Homotopy Operations on Simplicial Algebras over an Operad

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2.1.1 The Dold-Kan correspondence

Let k be a field. A *simplicial k-vector space* is a simplicial object in the category of k-vector spaces. We denote this category by $s\mathsf{Vect}$. Let $\mathrm{Ch}_{\geq 0}$ denote the category of positively-graded chain complexes over the field k. Since k will be a fixed field throughout, we suppress it from the notation.

Theorem 2.1 (Dold-Kan). *The category of simplicial k-vector spaces is equivalent to the category of positively-graded chain complexes over k.*

The Dold-Kan correspondence works equally well for simplicial (left)-modules over any ring, not just a field. We will need it for modules over the group ring kG where G is a finite group and k is our fixed field.

Given a simplicial vector space V, the corresponding chain complex is the *normalized chain complex* NV. It will be easier for us to think of this as given by

$$NV = CV/DV$$

where $CV_n = V_n$, DV_n is the sum of the images of the degeneracy maps $s^j : V_{n-1} \to V_n$ and the differential is the alternating sum of the face maps

$$d = \sum_{i=0}^{n} (-1)^n d^i : CV_n \to CV_{n-1}.$$

The resulting chain complex is naturally isomorphic to that obtained by intersecting the kernels of the face maps (which is the usual definition of NV).

Let us denote the inverse equivalence from chain complexes to simplicial vector spaces by K. Its definition is:

$$KC_n = \oplus_{\mathbf{n} \twoheadrightarrow \mathbf{i}} C_i$$

where $\mathbf{n}$ is the partially ordered set $\{0, \dots, n\}$ and the sum is taken over all surjective order-preserving maps. The face and degeneracy maps are a bit tricky to write down. The important calculation is the following.

Lemma 2.2. *Write $[n]$ for the chain complex with the one-dimensional vector space k concentrated in position n. The simplicial vector space $K[n]$ is isomorphic to the free vector space on the simplicial n-sphere $\Delta^n/\partial\Delta^n$ modulo the basepoint.*

Proof. There is a k-basis for the vector space of q-simplices in $K[n]$ given by the set of surjective order-preserving maps $\mathbf{q} \twoheadrightarrow \mathbf{n}$. The q-simplices in Δ^n correspond to all order-preserving maps $\mathbf{q} \to \mathbf{n}$. It then turns out that a q-simplex in Δ^n lies in the boundary if and only if the corresponding map is not surjective. This shows that we have an isomorphism on q-simplices and it is a simple matter to check that the face and degeneracy maps agree. □

The basis for $K[n]_q$ given here will be important later on. We can describe it in another way by associating to the map $f : \mathbf{q} \twoheadrightarrow \mathbf{n}$, the ordered n-tuple of values from $\mathbf{q}$ where the function f jumps from one element of $\mathbf{n}$ to the next. So we can index the basis by the n-tuples $(i_1, \dots, i_n)$ where $1 \le i_1 \le \cdots \le i_n \le q$.

2.1.2 Homotopy theory

Homotopy theory notions exist independently on the categories of simplicial vector spaces and of chain complexes. A simplicial vector space V is a simplicial set and in fact a Kan complex. Therefore it has homotopy groups $\pi_n V$. The abelian group structure on V induces abelian group structures on the groups $\pi_n V$ (for all $n \geq 0$) which agree with the usual group structures. The scalar multiplication on V then makes each $\pi_n V$ into a k-vector space.

On the other hand there is the notion of chain homotopy on chain complexes. The 'homotopy' groups in this case are just the usual homology groups of the complex. For a chain complex C of k-vector spaces, its homology groups $H_n C$ are themselves k-vector spaces.

It turns out that these two notions are equivalent via the Dold-Kan correspondence. So for a simplicial vector space V we have

$$\pi_n V \cong H_n N V$$

where NV is the normalised chain complex associated to V by the Dold-Kan correspondence. We will say that a map of simplicial vector spaces is a *weak equivalence* if it induces an isomorphism of homotopy groups. This is of course the same thing as saying the induced map of chain complexes is a quasi-isomorphism.

Proposition 2.3. *The functor $V \mapsto \pi_n V$ on simplicial vector spaces is represented by $K[n]$. To be precise, $\pi_n V$ is naturally isomoprhic to the vector space of simplicial homotopy classes of maps $K[n] \to V$.*

The homotopy theories for simplicial vector spaces and for chain compexes come from underlying model category structures and these also correspond under Dold-Kan (see [GJ]). Moreover, we have the following observation.

Proposition 2.4. *The homotopy category of simplicial k-vector spaces (or chain complexes over k) is equivalent to the category of graded k-vector spaces.*

Proof. The homology groups functor H_* is a functor from the homotopy category of chain complexes to the category of graded vector spaces. Given a graded vector space U we can treat U as a chain complex with zero differential. We claim that these functors are inverse equivalences. The only non-trivial part is seeing that a chain complex C is quasi-isomorphic to $H_* C$ with zero differential. We construct a chain map $C \to H_* C$ using the fact that every $H_i V$ is a k-vector space and so in particular an injective k-module. □

2.1.3 Tensor Products and the Eilenberg-Zilber Theorem

In order to consider operads on simplicial vector spaces we need to decide on the appropriate symmetric monoidal structure on this category. There is a natural notion of tensor product both for simplicial vector spaces and for chain complexes and in general these do not coincide under the Dold-Kan correspondence.

If V and W are simplicial k-vector spaces, define the tensor product $V \otimes W$ by

$$(V \otimes W)_n = V_n \otimes W_n$$

with face and degeneracy maps acting on each factor separately.

If A and B are chain complexes, we define $A \otimes B$ by

$$(A \otimes B)_n = \bigoplus_{i=0}^{n} A_i \otimes B_{n-i}$$

with differential given by

$$d(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d(b)$$

Whilst these two notions are not exactly the same, the Eilenberg-Zilber Theorem tells us that they are the same up to homotopy.

Theorem 2.5 (Eilenberg-Zilber). *Let V and W be simplicial k-vector spaces. Then there is a natural (in V and W) chain homotopy equivalence*

$$f : N(V) \otimes N(W) \to N(V \otimes W).$$

This extends to the tensor product of any number of simplicial vector spaces.

To work on operads we will want to understand the action of the symmetric group Σ_n on the tensor powers. The Eilenberg Zilber Theorem says that $N(V^{\otimes n})$ is chain homotopically equivalent to $N(V)^{\otimes n}$, but this is not a Σ_n-equivariant equivalence. While the map f from the Theorem is naturally Σ_n-equivariant, its chain homotopy inverse is not.

The two tensor products give us two different symmetric monoidal structures on the model category $\mathcal{M}$ of simplicial vector spaces / chain complexes. Each of these makes $\mathcal{M}$ into a symmetric monoidal model category in the sense of [Hovey]. We then get induced symmetric monoidal structures on the homotopy category $\operatorname{Ho}\mathcal{M}$ (which we recall is just the category of graded vector spaces. These induced monoidal products are both just the usual graded tensor product but the symmetry isomorphisms $U\otimes V \to V\otimes U$ are different. For the chain complex product, we get

$$u \otimes v \mapsto v \otimes u$$

but for the simplicial product, it is

$$u \otimes v \mapsto (-1)^{|u||v|} v \otimes u.$$

While these are equivalent monoidal structures, they are not equivalent as *symmetric monoidal structures.*

2.2 Representation Theory of Finite Groups

2.2.1 Coinvariants and Invariants of kG-modules

Let k be a field and let G be a finite group. Let kG be the k-algebra whose underlying vector space is the free vector space on the elements of G. Define the multiplication in kG by

$$[g][h] = [gh]$$

where [g] is the basis element corresponding to $g \in G$. We usually just write this element as g. A (left/right) module over the ring kG is the same thing as a representation of the group G over the field k, that is, a vector space V together with a (left/right) action of the group G by linear maps.

There is a natural augmentation of kG as a k-algebra, that is, a map $kG \to k$ given by

$$\sum_{g\in G} \lambda_g[g] \mapsto \sum g \in G\lambda_g.$$

This makes k into a kG-bimodule with trivial G-actions. We will call this *the trivial kG-module.*

If M is a left kG-module, the tensor product

$$M_G = k \otimes_{kG} M$$

is called the *(module of) coinvariants* of M. We can also write it as the quotient of M by the submodule generated by

$$\{gm - m | m \in M, g \in G\}.$$

Notice that G acts trivially on the module of coinvariants and it can be characterized as the largest quotient of M for which this is true (see [Weibel]). Alternatively, the coinvariants functor from kG-modules to k-vector spaces is left adjoint to the functor that gives a vector space the trivial G-action.

The vector space

$$M^G = \mathrm{Hom}_{kG}(k, M)$$

is called the *(module of) invariants* of M. It can also be thought of as the submodule of M consisting of elements fixed by the action of G. Alternatively, the invariants functor is right adjoint to the trivial G-action functor.

There is a natural well-defined map from the coinvariants to the invariants, called the *trace map* (also called the norm map) given by

$$\mathrm{Tr} : M_G \to M^G$$

$$[m] \mapsto \sum_{g \in G} gm$$

where $[m]$ denotes the equivalence class of m in the quotient (so $[m]$ is the element $1 \otimes m$ in the tensor product for M_G).

2.2.2 Derived Functors and Group Cohomology

The coinvariants functor is a left adjoint and so is right exact. We can take its left derived functors in the usual homological sense and because the coinvariants is a tensor product, these are Tor-groups. These groups are the *homology groups of G with coefficients in M*:

$$H_i(G, M) = \mathrm{Tor}_i^{kG}(k, M) = L(-)_G(M).$$

In particular, $H_0(G, M) = M_G$.

The invariants functor is left exact and its right derived functors are *Ext*-groups. These are the *cohomology groups of G with coefficients in M*:

$$H^i(G, M) = \mathrm{Ext}^i_{kG}(k, M) = R(-)^G(M)$$

with $H^0(G, M) = M^G$.

Even though the coinvariants functor is not left exact we can still define its right derived functors (what you get when you take an injective resolution for M, apply the coinvariants and take homology).

Lemma 2.6. *The trace map* $\mathrm{Tr} : I_G \to I^G$ *is an isomorphism for injective* kG*-modules* I*.*

Proof. For the free module kG, each side is a one-dimensional vector space over k and the trace is an isomorphism. This means it is an isomorphism for any free module. If M is a direct summand of the free module F then M_G and M^G are direct summands of F_G and F^G respectively and the trace for M is the restriction of that for F. Therefore, Tr is an isomorphisms for projective kG-modules. But kG is a Frobenius algebra so injective modules are the same thing as projective modules. □

Corollary 2.7. *The right derived functors of coinvariants are naturally isomorphic to the right derived functors of invariants, that is, to the cohomology groups.*

Lemma 2.8. *If the characteristic of* k *does not divide the order of* G *(or is zero) then the trace map* $\mathrm{Tr} : M_G \to M^G$ *is an isomorphism for any* kG*-module* M*.*

Proof. Define an inverse $D : M^G \to M_G$ by

$$m \mapsto |G|^{-1}[m].$$

The composite $\mathrm{Tr} \circ D$ is

$$m \mapsto |G|^{-1}[m] \mapsto |G|^{-1} \sum_{g \in G} gm = |G|^{-1}|G|m = m$$

since m is a G-invariant. The composite $D \circ \mathrm{Tr}$ is

$$[m] \mapsto \sum_{g \in G} gm \mapsto |G|^{-1} \sum_{g \in G} [gm] = |G|^{-1}|G|[m] = [m]$$

since $[m] = [gm]$ for any g and m. □

2.2.3 Transfer and Restriction Maps

We now study how to relate the coinvariants and invariants constructions for different groups. Suppose that H is a subgroup of G and let M be a (left) kG-module. We will also write M for the kH-module obtained by restricting the action of G to that of H.

There is then a natural injection $\rho : M^G \to M^H$ and surjection $\rho : M_H \to M_G$. We will call these the *restriction* maps. We can also define maps in the other directions by

$$\begin{aligned} \tau :\ M^H &\longrightarrow M^G \\ m &\longmapsto \sum_{g_i \in G/H} g_i m \end{aligned}$$

and

$$\begin{aligned} \tau :\ M_G &\longrightarrow M_H \\ [m] &\longmapsto \sum_{g_i \in G/H} [g_i^{-1} m] \end{aligned}$$

where each sum runs over a set $\{g_i\}$ of left coset representatives for H in G. These maps are well-defined and natural in kG-modules M. We call them the *transfer* maps.

Lemma 2.9. *The transfer and restriction maps commute with the trace map. That is, the following diagram commutes:*

$$\begin{array}{ccccc} M_G & \xrightarrow{\tau} & M_H & \xrightarrow{\rho} & M_G \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ M^G & \underset{\rho}{\hookrightarrow} & M^H & \xrightarrow[\tau]{} & M^G \end{array}$$

Proof. This just requires working through the definitions of the maps. □

We can actually say more about the horizontal compositions in the above diagram.

Lemma 2.10. *The compositions*

$$M_G \xrightarrow{\tau} M_H \xrightarrow{\rho} M_G$$

$$M^G \underset{\rho}{\hookrightarrow} M^H \xrightarrow[\tau]{} M^G$$

are each multiplication by $[G : H]$ the index of H in G.

Proof. Again, we just work through the definitions. □

Corollary 2.11. *If the characteristic of the field k divides $[G : H]$ then the above compositions are zero. If not, then $\tau : M_G \to M_H$ is injective and $\tau : M^H \to M^G$ is surjective and, moreover, M_G is naturally a direct summand of M_H and M^H is naturally a direct summand of M^G.*

Since they are natural in kG-modules, the transfer and restriction maps induced natural transformations of the left and right derived functors. Taking the left derived functors of the maps on coinvariants, we get maps

$$H_i(G, M) \xrightarrow{\tau} H_i(H, M) \xrightarrow{\rho} H_i(G, M).$$

Taking the right derived functors of the maps on invariants we get maps

$$H^i(G, M) \xrightarrow{\rho} H^i(H, M) \xrightarrow{\tau} H^i(G, M).$$

These maps will be called the transfer and restriction maps for homology and cohomology respectively. We could also take the right derived functors of coinvariants, but since the trace map induces an isomorphism on these derived functors we would get (by Lemma 2.9) the same maps on cohomology.

2.2.4 Sign Representations

Here we mention some notation and calculations that will be important later. Suppose that G is a subgroup of a symmetric group Σ_n. Then let ϵ denote the *sign representation* of G, that is, the one-dimensional vector space on which the elements of G act as multiplication by the sign of the corresponding permuation of $\{1, \ldots, n\}$. For us, the importance of this representation is as follows. If $[r]$ denotes the graded vector space with just a one-dimensional vector space in position r, then $[r]^{\otimes n}$ has just a one-dimensional vector space in position nr. If G acts on $[r]^{\otimes n}$ by permuting the factors then as a kG-module, that one-dimensional vector space is $\epsilon^{\otimes r}$. We will often by concerned with $M \otimes \epsilon^{\otimes r}$ for some kG-module M and we will usually denote this by $M^{\pm}$ when r is clear from the context.

In particular, if $K[r]$ is the simplicial vector space described in a previous section, then by the Eilenberg-Zilber Theorem

$$\pi_q(M \otimes K[r]^{\otimes n}) = \begin{cases} M^{\pm}, & \text{if } q = nr; \\ 0, & \text{otherwise.} \end{cases}$$

We will also need an extension of this that will calculate $\pi_*(M \otimes K[r_1] \otimes \ldots \otimes K[r_n])$ as a graded kG-module when G is a subgroup of Σ_n that preserves the sequence of integers $(r_1, \ldots, r_n)$ under permutations. To calculate the G-action we take $g \in G$ and break it into a disjoint union of cycles $g_1 \ldots g_t$. Each g_s acts on some subset of the r_j which must all be the same. Denote this common value by $r_{(s)}$. Then let $g \in G$ act on the one-dimensional vector space as multiplication by the products of the signs of the g_s to the powers $r_{(s)}$. Denote the resulting kG-module by $\epsilon^{(r_1,\ldots,r_n)}$ and write $M^\pm = M \otimes \epsilon^{(r_1,\ldots,r_n)}$ when $r_1, \ldots, r_n$ are clear from the context.

The reason for these definitions is that

$$\pi_q(M \otimes K[r_1] \otimes \ldots \otimes K[r_n]) = \begin{cases} M^\pm, & \text{if } q = r_1 + \cdots + r_n; \\ 0, & \text{otherwise.} \end{cases}$$

again by the Eilenberg-Zilber Theorem.

3 Homotopy Operations for Vector Spaces

3.1 Homotopy Operations and Derived Functors

3.1.1 Homotopy T-operations

Let T be a functor from the category of simplicial k-vector spaces to itself. Eventually we will take T to be a monad and we will be interested in operations on the homotopy groups of T-algebras. But for much of the time we won't care about the monad structure. Therefore we make the following definition. A *homotopy T-operation* is a natural transformation

$$\pi_n V \to \pi_* TV$$

of functors on the category of simplicial vector spaces, where these functors are taking values in the category of sets. This means that the maps $\pi_n V \to \pi_m TV$ need not be linear even though the homotoy groups of a simplicial vector space are naturally themselves vector spaces. Despite this, there is a natural vector space structure on the set of all homotopy T-operations, given by

$$(\lambda\delta + \mu\epsilon)(\alpha) = \lambda\delta(\alpha) + \mu\epsilon(\alpha)$$

where δ, ϵ are homotopy T-operations, $\lambda, \mu \in k$ and $\alpha \in \pi_n V$.

Because the homotopy group functors are representable, we should easily be able to describe all homotopy T-operations by the Yoneda Lemma. For this to work however, we need T to be a functor on the homotopy category of simplicial vector spaces. We say that T is *homotopy-preserving* if it preserves weak equivalences of simplicial vector spaces. If this is the case, T induces a functor T_* on the homotopy category of simplicial vector spaces which is equivalent to the category of graded vector spaces. We call T_* the *derived functor* of T.

Lemma 3.1. *Let T be a homotopy-preserving functor of simplicial vector spaces. The vector space of homotopy T-operations $\pi_n V \to \pi_* TV$ is (naturally in T) isomorphic to $\pi_* TK[n]$.*

Proof. This is precisely the Yoneda Lemma. If δ is a homotopy T-operation, we can apply it to $1 \in \pi_n K[n] = [n]$ to get an element of $\pi_* TK[n]$. Conversely, let u be an element of $\pi_* TK[n]$. An element $\alpha \in \pi_n V$ is equivalent to a homotopy class of maps $K[n] \to V$. Since T is homotopy-preserving, this gives us a homotopy class of maps $TK[n] \to TV$. This then induces a map $\pi_* TK[n] \to \pi_* TV$ and we take $\delta(\alpha)$ to be the image of u under this map. □

To make this lemma useful, we need to know some homotopy-preserving functors.

Lemma 3.2 (Dold). *Let T be a functor on the category of k-vector spaces. We can* prolong *T to simplicial vector spaces by defining*

$$(TV)_n = T(V_n)$$

with face and degeneracy maps in TV got by applying T to those for V. Then T is homotopy-preserving. In this case, T_* will denote the derived functor of the prolongation of T.

3.1.2 Homotopy operations for monads

A *monad* (or *triple*) on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to mathcalC$ and natural transformations $T^2 \to T$ and $1_{\mathcal{C}} \to T$ satisfying certain relations that make T a monoid in the monoidal category of functors on $\mathcal{C}$ with product given by composition. An *algebra* over the monoid is an object $V \in \mathcal{C}$

together with a map $TV \to V$ satisfying certain other relations (that make the constant functor to V a module over the monoid T).

If T is a monad on simplicial vector spaces, we define a *homotopy operation for T-algebras* to be a natural transformatoin

$$\pi_n V \to \pi_* V$$

of functors on the category of T-algebras, where again these only have to be maps of sets.

Proposition 3.3. *Let T be a monad on the category of simplicial k-vector spaces. Then there is a 1-1 correspondence between homotopy T-operations and homotopy operations for T-algebras.*

Proof. If V is a T-algebra we can compose a homotopy T-operation $\pi_n V \to \pi_* TV$ with the map induced by $TV \to V$ to get a natural homotopy operation on V. Conversely, if V is any simplicial vector space, then TV is a T-algebra and the unit natural transformation for the monad gives us a map $V \to TV$. Composing the induced map on π_n with a homotopy operation for TV gives us a homotopy T-operation. The monad axioms tell us that these processes are inverses. □

If T is homotopy-preserving as well as being a monad, we have a way to calculate the homotopy T-operations and hence the operations for T-algebras. For a particular case of this, notice that if T is a monad on vector spaces, there is an obvious monad structure on the prolongation of T to simplicial vector spaces.

Corollary 3.4. *Let T be a monad on k-vector spaces. Then the homotopy operations for simplicial T-algebras (the same thing as algebras for the prolongation of T) are given by the vector spaces $\pi_* TK[n] = T_*[n]$ for $n = 0, 1, 2, \ldots$.*

Proof. The prolonged functor T is a homotopy-preserving monad so its homotopy operations are as claimed. □

It is clear that we can also compose homotopy operations over a monad. If $u : \pi_r V \to \pi_s V$ and $v : \pi_s V \to \pi_t V$ are two such operations, their composite vu is another natural operation. If T is homotopy-preserving there is a corresponding composition map

$$\pi_s TK[r] \times \pi_t TK[s] \to \pi_t TK[r]$$
$$(u, v) \mapsto vu.$$

There are several equivalent ways to view this map. For example, we can consider v as a homotopy T-operation which when applied to u gives us an element of $\pi_t T^2 K[r]$. Applying the monad structure map $T^2 \to T$ to this yields vu.

3.2 Cross Effects and Homotopy Operations

3.2.1 Cross Effect Functors

Let T be a functor from the category of k-vector spaces to itself such that $T(0) = 0$ (this condition is not essential but makes things simpler and is the only case we need to consider). Suppose we wish to calculate $T(V \oplus W)$. A first approximation to this is $T(V) \oplus T(W)$. It is easy to see that this is naturally a direct summand of $T(V \oplus W)$. So there is a functor from pairs of vector spaces to vector spaces, which we shall write $\chi_2 T$ such that

$$T(V \oplus W) \cong T(V) \oplus T(W) \oplus \chi_2 T(V, W)$$

naturally in V and W. This functor $\chi_2 T$ is the *second cross-effect functor* of T. If we now want to know about $T(U \oplus V \oplus W)$ we have

$$T(U \oplus V \oplus W) = T(U \oplus V) \oplus T(W) \oplus \chi_2 T(U \oplus V, W)$$

which we can further expand as

$$T(U) \oplus T(V) \oplus T(W) \oplus \chi_2 T(U, V) \oplus \chi_2 T(U, W) \oplus \chi_2 T(V, W) \oplus ?$$

where the ? is in some way the second cross-effect of the second cross-effect. We denote it by $\chi_3 T(U, V, W)$ and call it the *third cross-effect* of T. In a similar way we can define the n^{th} *cross-effect* for any $n \geq 2$ in such a way that

$$T(V_1 \oplus \cdots \oplus V_n) = T(V_1) \oplus \cdots \oplus T(V_n) \oplus \chi_2 T(V_1, V_2) \oplus \cdots \oplus \chi_n T(V_1, \ldots, V_n)$$

where there is one term for each non-empty subset of $\{V_1, \ldots, V_n\}$.

For an element $\sigma \in \Sigma_n$ there is a isomorphism

$$V_1 \oplus \cdots \oplus V_n \to V_{\sigma(1)} \oplus \cdots \oplus V_{\sigma(n)}$$

given by permuting the terms. This induces an isomorphism

$$\chi_n T(V_1, \ldots, V_n) \to \chi_n T(V_{\sigma(1)}, \ldots, V_{\sigma(n)}).$$

This means that the cross-effect $\chi_n T$ is a symmetric function of n variables. If we take $V_1 = \cdots = V_n = V$ then this means that

$$\chi_n T(V) := \chi_n T(V, \ldots, V)$$

comes with a (left) action of the symmetric group Σ_n. More generally, if some but not all of the V_i are the same, then suitable subgroups of Σ_n act on the n^{th} cross-effect. When we start calculating some cross-effects it will be important to keep track of these action.

This definition of the cross-effects of T extends to functors on any abelian category. In particular it works just the same for simplicial vector spaces. Moreover, if T is a functor on vector spaces prolonged to simplicial vector spaces then the cross-effects of the prolongation are just the prolongations of the cross-effects.

Since the functor given by taking homotopy groups commutes with direct sums, it also commutes with cross-effects. Hence

$$\pi_* \chi_2 T(V, W) = \chi_2(\pi_* T)(V, W)$$

for simplicial vector spaces V, W. Put another way, the derived functors of the cross-effects are the cross-effects of the derived functors.

3.2.2 Diagonal and Codiagonal Maps

Let T be a functor on k-vector spaces (or more generally on any abelian category). If V is a vector space, we have maps

$$\Delta : V \to V \oplus V$$

$$\nabla : V \oplus V \to V$$

called the *diagonal* and *codiagonal* respectively (since the direct sum $\oplus$ is both a product and a coproduct in an abelian category). We can apply T to these maps and decompose $T(V \oplus V)$ using the second cross-effect:

$$\Delta : T(V) \to T(V) \oplus T(V) \oplus \chi_2 T(V, V)$$

$$\nabla : T(V) \oplus T(V) \oplus \chi_2 T(V, V) \to T(V)$$

On the two $T(V)$ terms in $T(V \oplus V)$, these maps are the identity. Restricting to the cross-effect term we get

$$\Delta : T(V) \to \chi_2 T(V, V)$$

$$\nabla : \chi_2 T(V,V) \to T(V)$$

We can extend these constructions to higher cross-effects. For $i = 0,\ldots,n$ there is a natural map

$$\Delta_i : V^{\oplus n} \to V^{\oplus n+1}$$

that misses the i^{th} factor and maps as the identity on everything else in order and for $j = 0,\ldots,n$ a natural map

$$\nabla_j : V^{\oplus n} \to V^{\oplus n-1}$$

that is the codiagonal on the j^{th} and $j+1^{\text{th}}$ factors for $j \neq 0, n$, kills the first factor for $j = 0$ and kills the last factor for $j = n$ (and are all the identity on the other factors in order). These are the face and degeneracy maps in a simplicial k-vector space $V^{\oplus}$ with $V^{\oplus}_n = V^{\oplus n}$. In fact, it is easy to see that $V^{\oplus} = V \otimes K[1]$ (where in the latter case we are considering V as the discrete simplicial vector space).

Lemma 3.5 (Dold-Puppe). *The normalized chain complex*

$$N(T(V \otimes K[1]))$$

(with the functor T prolonged to simplicial vector spaces) has terms

$$N(T(V \otimes K[1]))_n \cong \chi_n T(V,\ldots,V)$$

and differential

$$d = \sum_{j=0}^{n} (-1)^j \nabla_j : \chi_n T(V,\ldots,V) \to \chi_{n-1} T(V,\ldots,V).$$

Proof. We can write $T(V^{\oplus n})$ as the direct sum over all non-empty subsets of $\{1,\ldots,n\}$ of the appropriate cross-effect. The image of the degeneracy map $\Delta_i : T(V^{\oplus n-1}) \to T(V^{\oplus n})$ is the direct sum of those terms corresponding to subsets not containing i. Therefore the degenerate group $D(V \otimes K[1])_n$ consists of everything except the term corresponding to the whole of $\{1,\ldots,n\}$, that is, the n^{th} cross-effect $\chi_n T(V,\ldots,V)$. □

This Lemma applies in any abelian category, in particular, to simplicial vector spaces themselves. If V is a simplicial vector space then $V^{\oplus} = V \otimes K[1]$ is the bisimplicial object

$$V^{\oplus}_{p,q} = V_p \otimes K[1]_q.$$

3.2.3 Multivariable Homotopy Operations

Let T be a homotopy-preserving functor on simplicial vector spaces. We have seen how to describe unary homotopy T-operations. We will use cross-effects to study operations on more than one input. We define an *r-ary homotopy T-operation* to be a natural transformation of sets

$$\pi_{i_1}V \times \cdots \times \pi_{i_r}V \to \pi_* TV$$

defined on simplicial vector spaces V.

Proposition 3.6. *The r-ary homotopy T-operations correspond to the elements of the vector spaces*

$$\pi_* T(K[i_1] \oplus \cdots \oplus K[i_r]).$$

Proof. An r-tuple of elements $\{\alpha_k \in \pi_{i_k}V | k = 1, \ldots, r\}$ corresponds to a homotopy class of maps from $K[i_1] \oplus \cdots \oplus K[i_r]$ to V. The result follows by a similar argument to the one for unary operations. $\square$

Inside $\pi_* T(K[i_1] \oplus \cdots \oplus K[i_r])$ we have the r^{th} cross-effect

$$\pi_* \chi_r T(K[i_1], \ldots, K[i_r]) = \chi_r \pi_* T(K[i_1], \ldots, K[i_r]).$$

If an r-ary homotopy T-operation corresponds to an element of this subspace, we will say it is *essential*.

Corollary 3.7. *Let σ be an r-ary homotopy T-operation. Then there are essential operations σ_I, one for each non-empty subset I of $\{1, \ldots, r\}$ of arity equal to the size of I, such that*

$$\sigma(i_1, \ldots, i_r) = \sum_I \sigma_I(i_I)$$

where each operation is acting on the appropriate subset of $\{i_1, \ldots, i_r\}$. Moreover, this decomposition is unique.

Proof. This follows from the cross-effect decomposition for $\pi_* T(K[i_1] \oplus \cdots \oplus K[i_r])$. $\square$

Recall that if some of the i_j are the same and G is the subgroup of Σ_r of elements that fix $(i_1, \ldots, i_r)$ under permutation, then there is an action of G on the cross-effect $\pi_*\chi_r T(K[i_1], \ldots, K[i_r])$. This corresponds to the process of permuting the inputs to an r-ary homotopy T-operation.

We can now begin to express relationships between our operations. For example, suppose that $\delta : \pi_n V \to \pi_* TV$ is a unary homotopy T-operation. The operation

$$(x, y) \mapsto \delta(x + y)$$

is a binary homotopy T-operation. Therefore, we can decompose it as

$$\delta(x + y) = \delta_1(x) + \delta_2(y) + \delta_{12}(x, y)$$

where δ_1, δ_2 are some unary homotopy T-operations and δ_{12} is an essential binary homotopy T-operation.

The operation δ is given by an element $u \in \pi_* TK[n]$ such that $\delta(x) = x_* u$ where x_* is the homomorphism induced by the homotopy class of maps $K[n] \to V$ corresponding to $x \in \pi_n V$. If x, y are elements in $\pi_n V$, the pair (x, y) corresponds to a homotopy class of maps $K[n] \oplus K[n] \to V$. The sum $x + y$ then corresponds to the composite

$$K[n] \to K[n] \oplus K[n] \to V$$

where the first map is the diagonal Δ. Then we have

$$\delta(x + y) = (x, y)_* \Delta(u).$$

This means that the binary operation $(x, y) \mapsto \delta(x + y)$ is represented by the element $\Delta(u) \in \pi_* T(K[n] \oplus K[n])$. To find $\delta_1, \delta_2, \delta_{12}$ above we must decompose this into cross-effects

$$\Delta(u) \in \pi_* T(K[n] \oplus K[n]) = \pi_* TK[n] \oplus \pi_* TK[n] \oplus \chi_2 T(K[n], K[n]).$$

The definition of the diagonal Δ tells us that the first two terms are both u. Therefore we have the formula

$$\delta(x + y) = \delta(x) + \delta(y) + \delta_\Delta(x, y)$$

where we have written δ_Δ for the essential operation corresponding to $\Delta(u) \in \chi_2 T(K[n], K[n])$. We can of course generalise this to the sum of r elements:

Proposition 3.8. *Let δ be a homotopy T-operation $\pi_n V \to \pi_* TV$ given by $u \in \pi_* TK[n]$ and $x_1, \ldots, x_r \in \pi_n V$. Then we have*

$$\delta(\sum_i x_i) = \sum_i \delta(x_i) + \sum_{|I|\geq 2} \delta_I(x_I)$$

where the final sum is taken over all subsets I of $\{1, \ldots, r\}$ of size at least two, $\delta_{|I|}$ denotes the $|I|$-ary operation given by the image of u under the diagonal map

$$\Delta : \pi_* TK[n] \to \pi_* \chi_{|I|} T(K[n], \ldots, K[n])$$

and x_I denotes the appropriate subset of $\{x_1, \ldots, x_r\}$.

We could generalise this further to applications of higher arity operations to sums but the notation would become horrible and the pattern can readily be reproduced if necessary.

We have seen one way to relate operations of different arities. Another is given by repeating the inputs to an operation. For example, if σ is a binary homotopy T-operation $\pi_n V \times \pi_n V \to \pi_* TV$, then

$$x \mapsto \sigma(x, x)$$

gives us a unary operation. Suppose σ is given by the element $u \in \pi_* T(K[n] \oplus K[n])$. If the map $K[n] \to V$ represents $x \in \pi_n V$ then the composite

$$K[n] \oplus K[n] \to K[n] \to V$$

represents the pair (x, x) where $K[n] \oplus K[n] \to K[n]$ is the codiagonal ∇. Therefore the unary operation $\sigma(x, x)$ corresponds to the element $\nabla(u) \in \pi_* TK[n]$. This generalizes to higher arities in the obvious way.

As with unary operations, if T is a monad, higher arity homotopy T-operations correspond to homotopy operations on T-algebras. We can define composition for these higher arity operations and relate it to the groups $\pi_* T(K[i_1] \oplus \cdots \oplus K[i_r])$ that define them. It doesn't seem worth the notation to write down a general statement of this form and we will go into more details when we work with operads later on.

3.3 Cohomotopy operations

We would like to talk about operations for comonads as well as monads and for this we need to develop a dual theory. Here we summarize this. First we define the *cohomotopy groups* of a simplicial k-vector space V to be the duals of the homotopy groups

$$\pi^n V = \mathrm{Hom}_k(\pi_n V, k).$$

The homotopy group functors are represented by the spheres $K[n]$ and it turns out that the cohomotopy group functors are corepresented bvy the spheres.

Lemma 3.9. *There is a 1-1 correspondence between elements of $\pi^n V$ and homotopy classes of maps $V \to K[n]$.*

Proof. The cohomotopy groups of V are equal to the cohomology groups of the dual cochain complex of NV. The elements of these are given by chain homotopy classes of maps $NV \to [n]$ and the claim then follows from the Dold-Kan correspondence. □

Now let T be a functor on simplicial k-vector spaces. Then we define a *cohomotopy T-operation* to be a natural transformation of sets

$$\pi^n V \to \pi^* TV.$$

If T is homotopy-preserving, we can show that these correspond to elements of the cohomotopy groups $\pi^* TK[n]$. To be explicit, if $u \in \pi^* TK[n]$ and we take $\alpha \in \pi^n V$, then thinking of α as a map $V \to K[n]$ we get an induced map

$$\pi^* TK[n] \to \pi^* TV.$$

The image of α under the operation corresponding to u is then defined to be the image of u under the this map. This shows that the vector space of cohomotopy T-operations is dual to the vector space of homotopy T-operations.

Now suppose that T is also a comonad. That is, it comes equipped with natural transformations $T \to T^2$ and $T \to 1$ that make it into a comonoid in the monoidal category of functors on simplicial vector spaces and composition. A T-coalgebra V then comes with a map $V \to TV$. Taking the cohomotopy of this we get a map $\pi^* TV \to \pi^* V$. Composing this with a cohomotopy T-operation we get a map

$$\pi^n V \to \pi^* V$$

that is natural in T-coalgebras V and so can be called a *cohomotopy operation for T-coalgebras*. Notice that because our operations need not be linear, we can not get a homotopy operation (or more precisely a homotopy cooperation) just by taking the dual.

We can go to higher arities for cohomotopy operations as well. A pair of cohomotopy classes (x, y) with $x \in \pi^m V$ and $y \in \pi^n V$ corresponds to a homotopy class of maps $V \to K[m] \oplus K[n]$. Therefore, elements of $\pi^* T(K[m] \oplus K[n])$ define binary cohomotopy T-operations

$$\pi^m V \times \pi^n V \to \pi^* TV$$

and so on. As in the homotopy, we can define essential operations to be those that correspond to elements in the highest cross-effect and we have a similar decomposition theorem. If T is a comonad, these higher arity cohomotopy T-operations give us higher arity cohomotopy operations for T-coalgebras.

4 Homotopy Operations for Operads

4.1 Operads on k-vector spaces

4.1.1 Operads and Cooperads

Let k be a field. An *operad over k* consists of a sequence $\mathcal{P}(n)$ of k-vector spaces together with:

- an action of the symmetric group Σ_n on $\mathcal{P}(n)$;
- a *unit element* $1 \in \mathcal{P}(1)$;
- *composition maps*

  $$\mathcal{P}(r) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_r) \to \mathcal{P}(i_1 + \cdots + i_r)$$

  for any integers $r, i_1, \ldots, i_r \geq 0$ (all tensor products are over k unless otherwise specified);

such that:

1. the composition maps are $\Sigma_{i_1} \times \cdots \times \Sigma_{i_r}$-equivariant (where this group acts on $\mathcal{P}(i_1 + \cdots + i_r)$ by the natural inclusion into $\Sigma_{i_1+\cdots+i_r}$;

2. the composition maps and unit element satisfy certain 'natural' identities.

An operad is meant to encode some kind of 'algebraic structure' that might exist on a vector space. The elements of $\mathcal{P}(n)$ desribe the various n-ary operations that exist within the structure. The unit element represents the identity operation. The composition maps describe how the operations of different arity relate to one another. The 'natural' identities are the ones you would expect to hold given this interpretation. The Σ_n action on $\mathcal{P}(n)$ tells us how the operations react when we permute the inputs.

We can define operads on any symmetric monoidal category, just be replacing the tensor product of vector spaces with the given monoidal product (and the unit element with a map from the unit of the monoidal structure to $\mathcal{P}(1)$).

There is a notion of a cooperad is dual to that of an operad. A cooperad $\mathcal{Q}$ consists of a sequence of vector spaces $\mathcal{Q}(n)$ together with a Σ_n action on $\mathcal{Q}(n)$, a counit map $\mathcal{Q}(1) \to k$ and $\Sigma_{i_1} \times \cdots \times \Sigma_{i_r}$-equivariant cocomposition maps

$$\mathcal{Q}(i_1 + \cdots + i_r) \to \mathcal{Q}(r) \otimes \mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_r).$$

If $\mathcal{P}$ is an operad then there is a corresponding cooperad $\mathcal{P}^*$ with

$$\mathcal{P}^*(n) = \mathcal{P}(n)^*.$$

4.1.2 Operads as monoids for symmetric sequences

There is a more abstract way to define an operad that will help us understand how they give rise to monads. First let us define the category of symmetric sequences. We will work over the category of k-vector spaces but all this works in a general symmetric monoidal category (with sufficient limits/colimits). A *symmetric sequence* is a sequence of vector spaces $\mathcal{P}(n)$ for $n \geq 0$ together with an action of Σ_n on $\mathcal{P}(n)$. Thus an operad is a symmetric sequence with extra structure given by the unit and composition maps. Maps between symmetric sequences are sequences of Σ_n-equivariant maps $\mathcal{P}(n) \to \mathcal{Q}(n)$. This gives us a category.

Define the tensor product of two symmetric sequences by

$$(\mathcal{P} \otimes \mathcal{Q})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \mathcal{P}(i) \otimes \mathcal{Q}(j)$$

where Ind denotes the induced representation. The tensor product gives us a symmetric monoidal product on the category of symmetric sequences. The tensor power of a symmetric sequence is then

$$\mathcal{Q}^{\otimes r}(n) = \bigoplus_{i_1+\cdots+i_r=n} \mathrm{Ind}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_r}}^{\Sigma_n} \mathcal{Q}(i_1)\otimes\ldots\otimes\mathcal{Q}(i_r)$$

and the symmetric group Σ_r acts on this by permuting the r factors in each tensor product (and hence likely sending a term in the direct sum to a different term). Of course there is also a more 'internal' Σ_n action that is Σ_r-equivariant. Using these tensor powers we define another product, the *composition product*, by

$$(\mathcal{P}\circ\mathcal{Q})(n) = \bigoplus_{r\geq 0}(\mathcal{P}(r)\otimes\mathcal{Q}^{\otimes r}(n))_{\Sigma_r}$$

where Σ_r has the diagonal action on $\mathcal{P}(r)\otimes\mathcal{Q}^{\otimes r}(n)$ and we are taking the coinvariants of this $k\Sigma_r$-module.

The composition product is a (nonsymmetric) monoidal product on the category of symmetric sequences with unit given by the symmetric sequence $\mathcal{I}$ that has $\mathcal{I}(1) = k$ and all other $\mathcal{I}(n) = 0$. An operad is precisely a monoid for this monoidal structure. The unit map $\mathcal{I}\to\mathcal{P}$ determines the unit element $1\in\mathcal{P}(1)$ and a product $\mathcal{P}\circ\mathcal{P}\to\mathcal{P}$ gives us the operad composition maps.

We can define another monoidal product on the category of symmetric sequences by taking the invariants instead of the coinvariants. Fresse calls this the *composition product with divided symmetries*

$$(\mathcal{P}\tilde{\circ}\mathcal{Q})(n) = \bigoplus_{r\geq 0}(\mathcal{P}(r)\otimes\mathcal{Q}^{\otimes r}(n))^{\Sigma_r}.$$

A cooperad is then precisely a comonoid for this monoidal product.

The trace map gives a natural transformation of symmetric sequences

$$\mathrm{Tr}:\mathcal{P}\circ\mathcal{Q}\to\mathcal{P}\tilde{\circ}\mathcal{Q}$$

(natural in $\mathcal{P}$ and $\mathcal{Q}$). Fresse shows that if $\mathcal{Q}$ is *connected* ($\mathcal{Q}(0) = 0$) then Tr is an isomorphism of symmetric sequences. Therefore, connected operads are also monoids for this monoidal structure.

4.1.3 The monads associated to an operad

Recall that a monad on $\mathcal{C}$ is a monoid for the category of functors on $\mathcal{C}$ with the composition product. We get monads from operads by defining a monoidal functor from symmetric sequences with composition product to functors on $\mathcal{C}$ with composition product. Monoids for the symmetric sequences (ie operads) will then map to monoids for functors (ie monads).

Given a symmetric sequence $\mathcal{P}$ (on the category of k-vector spaces), define a functor $S(\mathcal{P})$ on vector spaces by

$$S(\mathcal{P})(V) = \bigoplus_{n \geq 0} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

where we are taking the Σ_n-coinvariants. Notice that $S(\mathcal{I})$ is the identity functor. There is also a natural isomorphism

$$S(\mathcal{P} \circ \mathcal{P}) \cong S(\mathcal{P}) \circ S(\mathcal{P})$$

that is coherently associative and unital. That is, the diagrams

$$\begin{array}{ccc} S(P \circ P \circ P) & \longrightarrow & S(\mathcal{P} \circ \mathcal{P}) \circ S(\mathcal{P}) \\ \downarrow & & \downarrow \\ S(\mathcal{P}) \circ S(\mathcal{P} \circ \mathcal{P}) & \longrightarrow & S(\mathcal{P}) \circ S(\mathcal{P}) \circ S(\mathcal{P}) \end{array}$$

and

$$\begin{array}{ccc} S(\mathcal{I} \circ \mathcal{P}) & \longrightarrow & S(\mathcal{I}) \circ S(\mathcal{P}) \\ & \searrow & \| \\ & & S(\mathcal{P}) \end{array}$$

where the diagonal map is induced by the isomorphism $\mathcal{I} \circ \mathcal{P} \cong \mathcal{P}$.

This means that S gives us a monoidal functor from symmetric sequences to functors on vector spaces. If $\mathcal{P}$ is an operad applying S to its product and unit maps give us product and unit maps for $S(\mathcal{P})$ which is therefore a monad.

Similarly, we define for a symmetric sequence $\mathcal{Q}$ a functor $\Gamma(\mathcal{Q})$ by

$$\Gamma(\mathcal{Q})(V) = \bigoplus_{n \geq 0} (\mathcal{Q}(n) \otimes V^{\otimes n})^{\Sigma_n}$$

where now we take the Σ_n-invariants. Again we have $\Gamma(\mathcal{I})$ equal to the identity and a natural isomorphism

$$\Gamma(\mathcal{Q}\tilde{\circ}\mathcal{Q}) \cong \Gamma(\mathcal{Q}) \circ \Gamma(\mathcal{Q}).$$

Therefore Γ is a monoidal functor from the category of symmetric sequences with the product $\tilde{circ}$ to functors on vector spaces. So a cooperad structure on $\mathcal{Q}$ gives us a comonad structure on $\Gamma(\mathcal{Q})$.

If $\mathcal{P}$ is a connected operad, recall that the trace map is an isomorphism $\mathcal{P} \circ \mathcal{P} \cong \mathcal{P}\tilde{\circ}\mathcal{P}$. The composition

$$\Gamma(\mathcal{P}) \circ \Gamma(\mathcal{P}) \cong \Gamma(\mathcal{P}\tilde{\circ}\mathcal{P}) \cong \Gamma(\mathcal{P} \circ \mathcal{P}) \to \Gamma(\mathcal{P})$$

then gives $\Gamma(\mathcal{P})$ a monad structure. Similarly, if $\mathcal{Q}$ is a connected cooperad then $S(\mathcal{Q})$ has a comonad structure.

An *algebra* for the operad $\mathcal{P}$ is defined to be precisely an algebra for the monad $S(P)$, that is, a vector space V together with a linear map $S(\mathcal{P})(V) \to V$ satisfying certain conditions. For any vector space V, $S(\mathcal{P})$ is an algebra for $S(\mathcal{P})$ and is called the *free $\mathcal{P}$-algebra* on V. A *coalgebra* for the cooperad $\mathcal{Q}$ is a coalgebra for the comonad $\Gamma(\mathcal{Q})$. For any V, $\Gamma(\mathcal{Q})$ is the *free $\mathcal{Q}$-coalgebra* on V.

4.2 Homotopy Operations for Operads

We have seen how to describe homotopy operations for monads (and cohomotopy operations for comonads) on simplicial vector spaces. Let us apply this to the monads arising from operads (and comonads arising from cooperads). To ensure that all our functors are homotopy-preserving we will concentrate for now on prolongations of operads for vector spaces.

So let $\mathcal{P}$ be an operad on the category of k-vectors spaces. We have defined the associated monad

$$S(\mathcal{P})(V) = \bigoplus_{n \geq 0} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}.$$

We prolong this to a monad on simplicial vector spaces and (because the tensor product of simplicial vector spaces is just the prolongation of that for vector spaces) the same formula holds for simplicial vector spaces V.

The algebras over this prolongation are the same thing as simplicial objects in the category of algebras over the original operad. We will refer to them as *simplicial $\mathcal{P}$-algebras*. We saw earlier that to describe the natural operations on the homotopy groups of these algebras we have to calculate the graded vector spaces

$$\pi_* S(\mathcal{P})K[r].$$

Because taking homotopy groups preserves direct sums these break up into direct sums

$$\bigoplus_{n\geq 0} \pi_*((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}).$$

For the purposes of finding the homotopy operations, we can therefore concentrate on these terms one at a time. We will say that a homotopy operation is of *weight* n if it lives in the n^{th} term in this sum. When we come to want to know how to compose these operations, we will have to take into account the composition map in the operad which relates the different values of n.

To calculate the cohomotopy operations on simplicial coalgebras over a cooperad we have to calculate the same groups but with invariants taken instead of coinvariants.

To begin the study of these let us make the following definition. Let n be a fixed integer and G a subgroup of the symmetric group Σ_n. Let M be a (left) kG-module. Define functors on simplicial k-vector spaces by

$$M^G(V) = (M \otimes V^{\otimes n})^G$$

and

$$M_G(V) = (M \otimes V^{\otimes n})_G$$

where the coinvariants and invariants of a simplicial kG-module are the prolongations of the constructions for kG-modules. This is dangerous notation and we will try not to confuse the functor M^G with the group of invariants of M, also denoted M^G. To make things even more complicated we will also use M^G to denote the functor of n simplicial vector spaces given by

$$M^G(V_1, \ldots, V_n) = (M \otimes V_1 \otimes \ldots \otimes V_n)^G$$

where enough of the V_i are the same that G preserves the sequence $V_1, \ldots, V_n$ when it acts by permutation. Similarly for M_G.

For much of this paper we will be concerned with what can be said in general about the (co)homotopy groups of the simplicial vector spaces$M^G(K[r])$ and $M_G(K[r])$ and $M^G(K[r_1], \ldots, K[r_n])$ and $M_G(K[r_1], \ldots, K[r_n])$ and the maps between them induced by changing the group G or the module M.

4.2.1 Multivariable Operations

Recall that to describe homotopy operations of higher arities, we have to calculate the cross-effects of our chosen functor. In this section we will calculate the cross-effects of the functors

$$V \mapsto M^G V = (M \otimes V^{\otimes n})^G$$

and

$$V \mapsto M_G V = (M \otimes V^{\otimes n})_G$$

where, as before, G is a subgroup of Σ_n and M is a kG-module. This will therefore tell is how to find multivariable homotopy operations on simplicial algebras over an operad (or simplicial coalgebras over a cooperad).

The functors M^G and M_G are homogeneous of degree n in the sense of Bousfield and so their cross-effects have certain decompositions. We will see that these decompositions allow us to write the cross-effects in terms of functors M^H (and respectively M_H) for other subgroups H of G. We get these subgroups as follows.

Fix the subgroup G of Σ_n and fix a positive integer r and take an ordered n-tuple $I = \{i_1, \dots, i_n\}$ of elements from the set $\{1, \dots, r\}$. Say that such an n-tuple is *essential* if every integer from 1 up to r occurs at least once in it and denote the set of such by $\langle n, r\rangle$. (This use of the term 'essential' mirrors that used previously to describe multivariable homotopy operations.) Thus the essential sequences correspond to surjective functions from the set of the first n positive integers to the set of the first r. The group G acts (on the left) on $\langle n, r\rangle$ by permuting the entries of each I. This is really in some sense a right action by the inverse so we will write it $I\sigma^{-1} = \{i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(n)}\}$. Let us write $\mathrm{stab}_G(I)$ for the set of elements of G that fix I under this action. Denote by $G\langle n, r\rangle$ the set of orbits of this G-action.

Proposition 4.1. *Let G be a subgroup of Σ_n and let M be a kG-module. If the functors M^G and M_G are as defined previously then there are isomorphisms:*

$$\chi_r M^G(V_1, \dots, V_r) \cong \bigoplus_{[I]\in G\langle n,r\rangle} (M \otimes V_{i_1} \otimes \dots \otimes V_{i_n})^{\mathrm{stab}_G(I)}$$

and

$$\chi_r M_G(V_1, \dots, V_r) \cong \bigoplus_{[I]\in G\langle n,r\rangle} (M \otimes V_{i_1} \otimes \dots \otimes V_{i_n})_{\mathrm{stab}_G(I)}.$$

The sums here are taken over all orbits of the action of G on the set $\langle n, r\rangle$ of essential n-tuples and we have chosen a representative member I of each orbit $[I]$.

The isomorphisms we obtain depend on which representatives we choose for these orbits. When we write an element of one of these terms we will use a subscript to denote which particular I has been chosen to give us the isomorphism. For example,

$$[m \otimes v_1 \otimes \ldots \otimes v_n]_I$$

denotes the element in the $[I]$ term of the direct sum, corresponding to the element $[m \otimes \ldots \otimes v_n]$ of the group of coinvariants where I has been used to obtain the isomorphism. It will follow from the proof of the Proposition that if $J = I\sigma^{-1}$ for $\sigma \in G$ (so that I and J are in the same orbit) then we have

$$[m \otimes v_1 \otimes \ldots \otimes v_n]_I = [\sigma(m) \otimes v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}]_J.$$

The same formula holds for the groups of invariants. Notice that the corresponding stabilizers are conjugate subgroups so the groups of (co)invariants are isomorphic whatever representatives we choose for the orbits.

Corollary 4.2.

$$\chi_r M^G(V) \cong \chi_r M^G(V, \ldots, V) \cong \bigoplus_{[I]\in G\langle n,r\rangle} M^{\mathrm{stab}_G(I)}(V)$$

$$\chi_r M_G(V) \cong \chi_r M^G(V, \ldots, V) \cong \bigoplus_{[I]\in G\langle n,r\rangle} M_{\mathrm{stab}_G(I)}(V)$$

We can also describe the Σ_r action on these r^{th} cross-effects. The group Σ_r acts on $\langle n, r\rangle$ (on the left) by

$$\pi : (i_1, \ldots, i_n) \mapsto (\pi(i_1), \ldots, \pi(i_n)) = \pi I.$$

Note the difference between this and the action of Σ_n on $\langle n, r\rangle$ defined above. From the definition of this action it follows that

$$\mathrm{stab}_G(I) = \mathrm{stab}_G(\pi I).$$

The action of $\pi \in \Sigma_r$ on the r^{th} cross-effects is then given by the identity maps

$$x_I \mapsto x_{\pi I}.$$

Of course if the sequence πI was not one of those we chose to obtain the isomorphisms in the Corollary, we have to act by an element of G to match things up.

For example, let us suppose that $n = r$ and that $G = \Sigma_n$. Then the r^{th} cross-effects just have one term:

$$\chi_r M_{\Sigma_n}(V) \cong M \otimes V^{\otimes n}$$

where we need to have picked $I \in \langle n, n \rangle$ to get this isomorphism. The action of Σ_r on this is

$$[m \otimes v_1 \otimes \ldots \otimes v_n]_I \mapsto [m \otimes v_1 \otimes \ldots \otimes v_n]_{\pi^{-1}I} = [\sigma(m) \otimes v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}]_I$$

where $\sigma \in \Sigma_n$ is chosen such that $(\pi I)\sigma^{-1} = I$. The relationship between σ and π will depend on which I we chose, but if we make the natural choice $I = (1, 2, \ldots, n)$ then we get $\sigma = \pi$ and hence the action of Σ_r on $\chi_r M_{\Sigma_n}$ is given by

$$m \otimes v_1 \otimes \ldots \otimes v_n \mapsto \pi(m) \otimes v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(n)}$$

which happens to agree with the action of Σ_n we have been considering all along.

This Proposition tells us that, as advertised, the cross-effects of the functors M^G and M_G can be decomposed in terms of M^H and M_H for other groups H (that are in fact subgroups of G). Notice in particular that only the first n cross-effects are nonzero (in other words M^G and M_G are functors of degree n). Before giving the proof of this result, let us see some example of these decompositions.

Example 1 **($G = \Sigma_2$).** *Only the second cross-effect will be non-zero. The two essential sequences with $n = r = 2$ are 21 and 12 and G switches these. THerefore there is one orbit and the stabilizer is trivial. So we have:*

$$\chi_2 M^{\Sigma_2}(V, W) = M \otimes V \otimes W$$

and similarly for M_{Σ_2}.

Example 2 **(**$G = \Sigma_3$**).** *Now there are second and third cross-effects given by:*

$$\chi_2 M^{\Sigma_3}(V, W) = (M \otimes V^2 W)^{\Sigma_2 \times \Sigma_1} \oplus (M \otimes V W^2)^{\Sigma_1 \times \Sigma_2}$$

and

$$\chi_3 M^{\Sigma_3}(U, V, W) = M \otimes U \otimes V \otimes W$$

where in the first case we have supressed some of the tensor symbols.

Example 3 **(**$G = \Sigma_2 \times \Sigma_2 \subset \Sigma_4$**).** *To get the second cross-effect we consider 4-tuples of elements that are either* 1 *or* 2*. There are 14 of these in which* 1 *and* 2 *each appears at least once. Under the action of* G *there are 7 orbits with representatives and stabilizers as follows:*

1112	$\langle(12)\rangle$
1211	$\langle(34)\rangle$
1122	G
2211	G
1212	$\{e\}$
1222	$\langle(34)\rangle$
2212	$\langle(12)\rangle$

Therefore the second cross-effect is the direct sum of seven corresponding terms. For the third cross-effect there are twelve terms, three with stabilizer $\langle(12)\rangle$*, three with* $\langle(34)\rangle$ *and six with trivial stabilizer. The fourth cross-effect has six terms all with trivial stabilizer.*

This example shows that some of the groups appearing in our decomposition may be equal to G *itself.*

Proof of Proposition 4.1. This is a straightforward calculation. Multiplying out the tensor products we get:

$$(M \otimes (V_1 \oplus \cdots \oplus V_r)^{\otimes n} = \bigoplus_I (M \otimes V_{i_1} \otimes \ldots \otimes V_{i_n})$$

where the sum is taken over all ordered n-tuples of integers from the set $\{1, \ldots, r\}$. It is easy to see that the r^{th} cross-effect will be the sum of those terms in which every integer from 1 to r appears at least once, that is, over $I \in \mathcal{I}_r$. To see the action of G on this thing we must collect terms corresponding to I in the same G orbit. To apply the invariants we can use the identity

$$\left(\bigoplus_{J\in[I]} M\otimes V_{j_1}\otimes\ldots\otimes V_{j_n}\right)^G \cong (M\otimes V_{i_1}\otimes\ldots\otimes V_{i_n})^{\mathrm{stab}_G(I)}$$

induced by the projection from the direct sum to the term corresponding to I. Explicitly, an element of the LHS is an element of the tensor product for each J such that if the J term is

$$\sum m\otimes v_1\otimes\ldots\otimes v_n$$

then the $\sigma(J)$ term is equal to

$$\sum \sigma(m)\otimes v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}.$$

Picking out the I term gives us an isomorphism to the RHS.

For the corresponding result for coinvariants, we use an identity

$$\left(\bigoplus_{J\in[I]} M\otimes V_{j_1}\otimes\ldots\otimes V_{j_n}\right)_G \cong (M\otimes V_{i_1}\otimes\ldots\otimes V_{i_n})_{\mathrm{stab}_G(I)}$$

where something in the J term on the LHS maps to the RHS by applying $\sigma\in G$ such that $J\sigma^{-1}=I$.

In each case, summing over all the orbits gives the claimed result. □

We saw before that there were relations between the homotopy operations of different arities determined by the diagonal and codiagonal maps relating the cross-effects. The following result tells us what these are under the decompositions of Proposition 4.1.

Proposition 4.3. *The diagonal map $M^G(V)\to\chi_2M^G(V,V)$ is given by the sum of the restriction maps ρ*

$$(M\otimes V^{\otimes n})^G \to \bigoplus_{[I]\in G\langle n,2\rangle}(M\otimes V^{\otimes n})^{\mathrm{stab}_G(I)}$$

and the codiagonal $\chi_2M^G(V,V)\to M^G(V)$ by the sum of the transfers τ

$$\bigoplus_{[I]\in G\langle n,2\rangle}(M\otimes V^{\otimes n})^{\mathrm{stab}_G(I)}\to(M\otimes V^{\otimes n})^G.$$

The corresponding maps for the coinvariants functors are given by the transfers and restrictions respectively.

Example 4 ($G = \Sigma_2$). *Recall that the second cross-effect of M^G (and indeed of M_G) is just $(V, W) \mapsto M \otimes V \otimes W$. The diagonals are then the inclusion*

$$\Delta : (M \otimes V \otimes V)^{\Sigma_2} \hookrightarrow (M \otimes V \otimes V)$$

and the transfer

$$\begin{aligned}\Delta : (M \otimes V \otimes V)_{\Sigma_2} &\longrightarrow (M \otimes V \otimes V)\\ [m \otimes v \otimes v'] &\longmapsto (m \otimes v \otimes v') + (\sigma m \otimes v' \otimes v)\end{aligned}$$

where σ is the non-trivial element of Σ_2. The codiagonals are the transfer

$$\begin{aligned}\nabla : (M \otimes V \otimes V) &\longrightarrow (M \otimes V \otimes V)^{\Sigma_2}\\ (m \otimes v \otimes v') &\longmapsto (m \otimes v \otimes v') + (\sigma m \otimes v' \otimes v)\end{aligned}$$

and the projection

$$\nabla : (M \otimes V \otimes V) \longrightarrow (M \otimes V \otimes V)_{\Sigma_2}$$

To understand how these results affect our homotopy operations we need to know the maps induced by transfer and restriction on homotopy groups. This will need to wait for better calculations of the groups themselves.

4.2.2 Composition of Operations

The composition maps in an operad determine the composition natural transformation in the corresponding monad and this in turn tells us how to compose our homotopy operations. In this section we will describe the resulting composition in terms of the homotopy groups

$$\pi_*((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n})$$

that tell us what the homotopy operations are in the first place. Suppose we have homotopy operations determined by elements

$$u \in \pi_s((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}) \subset \pi_s S(\mathcal{P})(K[r])$$

and

$$v \in \pi_t((\mathcal{P}(m) \otimes K[s]^{\otimes m})_{\Sigma_m}) \subset \pi_t S(\mathcal{P})(K[s]).$$

By definition, this means that u is of weight n and v is of weight m. Then the composition vu is of weight mn and is determined as follows. We can think of u as a map (or rather a homotopy class of maps)

$$K[s] \to (\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}$$

which then induces a map

$$(\mathcal{P}(m) \otimes K[s]^{\otimes})_{\Sigma_m} \to (\mathcal{P}(m) \otimes ((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n})^{\otimes m})_{\Sigma_m}.$$

We can take the image of v under the induced map of homotopy groups to get an element of

$$\pi_t((\mathcal{P}(m) \otimes ((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n})^{\otimes m})_{\Sigma_m})$$

which we will call the *(pre-)composition* of u and v. This is as far as we can get without using the monad structure on $S(\mathcal{P})$ (that is, the operad structure on $\mathcal{P}$). This structure gives us a map from the above homotopy group (which is just a direct summand of $\pi_t S(\mathcal{P})^2(K[r])$) to $\pi_t S(\mathcal{P})(K[r])$ and the image of the pre-composition of u and v is an element that is really the composition of u and v as operations.

We can describe what is going on in terms of the composition maps of the operad $\mathcal{P}$. Firstly, there is an isomorphism

$$(\mathcal{P}(m) \otimes ((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n})^{\otimes m})_{\Sigma_m} \cong (\mathcal{P}(m) \otimes \mathcal{P}(n)^{\otimes m} \otimes K[r]^{\otimes mn})_{\Sigma_m \wr \Sigma_n}$$

where $\Sigma_m \wr \Sigma_n$ is the wreath product, the subgroup of Σ_{mn} generated by $\Sigma_n \times \cdots \times \Sigma_n$ together with the action of Σ_m on the m blocks of n in $\{1, \ldots, mn\}$.

Secondly, the operad structure gives us a $\Sigma_m \wr \Sigma_n$-equivariant map

$$\mathcal{P}(m) \otimes \mathcal{P}(n)^{\otimes m} \to \mathcal{P}(mn)$$

that induces a map

$$(\mathcal{P}(m) \otimes \mathcal{P}(n)^{\otimes m} \otimes K[r]^{\otimes nm})_{\Sigma_m \wr \Sigma_n} \to (\mathcal{P}(mn) \otimes K[r]^{mn})_{\Sigma_m \wr \Sigma_n}.$$

Thirdy, there is a restriction map

$$(\mathcal{P}(mn) \otimes K[r]^{\otimes mn})_{\Sigma_m \wr \Sigma_n} \to (\mathcal{P}(mn) \otimes K[r]^{\otimes mn})_{\Sigma mn}.$$

Lemma 4.4. *The map (induced by the monad structure on $S(\mathcal{P})$) that sends the pre-composition of u and v to the actual composition of u and v is the result of combining the three maps described above:*

$$\begin{array}{c}(\mathcal{P}(m) \otimes \mathcal{P}(n)^{\otimes m} \otimes K[r]^{\otimes nm})_{\Sigma_m \wr \Sigma_n} \\ \downarrow \\ (\mathcal{P}(mn) \otimes K[r]^{\otimes mn})_{\Sigma_m \wr \Sigma_n} \\ \rho \downarrow \\ (\mathcal{P}(mn) \otimes K[r]^{\otimes mn})_{\Sigma_{mn}}\end{array}$$

Proof. This is a reflection of how the monad structure of $S(\mathcal{P})$ relates to the composition maps for the operad $\mathcal{P}$. □

Similar arguments apply to the composition of homotopy operations of higher arities.

The simplicial vector spaces involved in the Lemma are all of the form $(M \otimes K[r]^{\otimes N})_G$, so we are again reduced to calculating the homotopy groups of such things and the maps on homotopy induced by

1. maps of kG-modules $M \to M'$;
2. transfer and reduction associated to group homomorphisms $G \to H$.

Recall that if $\mathcal{P}$ is a connected operad then there is another monad $\Gamma(\mathcal{P})$ associated to it with

$$\Gamma(\mathcal{P})(V) = \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})^{\Sigma_n}.$$

We get a similar answer when we try to calculate the composition of homotopy operations for this monad. Here instead we have the transfer for groups of invariants at the last stage.

There is a corresponding analysis for cooperads involving the restriction for groups of invariants (for the $\Gamma(\mathcal{Q})$ operations) and, if $\mathcal{Q}$ is connected, the transfer (for the $S(\mathcal{Q})$ operations).

4.3 Basic Calculations

4.3.1 The $\mathcal{P}$-algebra structure on a simplicial $\mathcal{P}$-algebra

Classically, it is the Eilenberg-Zilber Theorem that allows us to get a product on the homology groups of a space X when we are given a product $X \times X \to X$. In the same way, if V is a simplicial algebra over the operad $\mathcal{P}$ then there is a $\mathcal{P}$-algebra structure on the graded vector space $\pi_* V$. This gives us homotopy operations which we can interpret in terms of the groups $\pi_*((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}$.

Let $\mathcal{P}$ be an operad on k-vector spaces. The category of graded k-vector spaces has two standard choices of symmetric monoidal product. We choose the one for which the symmetry isomorphism includes a sign. That is,

$$T : V \otimes W \to W \otimes V$$

given by

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

The reason for this is that it makes the natural functor from graded vector spaces to chain complexes (given by taking zero differential) into a symmetric monoidal functor. We can then extend the operad $\mathcal{P}$ to an operad on graded k-vector spaces simply by treating each $\mathcal{P}(n)$ as a graded vector space concentrated in position zero.

Now let V be a simplicial k-vector space. Since k is a field, we can choose a map of chain complexes $\pi_* V \to NV$ ($\pi_* V$ considered as a chain complex with zero differential) that induces the identity map on homology. From this we get a Σ_n-equivariant map of chain complexes

$$\mathcal{P}(n) \otimes (\pi_* V)^{\otimes n} \to \mathcal{P}(n) \otimes N(V)^{\otimes n}$$

for each $n \geq 0$. We can compose this with the Eilenberg-Zilber map to get a Σ_n-equivariant map

$$\mathcal{P}(n) \otimes (\pi_* V)^{\otimes n} \to N(\mathcal{P}(n) \otimes V^{\otimes n}).$$

Taking the homology of the induced map on the groups of coinvariants we get

$$(\mathcal{P}(n) \otimes (\pi_* V)^{\otimes n})_{\Sigma_n} \to \pi_*((\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}).$$

These maps do not depend on the initial choice of $\pi_* V \to NV$ since any two such choices are chain homotopic. Summing over n we get

$$S(\mathcal{P})(\pi_* V) \to \pi_* S(\mathcal{P})(V)$$

where on the LHS, $S(\mathcal{P})$ stands for the monad associated to the extension of $\mathcal{P}$ to graded vector spaces, and on the RHS, for the monad associated to the prolongation of $\mathcal{P}$ to simplicial vector spaces.

Lemma 4.5. *The following diagrams commute:*

$$\begin{array}{ccccc} S(\mathcal{P})S(\mathcal{P})(\pi_* V) & \longrightarrow & S(\mathcal{P})\pi_* S(\mathcal{P})(V) & \longrightarrow & \pi_* S(\mathcal{P})S(\mathcal{P})(V) \\ \downarrow & & & & \downarrow \\ S(\mathcal{P})(\pi_* V) & & \longrightarrow & & \pi_* S(\mathcal{P})(V) \end{array}$$

where the vertical arrows are given by the composition for $S(\mathcal{P})$ and

$$\begin{array}{ccc} \pi_* V & = & \pi_* V \\ \downarrow & & \downarrow \\ S(\mathcal{P})(\pi_* V) & \longrightarrow & \pi_* S(\mathcal{P})(V) \end{array}$$

where the vertical arrows are given by the unit for $S(\mathcal{P})$.

Corollary 4.6. *Suppose V is a simplicial $\mathcal{P}$-algebra. Then the graded k-vector space $\pi_* V$ is a $\mathcal{P}$-algebra.*

This means that the elements of $\mathcal{P}(n)$ (which define n-ary operations on $\mathcal{P}$-algebras) define n-ary homotopy operations for simplicial $\mathcal{P}$-algebras and hence elements of the homotopy groups of the corresponding cross-effects. This is reflected in the fact that the n^{th} cross-effect of the functor

$$\mathcal{P}(n)_{\Sigma_n} : V \mapsto (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

is (isomorphic to)

$$(V_1, \ldots, V_n) \mapsto \mathcal{P}(n) \otimes V_1 \otimes \ldots \otimes V_n$$

Therefore the space of essential n-ary weight n homotopy operations

$$\pi_{r_1} V \times \cdots \times \pi_{r_n} V \to \pi_q \mathcal{P}(n)_{\Sigma_n}(V)$$

is

$$\pi_q(\mathcal{P}(n) \otimes K[r_1] \otimes \ldots \otimes K[r_n]) = \begin{cases} \mathcal{P}(n), & \text{if } q = r_1 + \cdots + r_n; \\ 0, & \text{otherwise.} \end{cases}$$

The arguments of this section apply equally well to algebras over the monad $\Gamma(\mathcal{P})$ when $\mathcal{P}$ is a connected operad and to coalgebras over the comonads $\Gamma(\mathcal{Q})$ and $S(\mathcal{Q})$ when $\mathcal{Q}$ is a cooperad (respectively a connected cooperad).

Furthermore, Fresse has shown that for suitable V, the maps

$$S(\mathcal{P})(\pi_* V) \to \pi_* S(\mathcal{P})(V)$$

and

$$\Gamma(\mathcal{P})(\pi_* V) \to \pi_* \Gamma(\mathcal{P})(V)$$

factorize as in the diagram

$$\begin{array}{ccc} S(\mathcal{P})(\pi_* V) & \longrightarrow & \pi_* S(\mathcal{P})(V) \\ {\scriptstyle \mathrm{Tr}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Tr}} \\ \Gamma(\mathcal{P})(\pi_* V) & \longrightarrow & \pi_* \Gamma(\mathcal{P})(V) \end{array} .$$

We will recover this result later on in this paper.

It follows that for suitable simplicial $\mathcal{P}$-algebras V, the graded k-vector space $\pi_* V$ is a $\Gamma(\mathcal{P})$-algebra. For example, the homotopy of a simplicial commutative algebra is a divided power algebra and the homotopy of a simplicial Lie algebra is a restricted Lie algebra.

4.3.2 Homotopy operations in characteristic zero

It turns out that in characteristic zero, there is no further structure on the homotopy of a simplicial $\mathcal{P}$-algebra than the $\mathcal{P}$-algebra structure defined above.

Lemma 4.7. *Suppose that k is a field of characteristic zero (or more generally that the characteristic does not divide the order of the group G). Then the map*

$$(M \otimes (\pi_* V)^{\otimes n})_G \to \pi_*((M \otimes V^{\otimes n})_G)$$

constructed in the previous section, is an isomorphism. By Lemma 2.8, the same result holds for groups of invariants.

Corollary 4.8. *Let k be a field of characteristic zero and $\mathcal{P}$ an operad on k-vector spaces. Then the only homotopy operations for simplicial $\mathcal{P}$-algebras V are those determined by the $\mathcal{P}$-algebra structure on $\pi_* V$ of the previous section.*

Proof of Lemma. We have a diagram

$$\begin{array}{ccc}
(M \otimes (\pi_* V)^{\otimes n})_G & \longrightarrow & \pi_*((M \otimes V^{\otimes n})_G) \\
\downarrow \tau & & \downarrow \tau \\
(M \otimes (\pi_* V)^{\otimes n}) & \longrightarrow & \pi_*(M \otimes V^{\otimes n}) \\
\downarrow \rho & & \downarrow \rho \\
(M \otimes (\pi_* V)^{\otimes n})_G & \longrightarrow & \pi_*((M \otimes V^{\otimes n})_G)
\end{array}$$

where the vertical maps are induced by the transfer and restriction maps related to the inclusion of the identity subgroup $\{e\}$ into G. By Lemma 2.10 the vertical compositions are each multiplication by the order of G which is an isomorphism by the hypothesis on the characteristic of k. It follows that the top horizontal map is a retract of the middle horizontal map. But the middle one is an isomorphism by the Eilenberg-Zilber Theorem so the top one is too. □

Taking $V = K[r]$ in the Lemma, we get

$$\pi_q(M \otimes K[r]^{\otimes n})_G = \begin{cases} (M^{\pm})_G, & \text{if } q = nr; \\ 0, & \text{otherwise.} \end{cases}$$

It's not hard to see that these unary operations are all obtained by applying the essential n-ary operations corresponding to the elements of M to the same input repeated n times. Another way we could say this is that the codiagonal map from the n^{th} cross-effect to these groups are surjective. In fact they are just the canonical projections

$$(M^{\pm}) \twoheadrightarrow (M^{\pm})_G.$$

4.4 Further Calculations

If k is a field of nonzero characteristic, there are in general many more homotopy operations on simplicial $\mathcal{P}$-algebras than those that come from the

$\mathcal{P}$-algebra structure. In this section we will first put some bounds on the dimensions in which these operations can exist. Then we will do some initial calculations of the operations in the possible range. Our approach will be to try to calculate directly the homotopy groups

$$\pi_*((M \otimes V^{\otimes n})_G)$$

that we are interested (and the corresponding thing for groups of invariants). The most direct way to calculate the homotopy groups of a simplicial vector space is to find is normalized chain complex and take its homology. In this section we see how far this naïve approach will take us.

4.4.1 The homotopy groups that must always be zero

Let G be a subgroup of Σ_n and M a kG-module. Both taking the invariants or coinvariants and tensoring with M commute with the normalized chain complex functor so we have

$$N((M \otimes K[r]^{\otimes n})_G) = (M \otimes N(K[r]^{\otimes n}))_G$$

and similarly for the invariants. So first we will investigate $N(K[r]^{\otimes n})$.

Lemma 4.9. *A k-basis for $K[r]_q$ is given by the set of sequences of integers $\{i_1, \ldots, i_r\}$ with $1 \leq i_1 < \cdots < i_r \leq q$. The degeneracy map $s^j : K[r]_q \to K[r]_{q+1}$ acts by increasing those i_l that are greater than j by 1 and leaving the others unchanged (Thus in the extreme cases s^0 increases everything and s^q changes nothing)) The face map $d^j : K[r]_q \to K[r]_{q-1}$ acts by decreasing those i_l that are greater than j by 1 and leaving the others unchanged. If the resulting sequence is no longer strictly increasing or contains either 0 or q it is set to zero. (In the extreme cases, d^0 decreases everything and d^q changes nothing.)*

Corollary 4.10. *A k-basis for $(K[r]^{\otimes n})_q = (K[r]_q)^{\otimes n}$ is given by ordered n-tuple of the basis elements for $K[r]_q$. The face and degeneracy maps act on each member of the n-tuple as in $K[r]$.*

Proposition 4.11. *A k-basis for the q^{th} group in the normalized chain complex $N(K[r]^{\otimes n})$ is given by ordered n-tuples of sequences $\{i_1, \ldots, i_r\}$ with $1 \leq i_1 < \cdots < i_r \leq q$ in each sequence and such that every integer from 1 to q appears in at least one of the sequences. The differential to the $(q-1)^{st}$*

group of such an object is obtained by taking for each j the object obtained by decreasing each integer larger than j by 1 (setting this equal to zero if it no longer satisfies the conditions) and forming the alternating sum of these objects with signs $(-1)^j$.

Proof. If the integer j does not appear in one of these objects then it is the image under the degeneracy map s^{j-1} fo something in the next level down. Conversely, anything in the image of s^{j-1} does not contain j. Therefore (using the expression for the normalized chain complex NV as CV/DV), the result follows. □

When we take the tensor product over k with M we see that $N_q(M \otimes K[r]^{\otimes})$ is (as a k-vector space) a direct sum of copies of M indexed by the basic elements described in this proposition. In particular we have:

Corollary 4.12.

$$N_q(M \otimes K[r]^{\otimes n}) = \begin{cases} 0, & \textit{if } q > nr; \\ M, & \textit{if } q = r; \\ 0, & \textit{if } q < r. \end{cases}$$

Proof. If $q > rn$ then no n-tuple of sequences of length r can cover every integer from 1 to q, so $N_q = 0$. If $q < r$, there are no strictly increasing sequences of length r consisting of integers from 1 to q so $N_q = 0$. If $q = r$ then there is precisely one basis element given by repeating the sequence $\{1, \ldots, r\}$ n times. □

When we take the G invariants or coinvariants and apply homotopy we will still have these zeros.

Corollary 4.13.

$$\pi_q((M \otimes K[r]^{\otimes n})_G) = \pi_q((M \otimes K[r]^{\otimes n})^G) = \begin{cases} 0, & \textit{if } q > nr; \\ 0, & \textit{if } q < r. \end{cases}$$

A similar analysis can be done to show the following.

Proposition 4.14.

$$N_q(M \otimes K[r_1] \otimes \ldots \otimes K[r_n]) = \begin{cases} 0, & \textit{if } q > r_1 + \cdots + r_n; \\ 0, & \textit{if } q < \max_i r_i. \end{cases}$$

So, for suitable values of $r_1, \ldots, r_n$, the corresponding homotopy groups of $M^G(K[r_1], \ldots, K[r_n])$ and $M_G(K[r_1], \ldots, K[r_n])$ are zero.

4.4.2 Some calculations of the not necessarily zero groups

We have found above a description of the q^{th} position in the chain complex

$$N(M \otimes K[r]^{\otimes n})$$

as a vector space but to take the G-invariants or coinvariants we have to know how G acts on this complex. This action is the diagonal of the given action on M and the permutation action on the tensor power of $K[r]$. Since the action on the tensor power is induced by the action of G on the basis elements of Proposition 4.11, we need to understand this better.

The important fact here is contained in the following result.

Proposition 4.15. *Let $o(g)$ denote the number of orbits of the element $g \in G$ on $\{1, \ldots, n\}$ and let $o(G)$ be the maximum value of $o(g)$ for elements $g \in G$ not equal to the identity. Then G acts freely on the given basis elements of $N_q(K[r]^{\otimes n})$ if and only if $q > o(G)r$.*

Proof. The action is free if and only if every non-identity $g \in G$ has no fixed points. Such a fixed point would be a n-tuple of sequences of length r in which the sequences in each orbit of g are the same. To be able to cover all the integers from 1 to q we would therefore need $o(g)r \geq q$. If this does hold, we can construct a fixed point for g by taking our sequences of length r to be the same across orbits and to overlap sufficiently to cover precisely the integers $1, \ldots, q$. □

Notice that for G non-trivial $o(G) = 1$ if and only if every element in G is an n-cycle. This holds if and only if n is a prime and G is the cyclic group C_n. The maximum possible value for $o(G)$ is $(n-1)$ so our chain complex always has at least r free kG-modules from position nr down (and of course is zero above that).

Corollary 4.16. *The kG-module*

$$N_q(M \otimes K[r]^{\otimes n})$$

is free for $q > o(G)r$. So for any G this is free for $q > (n-1)r$ and if n is prime and G is the cyclic group C_n, it is free for $q > r$.

Notice that if $G = C_n$ with n prime then the only (potentially) non-free module in the chain complex occurs at $q = r$ where it is M.

This fact allows us to calculate some homotopy groups.

Proposition 4.17. *Let G be a subgroup of Σ_n and let $o(G)$ be as defined as previously. Let M be a kG-module. Then we have*

$$\pi_q((M \otimes K[r]^{\otimes n})_G) = \begin{cases} 0, & \text{for } q > nr; \\ H^{nr-q}(G, M^{\pm}), & \text{for } o(G)r + 2 \leq q \leq nr; \\ 0, & \text{for } q < r. \end{cases}$$

where $H^i(G, M)$ denotes the i^{th} cohomology group of G with coefficients in M. The same result holds for the homotopy groups defined by invariants instead of coinvariants and furthermore, the group cohomology answer is then also valid for $q = o(G)r + 1$.

You'll notice that if $G = C_n$ for a prime n, this Proposition completes the calculation except for one or two values of q. In fact we can work out these remaining groups as well.

Proposition 4.18. *Let n be a prime number and M a kC_n-module. Suppose that we are not in the case $r = 1$ and $n = 2$. Then the only nonzero homotopy groups are*

$$\pi_q((M \otimes K[r]^{\otimes n})_{C_n}) = H^{nr-q}(C_n, M^{\pm}) \text{ for } r + 2 \leq q \leq nr.$$

In the case $r = 1$, $n = 2$, the only nonzero group is

$$\pi_2((M \otimes K[1]^{\otimes 2})_{C_2} = \ker(M \to M_{C_2}) = (1 - \sigma)M$$

where σ denotes the non-trivial element in C_2. For the functors defined by invariants the only nonzero homotopy groups are

$$\pi_q((M \otimes K[r]^{\otimes n})^{C_n}) = H^{nr-q}(C_n, M^{\pm}) \text{ for } r \leq q \leq nr.$$

Notice that if n is odd then $M^{\pm} = M$ since n-cycles are even permutations.

Since the cohomology groups of the cyclic groups with arbitrary coefficients are well-known, this completes the calculation of the homotopy T-operations for the functors

$$V \mapsto (M \otimes V^{\otimes n})_{C_n}$$

and

$$V \mapsto (M \otimes V^{\otimes n})^{C_n}$$

when n is a prime.

There is a version of Proposition 4.17 that will help us calculating cross-effects. This is

Proposition 4.19. *Let $r_1, \dots, r_n$ be a sequence of positive integers and let G be a subgroup of Σ_n under which $(r_1, \dots, r_n)$ is invariant. Then there is a integer $o(G, r_1, \dots, r_n)$ such that*

$$\pi_q((M \otimes K[r_1] \otimes \dots \otimes K[r_n])_G) = \begin{cases} 0, & \text{for } q > r_1 + \cdots + r_n; \\ H^{nr-q}(G, M^{\pm}), & \text{for } o(G, r_1, \dots, r_n) + 2 \le q \le r_1 + \cdots + r_n; \\ 0, & \text{for } q < \max_i r_i. \end{cases}$$

The integer $o(G, r_1, \dots, r_n)$ plays the role of $o(G)r$ in Proposition 4.17 and is defined as follows. Break a given $g \in G$ into disjoint cycles $g_1 \dots g_t$. The cycle g_s acts on some subset of the integers r_j all of which must be the same. Write $r_{(s)}$ for this common value. Then set

$$o(g, r_1, \dots, r_n) = \sum_{s=1}^{t} r_{(s)}.$$

This reduces to $o(g)r$ if all the r_j are equal to r. Finally take $o(G, r_1, \dots, r_n)$ to be the maximum of the $o(g, r_1, \dots, r_n)$ for non-identity elements $g \in G$.

Proof of Proposition 4.17. We write N_q for the kG-module $N_q(M \otimes K[r]^{\otimes n})$. Then the homotopy groups in question are found by taking the coinvariants of the chain complex N_* and then homology. As we have seen, this complex has the form

$$\cdots \to 0 \to N_{nr} \to N_{nr-1} \to \cdots \to N_{r+1} \to M \to 0 \to \dots$$

where the kG-modules N_q are free up to and including the point $N_{o(G)r+1}$.

Now recall that the Eilenberg-Zilber Theorem gives us a chain homotopy equivalence from $M \otimes N(K[r])^{\otimes n} = M \otimes [r]^{\otimes n}$ to this complex and that the map in this direction is G-equivariant. These complexes therefore have the same homology groups, with the same G-actions on those homology groups. But those homology groups are zero except for the nr^{th} which is M. However, recall from §2.2.4 that the G-action on this M is twisted, so as a kG-module, this homotopy group is $M \otimes \epsilon^{\otimes r} = M^{\pm}$.

Now kG is a Frobenius algebra for any finite group G so free modules are injective. Therefore the complex

$$0 \to N_{nr} \to N_{nr-1} \to \cdots \to N_{o(G)r+1}$$

is a complex of injective kG-modules whose homology is $M^{\pm}$ in the first position and zero elsewhere. That is, it is the beginning of an injective resolution for $M^{\pm}$. But the homotopy groups we want to find are just the homology groups of the complex obtained from N_* by applying coinvariants. These are then just the right derived functors of the coinvariants functor which are precisely the cohomology groups of G with coefficients in $M^{\pm}$. This gives the claimed result.

For the invariants version, the same argument applies. But in this case we can say something about the next homotopy group down as well. To see this, choose an injection $N_{o(G)r} \to I$ with I an injective kG-module. Then we can extend our injective resolution by

$$0 \to N_{nr} \to \cdots \to N_{o(G)r+1} \to I$$

where the final map is the composition of $N_{o(G)r+1} \to N_{o(G)r}$ with the injection $N_{o(G)r} \to I$. The invariants functor preserves injections and so $N^G_{o(G)r+1} \to I^G$ is the composition of $N^G_{o(G)r+1} \to N^G_{o(G)r}$ and an injection $N^G_{o(G)r} \to I^G$. But this means that the kernels of the maps

$$N^G_{o(G)r+1} \to I^G$$

and

$$N^G_{o(G)r+1} \to N^G_{o(G)r}$$

are the same. So the homology at position $o(G)r+1$ of the complex N_* is also equal to the given cohomology group of G with coefficients $M^{\pm}$. □

Proof of Proposition 4.18. The part of the complex we still have to conside looks like

$$N_{r+2} \to N_{r+1} \to M \to 0$$

with N_{r+2} and N_{r+1} not free (and hence injective) kG-modules. As long as $nr > r+1$ (which only fails in the case $r = 1$ and $n = 2$) this bit of the sequence is exact. Since the coinvariants functor is right exact, this means that

$$(N_{r+2})_{C_p} \to (N_{r+1})_{C_n} \to M_{C_n} \to 0$$

is also exact. But the homotopy groups we still need to calculate are the homology groups of this complex, which are therefore zero. So the only nonzero homotopy groups are the ones calculated in Proposition 4.17.

If $r = 1$ and $n = 2$, our complex looks like

$$0 \to N_2 \to M \to 0.$$

Using Proposition 4.11 we can say explicitly what N_2 is. It is the direct sum of two copies of M with the nontrivial element $\sigma \in C_2$ acting by

$$(m, n) \mapsto (\sigma n, \sigma m).$$

The differential $N_2 \to M$ is

$$(m, n) \mapsto m + n.$$

We we take the coinvariants of this complex we get

$$0 \to M \to M_{C_2} \to 0$$

where we have identified $(M \oplus M)_{C_2}$ with M by $[m, n] \mapsto (m + \sigma n)$. It follows that the map $M \to M_{C_2}$ is the usual quotient. Taking homology of this we get the claimed homotopy groups.

Finally, consider the invariants version. The only homotopy group left to calculate is that in position r. For this we meed to consider the complex

$$N_{r+1} \to M \to 0$$

where we know that N_{r+1} is injective. Let $M \to I_0 \to I_1$ be the beginning of an injective kC_p-resolution for M. Then we have a map of complexes

$$\begin{array}{ccccc} N_{r+1} & \longrightarrow & M & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \\ N_{r+1} & \longrightarrow & I_0 & \longrightarrow & I_1 \end{array}.$$

Taking invariants we get:

$$\begin{array}{ccccc} N_{r+1}^G & \longrightarrow & M^G & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \\ N_{r+1}^G & \longrightarrow & I_0^G & \longrightarrow & I_1^G \end{array}.$$

and since the invariants functor is left exact, the $\ker(I_0^G \to I_1^G) = M^G$. So this map of complexes induces an isomorphism on cycles in the middle position and hence an isomorphism in homology. But recall that the bottom complex computes the cohomology groups of G with coefficients in $M^{\pm}$. This completes the calculation of the only unknown homotopy group. □

Proof of Proposition 4.19. Analysis of the basis for $N_q(M \otimes K[r_1] \otimes \ldots \otimes K[r_n])$ determined by Lemma 4.9 tells us that these are free kG-modules for $q > o(G, r_1, \ldots, r_n)$. The Proposition then follows in the same way as Proposition 4.17. □

5 The Dwyer Map

In this section we will analyse more carefully the relationship between the homotopy groups

$$\pi_*((M \otimes K[r]^{\otimes n})_G)$$

and the group cohomology $H^*(G, M)$ suggested by Proposition 4.17. There is a map relating the two that generalizes a construction of Dwyer and in suitable cases this map is an isomorphism.

5.1 Derived Functors on Chain Complexes

To set up the Dwyer Map we recall material from §5 of [Dwyer]. We have up until now been working mainly in the category of simplicial k-vector spaces or equivalently the category of positively graded chain complexes over k (sometimes with an action of a finite group G). Now we will do something rather strange. Recall from Corollary 4.12 that the chain complex we are primarily interested in, that is, $N(M \otimes K[r]^{\otimes n})$ is bounded above. To get the Dwyer map we will forget about the positive grading and treat this as an object in the category of chain complexes that are bounded above (ie chain complexes C with $C_n = 0$ for sufficiently large n). Denote this category by Ch_-. The corresponding category of chain complexes of kG-modules will be denoted $\mathrm{Ch}_- G$.

Proposition 5.1. *There is a model category structure on* $\mathrm{Ch}_- G$ *in which the weak equivalences are quasi-isomorphisms, the cofibrations are injections and the fibrations are surjections whose kernel consists of injective kG-modules. Thus the fibrant objects are chain complexes of injective modules.*

An injective resolution for a kG-module M is a fibrant replacement for M in this model structure where M is considered as a chain complex concentrated in one dimension. For any chain complex $C \in \mathrm{Ch}_{\leq N} G$ there is a fibrant replacement $C \to I$ where this map is a quasi-isomorphism and I

is a chain complex of injectives. Such an I will also be called an *injective resolution* for C.

Now let F be a functor from $\mathrm{Ch}_- G$ to some other model category $\mathcal{C}$ that preserves weak equivalences between fibrant objects. That, is if $X \to Y$ is a quasi-isomorphism of injective chain complexes, then $F(X) \to F(Y)$ is a weak equivalence in $\mathcal{C}$. The standard theory of model categories then tells us that associated to F is a *right derived functor* $\mathrm{R}F$ from the homotopy category of $\mathrm{Ch}_- G$ to the homotopy category of $\mathcal{C}$. This can be constructed by taking a fibrant replacement (i.e. injective resolution) and applying F to that (and then looking in the homotopy category). The condition on F ensures that this is well-defined. There is also a natural transformation in the homotopy category of the form

$$F \to \mathrm{R}F$$

given by applying F to a functorial fibrant replacement. In terms of chain complexes, this is given by

$$F(C) \to F(I) = \mathrm{R}F(C)$$

where $C \to I$ is an injective resolution.

We will apply these constructions to the functors F given by taking invariants or coinvariants. For this to be possible we need the following result.

Lemma 5.2. *Let F be a functor from kG-modules to k-vector spaces. Then the degree-wise extension of F to a functor $\mathrm{Ch}_- G \to \mathrm{Ch}_-$ preserves weak equivalences between fibrant objects.*

In this case, the right derived functor of F as a functor of chain complexes extends the notion of right derived functors in homological algebra, which we shall write $\mathrm{R}^i F$. To be precise, $\mathrm{R}^i F(M) = H_{-i}\mathrm{R}F(M)$ where M is a kG-module treated as a chain complex concentrated in degree zero. The functors $H_i\mathrm{R}F$ are called the *hyper-derived functors* of F in [Weibel].

To calculate these hyper-derived functors there are two spectral sequences relating them to the basic derived functors.

Proposition 5.3. *Let C be a bounded above chain complex of kG-modules and F a functor from kG-modules to k-vector spaces. Then there are spectral sequences:*

$$E^2_{s,t} = (R^{-s}F)(H_t C) \implies H_{s+t}RF(C)$$

and
$$E^2_{s,t} = H_s(R^{-t}F)(C)) \implies H_{s+t}RF(C)$$
where in the second case, we apply $R^{-t}F$ to each term in the chain complex C and then take homology.

Because C is bounded above, each of these spectral sequences lies in the third quadrant (or could be translated to lie in the third quadrant).

5.2 The Generalized Dwyer Construction

We now apply the theory of the previous section to help us study the homotopy groups
$$\pi_*(M \otimes K[r]^{\otimes n})^G)$$
and
$$\pi_*(M \otimes K[r]^{\otimes n})_G).$$

Proposition 5.4. *Let G be a subgroup of Σ_n, M a kG-module and r a nonnegative integer. Let F be the functor from kG-modules to k-vector spaces given by taking either the invariants or coinvariants and let C be the chain complex*
$$N(M \otimes K[r]^{\otimes n}).$$
Then
$$H_q(RF)(C) \cong H^{nr-q}(G, M^{\pm}).$$

Proof. To see this we look at the first spectral sequence of Proposition 5.3. The chain comples C has homology

$$H_q(C) = \pi_q(M \otimes K[r]^{\otimes n}) = \begin{cases} M^{\pm}, & \text{if } q = nr; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the E^2 term only has one nonzero row. So the spectral sequence collapses and we are left with

$$H_q(\mathrm{R}F)(C) = (R^{nr-q}F)(M^{\pm}) = H^{nr-q}(G, M^{\pm}).$$

□

The natural transformations $F \to \mathrm{R}F$ then give us maps

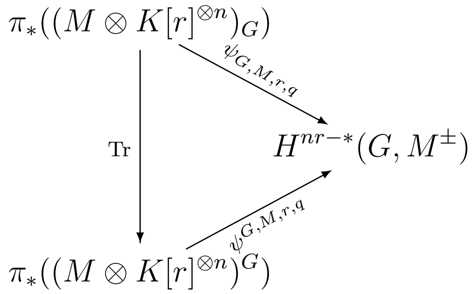
$$\pi_q((M \otimes K[r]^{\otimes n})^G) \longrightarrow H^{nr-q}(G, M^{\pm})$$

and

$$\pi_q((M \otimes K[r]^{\otimes n})_G) \longrightarrow H^{nr-q}(G, M^{\pm}).$$

We will call these maps the *Dwyer maps* and write them $\psi^{G,M,r,q}$ and $\psi_{G,M,r,q}$ respectively. Notice that for $q > nr$ both the domain and range of the Dwyer maps are zero. For $q < 0$ the domain is zero but the range can be nonzero.

Because the trace map is an isomorphism $M_G \to M^G$ for injective kG-modules M, it induces an isomorphism between the derived functors for coinvariants and invariants. We therefore have a commutative diagram



relating the Dwyer maps for coinvariants and invariants.

It is of course no coincidence that the domain and range of the Dwyer maps bear a striking resemblance to the groups involved in Proposition 4.17. We can deduce the results of §4.4.2 (at least for the invariants case) directly from the first spectral sequence of Proposition 5.3.

Proposition 5.5. *In Proposition 5.3, suppose that F is a left exact functor. Then the edge homomorphism in the second spectral sequence*

$$H_s(R^0F)(C)) \to H_sRF(C)$$

is just the map induced on homology by the natural transformation $F \to RF$.

Corollary 5.6. *The Dwyer map for invariants $\psi^{G,M,q,r}$ is an isomorphism for $o(G)r \leq q \leq nr$ and a monomorphism for $q = o(G)r - 1$. The Dwyer map for coinvariants $\psi_{G,M,q,r}$ is an isomorphism for $o(G)r + 2 \leq q \leq nr$. In the case $G = C_p$ for a prime p, all the Dwyer maps (with the exception of the coinvariants case $p = 2$, $r = 1$, $q = 2$) are either isomorphisms or the inclusion of zero according to Proposition 4.18.*

Proof. Since taking invariants is left exact, the Dwyer maps ψ^G are the edge homomorphisms of the spectral sequence

$$E^2_{s,t} = H_s(\mathrm{R}^{-t}F)(C) \implies H_{s+t}\mathrm{R}F(C)$$

where $C = N(M \otimes K[r]^{\otimes n})$. We know that C_q is a free kG-module for $q > o(G)r$ and so $H_s(\mathrm{R}^{-t}F)(C) = 0$ for $s > q$ and $t < 0$. The claimed results for ψ^G then follow from the structure of the spectral sequence. The results for ψ_G follow from the fact that the trace map is an isomorphism on free modules. □

The Dwyer maps are natural in the kG-module M but we would also like some kind of naturality in the group G.

Proposition 5.7. *Let H be a subgroup of G and treat the kG-module M as a kH-module by restriction. Then the following diagram commutes*

$$\begin{array}{ccccc}
\pi_*((M \otimes K[r]^{\otimes n})_G) & \xrightarrow{\mathrm{Tr}} & \pi_*((M \otimes K[r]^{\otimes n})^G) & \xrightarrow{\psi^G} & H^{nr-*}(G, M^{\pm}) \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
\pi_*((M \otimes K[r]^{\otimes n})_H) & \xrightarrow{\mathrm{Tr}} & \pi_*((M \otimes K[r]^{\otimes n})^H) & \xrightarrow{\psi^H} & H^{nr-*}(H, M^{\pm}) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\
\pi_*((M \otimes K[r]^{\otimes n})_G) & \xrightarrow{\mathrm{Tr}} & \pi_*((M \otimes K[r]^{\otimes n})^G) & \xrightarrow{\psi^G} & H^{nr-*}(G, M^{\pm})
\end{array}$$

where τ and ρ denote the transfer and restriction maps for coinvariants, invariants or cohomology as appropriate.

Corollary 5.8. *Suppose that the index $[G : H]$ is invertible in the field k. Then if a Dwyer map for H is an isomorphism, the corresponding Dwyer map for G is also an isomorphism. In particular, if k has characteristic p, this applies when H is a Sylow p-subgroup of G.*

Proof. If $[G : H]$ is invertible then by Lemma 2.10, the vertical compositions in Proposition 5.7 are the isomorphisms given by multiplication by $[G : H]$. It follows that the Dwyer map for G is a retract of that for H and the claim follows. □

Finally we can say something about homotopy operations for operads.

Corollary 5.9. *Let $\mathcal{P}$ be an operad over a field of characteristic p. The unary homotopy $\mathcal{P}$-operations of weights $\leq 2p-1$ areas follows. For $n = 0, \ldots, p-1$:*

$$\pi_{nr}((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}) \cong (\mathcal{P}(n)^{\pm})_{\Sigma_n}$$

and all other homotopy groups are zero. For $n = p, \ldots, 2p-1$ (unless $r = 0$ or $r = 1, p = 2$):

$$\pi_q((\mathcal{P}(n) \otimes K[r]^{\otimes n})_{\Sigma_n}) \cong H^{nr-q}(\Sigma_n, \mathcal{P}(n)^{\pm})$$

for $r + 2 \leq q \leq nr$ and all other groups are zero.

For homotopy $\Gamma(\mathcal{P})$-operations, we have

$$\pi_{nr}((\mathcal{P}(n) \otimes K[r]^{\otimes n})^{\Sigma_n}) \cong (\mathcal{P}(n)^{\pm})^{\Sigma_n}$$

for $n = 0, \ldots, p-1$ and

$$\pi_q((\mathcal{P}(n) \otimes K[r]^{\otimes n})^{\Sigma_n}) \cong H^{nr-q}(\Sigma_n, \mathcal{P}(n)^{\pm})$$

for $n = p, \ldots, 2p-1$ and $r \leq q \leq nr$. All other homotopy groups for these values of n are zero.

Proof. The Sylow p-subgroups of Σ_n for $n = 0, \ldots 2p-1$ are either trivial or the cyclic group C_p. These results therefore follow from Corollary 5.6. □

We remark here that the Dwyer maps are not always isomorphisms for the ranges of q in Corollary 5.9. Take $k = \mathbb{F}_2$ and M to be the $k(\Sigma_2 \times \Sigma_2)$-module isomorphic to $k \oplus k$ where each generator of $\Sigma_2 \times \Sigma_2$ swaps the factors. Direct calculation then shows that

$$\pi_1((M \otimes K[1]^{\otimes n})^{\Sigma_2 \times \Sigma_2}) = 0$$

but

$$H^3(\Sigma_2 \times \Sigma_2, M) \cong k.$$

In the next chapter we will see more directly how this examples arises.

Let $(r_1, \ldots, r_n)$ be a sequence of integers on which the group $G \subset \Sigma_n$ acts trivially. Then we can apply the Dwyer construction (right derived functors of invariants/coinvariants) to the chain complex

$$N(M \otimes K[r_1] \otimes \ldots \otimes K[r_n]).$$

The result is a more general Dwyer map

$$\pi_q((M \otimes K[r_1] \otimes \ldots \otimes K[r_n])^G) \to H^{r_1+\cdots+r_n-q}(G, M^{\pm}).$$

Proposition 4.19 tells us that this map is an isomorphism for q larger than $o(G, r_1, \ldots, r_n) + 1$.

We can now recover the result of [Fresse] mentioned earlier.

Corollary 5.10. *Let V be a 2-connected (that is, $\pi_0 V = \pi_1 V = 0$) simplicial k-vector space and $\mathcal{P}$ a connected operad over k. Then we have a commutative diagram (in particular the diagonal map exists):*

$$\begin{array}{ccc} S(\mathcal{P})(\pi_* V) & \longrightarrow & \pi_* S(\mathcal{P})(V) \\ {\scriptstyle \mathrm{Tr}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Tr}} \\ \Gamma(\mathcal{P})(\pi_* V) & \longrightarrow & \pi_* \Gamma(\mathcal{P})(V) \end{array}$$

In particular, the homotopy groups of a 2-connected simplicial $\mathcal{P}$-algebra have the structure of a graded $\Gamma(\mathcal{P})$-algebra.

Proof. This diagram breaks up into a direct sum of diagrams of the form

$$\begin{array}{ccc} (M \otimes (\pi_* V)^{\otimes n})_G & \longrightarrow & \pi_*((M \otimes V^{\otimes n})_G) \\ {\scriptstyle \mathrm{Tr}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Tr}} \\ (M \otimes (\pi_* V)^{\otimes n})^G & \longrightarrow & \pi_*((M \otimes V^{\otimes n})^G) \end{array} .$$

for groups G and kG-modules M.

Now suppose first that $\pi_* V$ is finite-dimensional. Then V is weakly equivalent to a finite direct sum of $K[r]$s. Using the cross-effect formulas from Proposition 4.1 we see that each of the above diagrams breaks up into a direct sum of diagrams of the form

$$\begin{array}{ccc} (M \otimes [r_1] \otimes \ldots \otimes [r_n])_G & \longrightarrow & \pi_*((M \otimes K[r_1] \otimes \ldots \otimes K[r_n])_G) \\ {\scriptstyle \mathrm{Tr}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Tr}} \\ (M \otimes [r_1] \otimes \ldots \otimes [r_n])^G & \longrightarrow & \pi_*((M \otimes K[r_1] \otimes \ldots \otimes K[r_n])^G) \end{array}$$

where G is a subgroup of Σ_n that acts trivially on $(r_1, \ldots, r_n)$. This is a diagram of graded vector spaces and except at position $r_1 + \cdots + r_n$ the two things on the LHS are zero. In that case there is certainly a factorization as required. So consider position $r_1 + \cdots + r_n$. Because V is 2-connected, each $r_i \geq 2$. It then follows from the definition that

$$o(G, r_1, \ldots, r_n) + 2 \leq r_1 + \cdots + r_n$$

and so the above diagram becomes just

$$\begin{array}{ccc} (M^{\pm})_G & \longrightarrow & H^0(G, M^{\pm}) \\ {\scriptstyle \mathrm{Tr}}\big\downarrow & \nearrow & \cong\big\downarrow{\scriptstyle \mathrm{Tr}} \\ (M^{\pm})^G & \xrightarrow[\cong]{} & H^0(G, M^{\pm}) \end{array} .$$

The required factorisation then certainly exists here too.

Putting together all the pieces in the direct sums we get a map

$$\Gamma(\mathcal{P})(\pi_* V) \to \pi_* S(\mathcal{P})(V)$$

which factorises the original diagram. It is easy to see that this will be natural in V. It remains only to extend this to the case where $\pi_* V$ is not finite-dimensional.

□

5.3 Examples

5.3.1 The Commutative Operad

The commutative operad Com is defined by

$$\mathsf{Com}(n) = k$$

for all $n \geq 0$ with the trivial Σ_n action. Suppose that $k = \mathbb{F}_2$ and let V be a simplicial Com-algebra, that is, a simplicial commutative $\mathbb{F}_2$-algebra in the usual sense. Then there are weight 2 homotopy operations

$$\delta_i : \pi_r V \longrightarrow \pi_{r+i} V$$

for $i = 2, \ldots, r$, determined by the nonzero elements of $H^{2r-(r+i)}(\Sigma_2, \mathbb{F}_2) \cong \mathbb{F}_2$. These are the higher divided square operations of Cartan, Bousfield and Dwyer.

If $k = \mathbb{F}_p$ for an odd prime p, then we get weight p homotopy operations on simplicial commutative $\mathbb{F}_p$-algebras:

$$\epsilon_? : \pi_r V \longrightarrow \pi_? V$$

corresponding to generators of the cohomology groups $H^{nr-?}(\Sigma_p, \mathbb{F}_p^{\pm}) \cong \mathbb{F}_p$.

Algebras over the monad $\Gamma(\mathcal{P})$ are precisely divided polynomial algebras. For homotopy operations on these over $\mathbb{F}_2$ we get, as well as the $\delta_2, \ldots, \delta_r$ of commutative algebras, also δ_0 and δ_1. Similarly for p odd.

5.3.2 The Lie operad

The Lie operad Lie is not as simple to define. We take $\mathsf{Lie}(n)$ to be the submodule of the free Lie algebra over k on n elements consisting of expressions containing each element exactly once, with the Σ_n action that permutes these elements. It is not hard to see, when we restrict to $\Sigma_{n-1} \subset \Sigma_n$, that $\mathsf{Lie}(n)$ is a free $k\Sigma_{n-1}$ module, but the Σ_n action is difficult to describe except for small n.

However, $\mathsf{Lie}(2)$ is isomorphic to k and if the $k = \mathbb{F}_2$ then the action of Σ_2 is trivial. Thus, there are exactly the same homotopy operations of weight 2 on simplicial Lie algebras as in the commutative algebra case. These are the operations

$$\lambda_i : \pi_r V \longrightarrow \pi_{r+i} V$$

of Priddy. Algebras over the monad $\Gamma(\mathsf{Lie})$ are restricted Lie algebras and so we also recover Priddy's operations λ_0 and λ_1 on simplicial restricted Lie algebras.

We can describe $\mathsf{Lie}(3)$ as follows. It is a two-dimensional vector space over k with generators that we shall write $1(23)$ and $2(13)$. The group Σ_3 acts by permuting the three digits subject to the usual Lie identities

$$x(yz) + x(zy) = 0$$

and

$$x(yz) + y(zx) + z(xy) = 0.$$

Over $\mathbb{F}_2$, there are three non-zero elements ($1 = 1(23) = 1(32)$, $2 = 2(13) = 2(31)$, $3 = 3(12) = 3(21) = 1 + 2$) with the Σ_3-action suggested by the notation.

Corollary 5.9 tells us that the weight 3 homotopy operations for Lie are given by $H^*(\Sigma_3, \mathsf{Lie}(3))$. In characteristic 2, these groups are subgroups of

$H^*(\Sigma_2, \mathsf{Lie}(3))$ which are trivial except for $* = 0$ since $\mathsf{Lie}(3)$ is free over Σ_2. But $H^0(\Sigma_3, \mathsf{Lie}(3))$ is the group of invariants of $\mathsf{Lie}(3)$ which is 0 by inspection. This means that all weight 3 homotopy operations for (restricted and ordinary) simplicial Lie algebras V over $\mathbb{F}_2$ are zero. In particular we get the identities

$$[\lambda_i(x), x] = 0$$

for $x \in \pi_r V$ where the bracket is that in the induced Lie algebra structure on $\pi_* V$. It's not hard to see that this equation generalizes to fields of higher characteristic p where we replace λ_i by any of the weight p homotopy operations classified by the elements of

$$H^*(\Sigma_p, \mathsf{Lie}(p)).$$

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