) \to M$$

recursively as follows. Firstly, m_0 and m_2 are the structure maps for M as in Definition 3.1 and for completeness we take m_1 to be the identity on M. Suppose that $n \geq 3$ and that we have defined the maps m_r for $0 \leq r \leq n-1$. Then we define m_n to be any of the possible composites

$$\begin{aligned} \mu_n(M, \dots, M) &\xrightarrow{\alpha_{n,l,r}} \mu_{n-r+1}(M, \dots, \mu_r(M, \dots, M), \dots, M) \\ &\xrightarrow{\mu_{n-r+1}(\dots, m_r, \dots)} \mu_{n-r+1}(M, \dots, M) \\ &\xrightarrow{m_{n-r+1}} M. \end{aligned}$$

Lemma 3.5. *The composite maps used in 3.4 to define m_n are all equal, so m_n is well-defined.*

Proof. In the case $n = 3$, the two possible composites are equal precisely by condition (1) of Definition 3.1. For $n > 3$ (supposing that the map m_r is well-defined for $0 \leq r \leq n-1$) we need to show that whichever sequence of copies of M, from among the n copies, one chooses to 'multiply' together first, one gets the same composite.

So consider two sequences, A and B, of length between 2 and $n-1$, from the n copies of M. Without loss of generality, suppose that the length of A is less than or equal to the length of B. If one sequence is contained within the other, the claim follows by condition (2) of Definition 1.1. If the sequences are disjoint, the claim follows by condition (1) of Definition 1.1. If the intersection of A and B is of length two or more, then the composites of both A and B are equal to that of $A \cap B$, so are equal. If the union of A and B is of length less than n, then the composites of both A and B are equal to that of $A \cup B$, so are equal. Finally, suppose that the union of A and B is of length n, and the intersection is of length 1. Since $n \geq 4$, we cannot have the length of the shorter sequence A equal to $n-1$. Let A' be the sequence A expanded by one element. Then the intersection of A' with B is of length 2, so the composite maps for A and B are both equal to that for A'. □

Remark 3.6. Recall that a normal oplax monoidal category is an example of a multicategory. A monoid in an arbitrary multicategory $\mathcal{M}$ consists of an object M together with multi-maps $m_2 : (M, M) \to M$ and $m_0 : () \to M$ that satisfy axioms of the same form as those of Definition 3.1. (Here m_0 is a map from the empty sequence of objects in $\mathcal{M}$ to the single object M.)

We can also talk about actions of a monoid in an oplax monoidal category.

Definition 3.7. Let M be a monoid in the normal oplax monoidal category $\mathcal{E}$. A *left M-module* consists of an object $L \in \mathcal{E}$ and a *left action map*

$$l_2 : \mu_2(M, L) \to L$$

such that the following diagrams commute

(1)

$$\begin{array}{ccccccc} & & \mu_2(\mu_2(M,M),L) & \xrightarrow{\mu_2(m_2,L)} & \mu_2(M,L) & & \\ & \nearrow^{\alpha_{3,0,2}} & & & & \searrow^{l_2} & \\ \mu_3(M,M,L) & & & & & & L \\ & \searrow_{\alpha_{3,1,2}} & & & & \nearrow_{l_2} & \\ & & \mu_2(M,\mu_2(M,L)) & \xrightarrow{\mu_2(M,l_2)} & \mu_2(M,L) & & \end{array}$$

(2)

$$\begin{array}{ccccc} L & \xrightarrow{\alpha_{1,0,0}} & \mu_2(I,L) & \xrightarrow{\mu_2(m_0,L)} & \mu_2(M,L) \\ & \searrow_{1_L} & & & \downarrow{l_2} \\ & & & & L \end{array}$$

The corresponding notion of *right M-module* consists of an object $R \in \mathcal{E}$ with *a right action map*

$$r_2 : \mu_2(R, M)$$

satisfying similar conditions.

Remark 3.8. The structure maps for a left M-module L determine well-defined maps

$$l_n : \mu_n(M, \ldots, M, L) \to L$$

for all $n \geq 1$ given by any sequence of ways to 'multiply' subsets of the entries in $\mu_n(M, \ldots, M, L)$ ending with L.

Definition 3.9. Let M, N be two monoids in a normal oplax monoidal category $\mathcal{E}$. An (M, N)-*bimodule* consists of an object $B \in \mathcal{E}$ together with a left M-module structure l_2 and a right N-module structure r_2 such that the following diagram commutes:

$$\begin{array}{ccccccc} & & \mu_2(\mu_2(M,B),N) & \xrightarrow{\mu_2(l_2,N)} & \mu_2(B,N) & & \\ & \nearrow^{\alpha_{3,0,2}} & & & & \searrow^{r_2} & \\ \mu_3(M,B,N) & & & & & & B \\ & \searrow_{\alpha_{3,1,2}} & & & & \nearrow_{l_2} & \\ & & \mu_2(M,\mu_2(B,N)) & \xrightarrow{\mu_2(M,r_2)} & \mu_2(M,B) & & \end{array}$$

We now turn back to our favourite example: the composition product of symmetric sequences. In this case a monoid is precisely an operad on the underlying symmetric monoidal category.

Definition 3.10. Let $\mathcal{C}$ be a symmetric monoidal category. An *operad* in $\mathcal{C}$ consists of a symmetric sequence P together with, for each function of finite sets $f : A \to I$, a *composition map*

$$P(f) : P(I) \wedge \bigwedge_{i \in I} P(f^{-1}(i)) \to P(A),$$

and a *unit map*

$$\eta : 1 \to P$$

where 1 is the unit symmetric sequence. These maps should satisfy the following conditions:

- (Naturality in I): If $\sigma : I \to I'$ is a bijection then the following diagram commutes

$$\begin{array}{ccc} P(I) \wedge \bigwedge_{i \in I} P(f^{-1}(i)) & \xrightarrow{P(f)} & P(A) \\ \Big\downarrow{\scriptstyle P(\sigma)\wedge\tau} & & \Big\| \\ P(I') \wedge \bigwedge_{i' \in I'} P((\sigma f)^{-1}(i')) & \xrightarrow{P(\sigma f)} & P(A) \end{array}$$

  where the left-hand vertical map is given by combining $P(\sigma) : P(I) \to P(I')$ with the permutation τ that matches up the term $P(f^{-1}(i))$ with $P((\sigma f)^{-1}(i'))$ for $i' = \sigma(i)$.
- (Naturality in A): If $\alpha : A \to A'$ is a bijection then the following diagram commutes

$$\begin{array}{ccc} P(I) \wedge \bigwedge_{i \in I} P(f^{-1}(i)) & \xrightarrow{P(f)} & P(A) \\ \Big\downarrow{\scriptstyle 1\wedge\bigwedge_{i\in I} P(\alpha)} & & \Big\downarrow{\scriptstyle P(\alpha)} \\ P(I) \wedge \bigwedge_{i \in I} P((f\alpha^{-1})^{-1}(i)) & \xrightarrow{P(f\alpha^{-1})} & P(A') \end{array}$$

  where the left-hand vertical map is given by the restrictions of the bijection α to the subsets $f^{-1}(i) \subseteq A$.
- (Associativity): For functions $f : A \to I$ and $g : I \to J$, the following diagram commutes

$$\begin{array}{ccc} P(J) \wedge \bigwedge_{j \in J} P(g^{-1}(j)) \wedge \bigwedge_{i \in I} P(f^{-1}(i)) & \xrightarrow{P(g)\wedge 1} & P(I) \wedge \bigwedge_{i \in I} P(f^{-1}(i)) \\ \Big\downarrow{\scriptstyle 1\wedge\bigwedge_{j\in J} P(f)} & & \Big\downarrow{\scriptstyle P(f)} \\ P(J) \wedge \bigwedge_{j \in J} P((gf)^{-1}(j)) & \xrightarrow{P(gf)} & P(A) \end{array}$$

  where the left-hand vertical map is given by the restrictions of f to the subsets $(gf)^{-1}(j) \subset A$.

- (Left unit): For any map $f : A \to \{*\}$ from A to a singleton set, the following diagram commutes

$$\begin{array}{ccc} S \wedge P(A) & \xrightarrow{\eta} & P(*) \wedge P(A) \\ & \searrow^{\cong} & \downarrow{\scriptstyle P(f)} \\ & & P(A) \end{array}$$

- (Right unit): For any identity map $f : A \to A$, the following diagram commutes

$$\begin{array}{ccc} P(A) \wedge \bigwedge_{a \in A} S & \xrightarrow{\eta^{\wedge A}} & P(A) \wedge \bigwedge_{a \in A} P(\{a\}) \\ & \searrow^{\cong} & \downarrow{\scriptstyle P(f)} \\ & & P(A) \end{array}$$

Proposition 3.11. *Let P be a symmetric sequence on the symmetric monoidal category $\mathcal{C}$. The structure of a monoid, for the oplax monoidal structure on symmetric sequences of Proposition 2.11, on P is exactly the same as the structure of an operad on P.*

Proof. The monoid structure maps $\mu_2(P, P) \to P$ and $I \to P$ determine, and are determined by, the structure maps in the definition of an operad and tell us they satisfy the naturality conditions of Definition 3.10. The associativity and unit conditions for a monoid then correspond to associativity and unit conditions for an operad. □

4. Simplicial bar constriction on a monoid

In this section we show that we can form the simplicial bar construction for a monoid in any normal oplax monoidal category. This generalizes the usual construction for monoids in monoidal categories.

Definition 4.1. Let M be a monoid with respect to a normal oplax monoidal structure on a category $\mathcal{E}$. Let L be a left M-module and R a right M-module. The *two-sided simplicial bar construction on M with coefficients in R and L* is the simplicial object $\mathcal{B}_\bullet(R, M, L)$ in $\mathcal{E}$ defined as follows:

$$\mathcal{B}_n(R, M, L) := \mu_{n+2}(R, \underbrace{M, \ldots, M}_{n}, L)$$

Let $\alpha : \underline{n} \to \underline{m}$ be an order-preserving map. Then we define

$$\alpha^* : \mu_{m+2}(R, M, \ldots, M, L) \to \mu_{n+2}(R, M, \ldots, M, L)$$

to be the composite

$$\begin{aligned} \mu_{m+2}(R, M, \ldots, M, L) &\to \mu_{n+2}(\mu_{\alpha_0}(R, M, \ldots, M), \mu_{\alpha_2}(M, \ldots, M), \ldots, \mu_{\alpha_{n+1}}(M, \ldots, M, L)) \\ &\to \mu_{n+2}(R, M, \ldots, M, L) \end{aligned}$$

where

$$\alpha_i := \begin{cases} \alpha(0) & \text{if } i = 0; \\ \alpha(i) - \alpha(i-1) & \text{if } 1 \leq i \leq n; \\ m - \alpha(n) & \text{if } i = n+1. \end{cases}$$

The first map in the composite comes from the normal oplax monoidal structure on $\mathcal{E}$, and the second map comes from the monoid and module action maps for M, R and L.

Theorem 4.2. *Definition 4.1 produces a natural well-defined simplicial object $\mathcal{B}_\bullet(R, M, L)$ in $\mathcal{E}$. If the normal oplax monoidal structure on $\mathcal{E}$ arises from a monoidal product $\otimes$ on $\mathcal{E}$ as in Example 1.2, then $\mathcal{B}_\bullet(R, M, L)$ is the usual simplicial bar construction on a monoid M in $\mathcal{E}$ with respect to right and left M-modules R and L.*

Proof. The associativity properties of the oplax monoidal structure, and of the monoid M, together imply that this is a simplicial object of $\mathcal{E}$. If the oplax monoidal structure on $\mathcal{E}$ arises from a monoidal product $\otimes$, then we have

$$\mathcal{B}_n(R, M, L) \cong R \otimes \underbrace{M \otimes \cdots \otimes M}_{n} \otimes L$$

and the face and degeneracy maps are exactly those determined by the monoid and module action maps for R, M and L. So this is isomorphic to the usual simplicial bar construction. □

Remark 4.3. If M is an augmented monoid for the pseudomonoidal structure on $\mathcal{E}$, then there is a *reduced* bar construction given by

$$\mathcal{B}_\bullet(M) := \mathcal{B}_\bullet(I, M, I)$$

with the coefficients on both sides being the unit object I. Similarly, there are one-sided constructions for a single left or right M-module, taking the coefficients on the other side to be I.

Example 4.4. Now let $\mathcal{E}$ be the category of symmetric sequences on a symmetric monoidal category $\mathcal{C}$, with the pseudomonoidal structure given by the composition product. Then the simplicial bar construction described in this section is the 'standard' simplicial bar construction on an operad P with respect to a left P-module L and right P-module R.

If we now replace $\mathcal{C}$ with its opposite category $\mathcal{C}^{op}$, together with the canonical symmetric monoidal structure, the simplicial bar construction on an operad in $\mathcal{C}^{op}$ gives us a cosimplicial cobar construction $\mathsf{cobar}(R, Q, L)^\bullet$ on a cooperad Q in $\mathcal{C}$ with respect to a right Q-comodule R and left Q-comodule L. We have

$$\mathsf{cobar}(R, Q, L)^n(A) := \lim_{f \in A/\mathsf{FinSet}[n]^{op}} (R, \underbrace{Q, \dots, Q}_{n}, L)(f).$$

References

1. Michael Ching, *Bar constructions for topological operads and the Goodwillie derivatives of the identity*, Geom. Topol. **9** (2005), 833–933 (electronic). MR MR2140994
2. Brian Day and Ross Street, *Lax monoids, pseudo-operads, and convolution*, Diagrammatic morphisms and applications (San Francisco, CA, 2000), Contemp. Math., vol. 318, Amer. Math. Soc., Providence, RI, 2003, pp. 75–96. MR 1973511 (2004c:18011)
3. Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR 2094071 (2005h:18030)

4. Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR 2003f:18011

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