Spherical Spline Solution to a PDE on the Sphere

Victoria Baramidze and Ming-Jun Lai

Abstract. We use splines on spherical triangulations to approximate the solution of a second order elliptic PDE over the unit sphere. We establish existence and uniqueness of weak solutions in spherical spline spaces and estimate convergence of the spline approximations. We present a computational algorithm and summarize numerical results on convergence rates.

§1. Introduction

Recently we have used spherical spline functions introduced in [1] to deal with spherical scattered data interpolation and fitting problems (cf. [6]). We studied the convergence (or resemblance) of the spherical splines fitting a given data set under the assumption that the data locations become dense over the sphere and that the data values are obtained from a smooth function (cf. [5]). In this paper we use splines on spherical triangulations to approximate the solution of an elliptic partial differential equation over the sphere.

Traditionally, the classical solution of an elliptic second order PDE on the sphere can be expressed in terms of spherical harmonics. As the degree of the spherical harmonic functions increases, the spherical harmonics become more complicated to compute. The computation of the convolution of functions with spherical harmonics becomes challenge. See, e.g., [7], [13], and [14]. It is therefore necessary to develop other approaches to approximate solutions of spherical PDEs.

Our study is partially motivated by the recent work in [8], [9], and [10] where a standard second order elliptic partial differential equation (PDE) on the unit sphere was solved using spherical radial basis functions. The PDE was discretized using the Galerkin method based on a linear combination of rotations of a spherical radial basis function. Some

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numerical experiments were presented to demonstrate the effectiveness of the method. The spherical basis method in [8] overcomes the disadvantage inherited from the zonal kernel method (cf. [11]) for the solution of pseudo-differential equations on sphere.

In this paper we shall explain how to use spherical spline functions to obtain numerical solutions of an elliptic second order PDE on the unit sphere. Although we shall use the standard Galerkin method, we will solve the spherical PDE without constructing finite element like functions. This is the main difference from the traditional finite element method for PDE. We will be able to use spherical polynomials of arbitrary degree d and arbitrary smoothness r with $d > r$ over an arbitrary triangulation of the sphere. Finally we shall present some numerical experiments to demonstrate the effectiveness and efficiency of the spherical spline method. We compute the numerical solutions of the PDE for several different right-hand side functions as in [8]. Numerical experiments show that spherical splines approximate the solution much better than the radial basis functions (see Remark 1 at the end of the paper).

§2. Spherical Harmonics

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. The 3D Laplace operator Δ restricted to the unit sphere is the well-known Laplace-Beltrami operator Δ^*. We consider the following partial differential equation over the sphere:

$$-\Delta^* u + \omega^2 u = f, \quad \text{on } \mathbb{S}^2. \tag{1}$$

The equation (1) typically arises from time discretization of the heat equation

$$\frac{\partial u(t,v)}{\partial t} - \Delta^* u(t,v) = g(t,v)$$

on $\mathbb{S}^2 \times [0,T]$, subject to the initial conditions $u(0,v) = h(v)$. Divided difference approximation of the time derivative

$$\frac{\partial u}{\partial t}(t_k, v) \approx \frac{u(t_k,v) - u(t_{k-1},v)}{t_k - t_{k-1}}$$

results in the implicit scheme

$$\frac{u_k - u_{k-1}}{t_k - t_{k-1}} - \Delta^* u_k = g_k,$$

or

$$-\Delta^* u_k + \omega^2 u_k = g_k + \omega^2 u_{k-1}.$$

This equation is to be solved at every time step t_k, and $\omega^2 = \frac{1}{t_k - t_{k-1}}$. Note that the time step in the derivative approximation of u has to be small to ensure higher accuracy, therefore we are mainly interested in solving the problem (1) with large ω.

Let Ω be a solid in $\mathbb{R}^3$ bounded by two concentric spheres of radii $0 < r_1 < 1 < r_2$. Suppose f is a continuous function on $\mathbb{S}^2$. Consider the problem

$$-\Delta\tilde{u}(v) + \omega^2\frac{\tilde{u}(v)}{|v|^2} = \frac{\tilde{f}(v)}{|v|^2}, v \in \Omega, \tag{2}$$

subject to the boundary conditions $\frac{\partial\tilde{u}}{\partial r}(v) = 0, v \in \partial\Omega$. Here $\tilde{f}(v) = f(\frac{v}{|v|})$ is the unique homogeneous extension of f to $\mathbb{R}^3\backslash\{0\}$ of degree 0. The boundary $\partial\Omega$ is smooth enough to guarantee the existence of a C^2 solution to the problem (2). The right-hand side of (2) is homogeneous of degree -2, therefore the solution $\tilde{u}$ must be homogeneous of degree 0. By the definition $\Delta^*u = \Delta u_0|_{\mathbb{S}^2}$, where u_0 is a homogeneous extension of u to $\mathbb{R}^3\backslash\{0\}$ of the degree 0. Therefore by the uniqueness of the homogeneous extensions $u = \tilde{u}|_{\mathbb{S}^2}$ solves

$$-\Delta^*u + \omega^2u = f,$$

since $\tilde{u} \in C^2(\Omega)$, $u \in C^2(\mathbb{S}^2)$. Uniqueness follows from the fact that $-\Delta^*u + \omega^2u = 0$ if and only if either u is a spherical harmonic Y_{jk}, in which case we must have $\omega^2 = -j(j+1)$, which is impossible, or $u = 0$. This argument generalizes naturally to conclude, that for every $f \in C^k(\mathbb{S}^2)$ there exists a unique solution $u \in C^{k+2}(\mathbb{S}^2)$ of the equation (1).

Expand f in terms of spherical harmonics by

$$f = \sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}\widehat{f}_{jk}Y_{jk},$$

where $Y_{jk}, k = 1, \cdots, 2j+1, j = 0, \cdots\infty$ are spherical harmonic functions which form an orthonormal basis for $L^2(\mathbb{S}^2)$, and $\widehat{f}_{jk}$ are computed by

$$\widehat{f}_{jk} = \int_{\mathbb{S}^2} f\bar{Y}_{jk}.$$

Let us write u in terms of spherical harmonics as well. That is,

$$u = \sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}\widehat{u}_{jk}Y_{jk}.$$

Since $\Delta^*Y_{jk} = -j(j+1)Y_{jk}$ we find that

$$-\Delta^*u + \omega^2u = \sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}\widehat{u}_{jk}(j(j+1) + \omega^2)Y_{jk}.$$

Comparing coefficients of u with coefficients of f we obtain

$$\widehat{u}_{jk} = \frac{\widehat{f}_{jk}}{j(j+1)+\omega^2}.$$

This defines an analytic expression for the solution of (1). To compute an approximate solution of u represented by an infinite sum, it is customary to cut off the tail of the series. There are several problems associated with this approach. The spherical harmonics Y_{jk} become increasingly complicated to compute as j increases and therefore it is not trivial to determine $\widehat{f}_{jk}$. The spherical harmonics Y_{jk} have global support on the sphere and become highly oscillating as j increases. To define partial sums u_N one needs to compute $(N+1)^2$ coefficients $\widehat{f}_{jk}$, and to evaluate u_N at a point $(N+1)^2$ evaluations of Y_{jk} have to be performed.

To establish the convergence rate in this approximation let us introduce appropriate Sobolev spaces and norms (cf. [8]).

The Sobolev space $H^s := \{f \in L^2 : \|f\|_{H^s} < \infty\}$, with the norm $\|\cdot\|_{H^s}$ defined as

$$\|f\|^2_{H^s} := \sum_{j=0}^{\infty}(1+j(j+1))^s \sum_{k=1}^{2j+1} |\widehat{f}_{jk}|^2.$$

We claim, that for a function $f \in H^s$

$$\|u-u_N\|_{L^2} < C\|f\|_{H^s}\frac{1}{N^{s+3/2}},$$

for a positive constant C depending on s. Indeed, suppose $f \in H^s$. Then

$$\|f\|^2_{H^s} := \sum_{j=0}^{\infty}(1+j(j+1))^s \sum_{k=1}^{2j+1} |\widehat{f}_{jk}|^2 < \infty,$$

therefore

$$(1+j(j+1))^s \sum_{k=1}^{2j+1} |\widehat{f}_{jk}|^2 \le \|f\|^2_{H^s},$$

for every j.

Since $u_N = \sum_{j=0}^{N}\sum_{k=1}^{2j+1} \frac{\widehat{f}_{jk}}{(\omega^2+j(j+1))} Y_{jk}$ we have

$$\begin{aligned}
\|u-u_N\|^2_{L^2} &= \sum_{j=N+1}^{\infty}\sum_{k=1}^{2j+1} \frac{|\widehat{f}_{jk}|^2}{(\omega^2+j(j+1))^2} \\
&\le \sum_{j=N+1}^{\infty}\sum_{k=1}^{2j+1} \frac{|\widehat{f}_{jk}|^2(1+j(j+1))^s}{(\beta+j(j+1))^{2+s}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|f\|_{H^s}^2 \sum_{j=N+1}^{\infty} \frac{1}{(\beta + j(j+1))^{2+s}} \\
&\leq \|f\|_{H^s}^2 \sum_{j=N+1}^{\infty} \frac{1}{(j^2)^{2+s}} \\
&\leq \|f\|_{H^s}^2 \int_N^{\infty} \frac{1}{x^{2s+4}} dx \\
&= \|f\|_{H^s}^2 \frac{1}{(2s+3)N^{2s+3}} \\
&\leq \frac{\|f\|_{H^s}^2}{2s+3} \frac{1}{N^{2s+3}}. \qquad (3)
\end{aligned}
$$

Here $\beta = \min\{1, \omega^2\}$.

We turn our attention to an alternative approach, namely spline approximation to a weak solution of (1). We can write a weak formulation of the equation (1) as

$$\langle -\Delta^* u + \omega^2 u, v\rangle = \langle f, v\rangle, \quad v \in H^1, \qquad (4)$$

where $\langle f, g\rangle = \int_{\mathbb{S}^2} fg d\sigma$, with $d\sigma$ being the Lebesgue measure on the unit sphere.

Lemma 1. *There exist positive constants α and β depending on ω such that*

$$\langle(-\Delta^* + \omega^2)u, v\rangle \leq \alpha\|u\|_{H^1}\|v\|_{H^1},$$

for all u and v in H^1, and

$$\langle(-\Delta^* + \omega^2)u, u\rangle \geq \beta\|u\|_{H^1}^2,$$

for all u in H^1.

Proof: By Parseval's equality, we have

$$\langle(-\Delta^* + \omega^2)u, v\rangle = \sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}(j(j+1) + \omega^2)\widehat{u}_{jk}\widehat{v}_{jk}.$$

By the Cauchy inequality, we conclude

$$
\begin{aligned}
\langle(-\Delta^* + \omega^2)u, v\rangle \leq & \left(\sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}(j(j+1) + \omega^2)|\widehat{u}_{jk}|^2\right)^{1/2} \\
& \left(\sum_{j=0}^{\infty}\sum_{k=1}^{2j+1}(j(j+1) + \omega^2)|\widehat{v}_{jk}|^2\right)^{1/2}
\end{aligned}
$$

$$\leq \quad \alpha \|u\|_{H^1} \|v\|_{H^1},$$

for $\alpha = \max\{1, \omega^2\}$. On the other hand,

$$\begin{aligned} \langle(-\Delta^* + \omega^2)u, u\rangle &= \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} (j(j+1) + \omega^2)|\widehat{u}_{jk}|^2 \\ &\geq \min\{1, \omega^2\} \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} (j(j+1) + 1)|\widehat{u}_{jk}|^2 \\ &= \beta \|u\|_{H^1}^2. \end{aligned}$$

This completes the proof. □

We are now able to use the well-known Lax-Milgram theorem to conclude that the weak formulation has a unique solution for any given $f \in L^2$.

Theorem 1. *Suppose that $\omega > 0$. There exists a unique $\mathbf{u} \in H^1$ satisfying (4).*

Note that we can be more precise in relating the inner product $\langle(-\Delta^* + \omega^2)u, u\rangle$ to the H^1 and L^2 norms of u as follows:

$$\begin{aligned} & \langle(-\Delta^* + \omega^2)u, u\rangle \\ = & \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} (j(j+1) + \omega^2)|\widehat{u}_{jk}|^2 \\ = & \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} (j(j+1) + 1)|\widehat{u}_{jk}|^2 + (\omega^2 - 1) \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} |\widehat{u}_{jk}|^2 \\ = & \|u\|_{H^1}^2 + (\omega^2 - 1)\|u\|_{L^2}^2. \end{aligned} \tag{5}$$

Using the considerations above and Theorem 1, we can bound the solution $\mathbf{u}$ in L^2 norm by that of f.

Lemma 2. *Let $\mathbf{u}$ be the solution of (4). Then*

$$\|\mathbf{u}\|_{L^2} \leq \|\mathbf{u}\|_{H^1} \leq \frac{1}{\beta}\|f\|_{L^2}.$$

In addition when $\omega > 1$ we have

$$\|\mathbf{u}\|_{L^2} \leq \frac{1}{\omega^2}\|f\|_{L^2}.$$

Proof: Clearly, since $\|\mathbf{u}\|_{L^2}^2 = \sum_{j=0}^{\infty}\sum_{k=1}^{2j+1} |\widehat{\mathbf{u}}_{jk}|^2$, we have $\|\mathbf{u}\|_{L^2} \le \|\mathbf{u}\|_{H^1}$. By Lemma 1

$$\begin{aligned}\|\mathbf{u}\|_{L^2}^2 &\le \|\mathbf{u}\|_{H^1}^2 \le \frac{1}{\beta}\langle -(\Delta^* + \omega^2)\mathbf{u}, \mathbf{u}\rangle \\ &= \frac{1}{\beta}\langle f, \mathbf{u}\rangle \le \frac{1}{\beta}\|f\|_{L^2}\|\mathbf{u}\|_{L^2} \\ &\le \frac{1}{\beta}\|f\|_{L^2}\|\mathbf{u}\|_{H^1}\end{aligned}$$

by Cauchy's inequality. Note that if $\omega < 1$, $\beta = \omega^2$, and if $\omega > 1$, $\beta = 1$ and $\|\mathbf{u}\|_{L^2} \le \|f\|_{L^2}$. We can improve this bound for $\omega > 1$. Consider

$$\begin{aligned}\|\mathbf{u}\|_{L^2}^2 \le \|\mathbf{u}\|_{H^1}^2 &= \langle -(\Delta^* + \omega^2)\mathbf{u}, \mathbf{u}\rangle - (\omega^2 - 1)\|\mathbf{u}\|_{L^2}^2 \\ &= \langle f, \mathbf{u}\rangle - (\omega^2 - 1)\|\mathbf{u}\|_{L^2}^2 \\ &\le \|f\|_{L^2}\|\mathbf{u}\|_{L^2} - (\omega^2 - 1)\|\mathbf{u}\|_{L^2}^2.\end{aligned}$$

Then

$$\|\mathbf{u}\|_{L^2} \le \|f\|_{L^2} - (\omega^2 - 1)\|\mathbf{u}\|_{L^2},$$

and therefore

$$\|\mathbf{u}\|_{L^2} \le \frac{1}{\omega^2}\|f\|_{L^2}.$$

□

Define $S_d^r(\Delta)$ to be the space of homogeneous spherical splines of degree $d \ge 3r + 2$ and smoothness $r \ge 1$ on a spherical triangulation Δ (cf. [1]). Let $\mathcal{H}^d$ denote the space of trivariate homogeneous polynomials of degree d restricted to the unit sphere. Then

$$S_d^r(\Delta) := \{s \in C^r(\mathbb{S}^2) : s|_\tau \in \mathcal{H}^d, \tau \in \Delta\}.$$

We use this finite dimensional vector space to find an approximation of the weak solution $\mathbf{u}$, i.e. we seek $\bar{u} \in S_d^r(\Delta)$ satisfying

$$\langle -\Delta^* \bar{u} + \omega^2 \bar{u}, v\rangle = \langle f, v\rangle, \quad \forall v \in S_d^r(\Delta), \tag{6}$$

for every $v \in S_d^r(\Delta)$.

We first note that spherical splines belong to H^1. Let $W^{k,p}(\mathbb{S}^2)$ denote the spaces of L^2 functions defined in [12] with the semi-norms

$$|g|_{k,p,\mathbb{S}^2} = \sum_{|\alpha|=k} \|D^\alpha g_{k-1}\|_{L^p}$$

and

$$|g|'_{k,p,\mathbb{S}^2} = \sum_{|\alpha|=k} \|D^\alpha g_{k-2}\|_{L^p}$$

defined in [4]. Here g_{k-1} denotes the homogeneous extension of g to $\mathbb{R}^3\backslash\{0\}$ of degree $k-1$, and similarly for g_{k-2}.

Lemma 3. *$W^{2,2}(\mathbb{S}^2) \subset H^1 \subset L^2$. In particular*

$$\|g\|_{H^1} \le \sqrt{3}|g|'_{2,2,\mathbb{S}^2} + \|g\|_{L^2},$$

for $g \in W^{2,2}(\mathbb{S}^2)$.

Proof: The right inclusion follows from the definition of H^1. Let $g \in W^{2,2}(\mathbb{S}^2)$. Then $\|g\|_{L^2} < \infty$ and $|g|'_{2,2,\mathbb{S}^2} < \infty$. By (5)

$$\begin{aligned}\|g\|^2_{H^1} &= \langle(-\Delta^* + \omega^2)g, g\rangle - (\omega^2-1)\|g\|^2_{L^2} \\ &\le |\int_{\mathbb{S}^2} \Delta^* g g| + \omega^2 \int_{\mathbb{S}^2} |g|^2 - (\omega^2-1)\|g\|^2_{L^2} \\ &\le (\int_{\mathbb{S}^2} |\Delta^* g|^2)^{1/2} (\int_{\mathbb{S}^2} |g|^2)^{1/2} + \|g\|^2_{L^2} \\ &\le (\|\Delta^* g\|_{L^2} + \|g\|_{L^2})\|g\|_{L^2}.\end{aligned}$$

Since $\|g\|_{L^2} \le \|g\|_{H^1}$ by above we get

$$\|g\|_{H^1} \le \|\Delta^* g\|_{L^2} + \|g\|_{L^2}.$$

Consider

$$\begin{aligned}\int_{\mathbb{S}^2} |\Delta^* g|^2 &= \int_{\mathbb{S}^2} |D_{xx} g_0 + D_{yy} g_0 + D_{zz} g_0|^2 \\ &\le 3 \sum_{|\alpha|=2} \int_{\mathbb{S}^2} |D^\alpha g_0|^2,\end{aligned}$$

therefore

$$\|\Delta^* g\|_{L^2} \le \sqrt{3}|g|'_{2,2,\mathbb{S}^2}.$$

We conclude that

$$\|g\|_{H^1} \le \sqrt{3}|g|'_{2,2,\mathbb{S}^2} + \|g\|_{L^2} < \infty.$$

Therefore $g \in H^1$. □

Since $S^r_d(\Delta) \subset W^{2,2}(\mathbb{S}^2)$ for $r \ge 1$, $d \ge 3r+2$, we can prove the following

Theorem 2. *Suppose that $\omega > 0$, $r \ge 1$ and $d \ge 3r+2$. There exists a unique $\bar{u} \in S^r_d(\Delta)$ satisfying (6).*

Proof: Since $S^r_d(\Delta)$ is a finite dimensional vector space we can write $v = \sum_i v_i \phi_i$ for any function $v \in S^r_d(\Delta)$ in terms of some basis $\{\phi_i\}_i$. The solution exists if we can find a set of coefficients u_i such that $\sum_i u_i \phi_i$ satisfies (6) for every basis function ϕ_j, i.e.

$$\langle(-\Delta^* + \omega^2)\sum_i u_i \phi_i, \phi_j\rangle = \langle f, \phi_j\rangle.$$

This leads to the system of linear equations $M\vec{u} = \vec{f}$ where $M(i,j) = \langle(-\Delta^* + \omega^2)\phi_i, \phi_j\rangle$, $\vec{u}(i) = u_i$ and $\vec{f} = \langle f, \phi_j\rangle$. We claim that M is positive definite and therefore there exists a unique solution of (6). By Lemma 3 $S_d^r(\Delta) \subset H^1$. Then

$$\vec{u}^t M\vec{u} = \langle(-\Delta^* + \omega^2)u, u\rangle \geq \beta\|u\|_{H^1},$$

by Lemma 1. Therefore $\vec{u}^t M\vec{u} \geq 0$ and equality holds if and only if $u = 0$. □

We now discuss how well $\bar{u}$ approximates $\mathbf{u}$. The following is a spherical version of Céa lemma.

Lemma 4. *There exists a constant C_1 depending on ω such that*

$$\|\mathbf{u} - \bar{u}\|_{H^1} \leq C_1 \inf_{v\in S_d^r(\Delta)} \|\mathbf{u} - v\|_{H^1}.$$

Proof: Note that

$$\langle(-\Delta^* + \omega^2)(\mathbf{u} - \bar{u}), v\rangle = 0, \tag{7}$$

for all $v \in S_d^r(\Delta)$. Consider $v = \mathbf{u} - \bar{u} - (\mathbf{u} - u_b) \in S_d^r(\Delta)$ for any approximation u_b of $\mathbf{u}$. Then we have

$$\langle(-\Delta^* + \omega^2)(\mathbf{u} - \bar{u}), \mathbf{u} - \bar{u}\rangle = \langle(-\Delta^* + \omega^2)(\mathbf{u} - \bar{u}), \mathbf{u} - u_b\rangle. \tag{8}$$

By Lemma 1 above,

$$\begin{aligned} \beta\|\mathbf{u} - \bar{u}\|^2_{H^1} &\leq \langle(-\Delta^* + \omega^2)(\mathbf{u} - \bar{u}), \mathbf{u} - \bar{u}\rangle \\ &= \langle(-\Delta^* + \omega^2)(\mathbf{u} - \bar{u}), \mathbf{u} - u_b\rangle \\ &\leq \alpha\|\mathbf{u} - \bar{u}\|_{H^1}\|\mathbf{u} - u_b\|_{H^1}. \end{aligned}$$

It thus follows

$$\beta\|\mathbf{u} - \bar{u}\|_{H^1} \leq \alpha\|\mathbf{u} - u_b\|_{H^1},$$

for any approximation u_b in $S_d^r(\Delta)$.

Let u_b be the quasi-interpolation of $\mathbf{u}$ in $S_d^r(\Delta)$ constructed in [12], with $d \geq 3r + 2$ and Δ being a quasi-uniform triangulation of the unit sphere. We need the following result (cf. [4]).

Theorem 3. *Let $d \geq 3r + 2$ and $r \geq 1$. Suppose $\mathbf{u}$ is in $W^{m+1,2}(\mathbb{S}^2)$, for some m between 0 and d with $(d-m)$ mod $2 = 0$. There exists a constant C_2 depending only on d and the smallest angle in Δ such that*

$$|\mathbf{u} - u_b|'_{k,2,\mathbb{S}^2} \leq C_2 \sum_{\ell=0}^{k} (\tan\frac{|\Delta|}{2})^{m+1-\ell} |\mathbf{u}|_{m+1,2,\mathbb{S}^2}, \tag{9}$$

for all $0 \leq k \leq \delta := \min\{r+1, m+1\}$. Here $|\Delta|$ denotes the size of the largest triangle in Δ which is assumed to be bounded by 1.

The following theorem provides an error bound on the difference between the weak solution of (1) and its spherical spline approximation in H^1 norm.

Theorem 4. *Let $S^r_d(\Delta)$ be a homogeneous spline space with $d \geq 3r+2$, $r \geq 1$ and $|\Delta| \leq 1$. Suppose that the solution $\mathbf{u} \in W^{m+1,2}(\mathbb{S}^2), 1 \leq m \leq d$. Then there exists a constant $C_3 > 0$ depending on $d, |\Delta|$, the smallest angle of Δ and ω such that*

$$\|\mathbf{u} - \bar{u}\|_{H^1} \leq C_3 (\tan \frac{|\Delta|}{2})^{m-1} |\mathbf{u}|_{m+1,2,\mathbb{S}^2}.$$

Here m must satisfy $(d-m)$ mod $2 = 0$.

Proof: Since $\mathbf{u} \in W^{m+1,2}(\mathbb{S}^2), m \geq 1$, and $u_b \in W^{r+1,2}(\mathbb{S}^2)$, we have $\mathbf{u} - u_b \in W^{\delta,2}(\mathbb{S}^2)$ with $\delta \geq 2$. By Lemma 3

$$\|\mathbf{u} - u_b\|_{H^1} \leq \sqrt{3} |\mathbf{u} - u_b|'_{2,2,\mathbb{S}^2} + \|\mathbf{u} - u_b\|_{L^2}.$$

By Theorem 3, we have

$$|\mathbf{u} - u_b|'_{2,2,\mathbb{S}^2} \leq C_2 \sum_{\ell=0}^{2} (\tan \frac{|\Delta|}{2})^{m+1-\ell} |\mathbf{u}|_{m+1,2,\mathbb{S}^2},$$

and

$$\|\mathbf{u} - u_b\|_{L^2} = |\mathbf{u} - u_b|'_{0,2,\mathbb{S}^2} \leq C_2 (\tan \frac{|\Delta|}{2})^{m+1} |\mathbf{u}|_{m+1,2,\mathbb{S}^2}.$$

Therefore

$$\begin{aligned} \|\mathbf{u} - u_b\|_{H^1} \quad &\leq \quad C_2 (\sqrt{3} + \sqrt{3} \tan \frac{|\Delta|}{2} + (\sqrt{3}+1)(\tan \frac{|\Delta|}{2})^2) \times \\ & \qquad \left(\tan \frac{|\Delta|}{2}\right)^{m-1} |\mathbf{u}|_{m+1,2,\mathbb{S}^2}. \end{aligned}$$

Now we use Lemma 4 to conclude the result. □

Note that the constant C_3 is a multiple of $\frac{\alpha}{\beta}$. For $\omega \gg 1$ C_3 is of order ω^2, for $w \ll 1$ C_3 is of order $\frac{1}{\omega^2}$. Therefore C_3 is very large for very large and very small ω. While $\bar{u}$ is expected to converge to $\mathbf{u}$ as the size of the triangulation Δ is decreasing, Theorem 4 implies that best results are expected when $w \sim 1$. As our numerical data suggests this is not the case. There is evidence that the error in the approximation decreases as ω grows, moreover the rate of convergence increases as well. The following theorem provides some explanation of our numerical results.

Theorem 5. *Let $S^r_d(\Delta)$ be a homogeneous spline space of degree d and smoothness r. Suppose $\omega > 1$ and let $\bar{u} \in S^r_d(\Delta)$ be the solution of (6). Then*

$$\|\mathbf{u} - \bar{u}\|_{L^2} \leq \frac{2}{\omega^2} \|f\|_{L^2}. \tag{10}$$

Proof: Follow the proof of Lemma 2 to conclude that $\|\bar{u}\|_{L^2} \le \frac{1}{\omega^2}\|f\|_{L^2}$.
□

In the case when the function f is described as a set of values at finitely many discrete locations, f too can be approximated by spherical splines. In this case we solve

$$\langle -\Delta^* u + \omega^2 u, v\rangle = \langle \tilde{f}, v\rangle, \quad \forall v \in S_d^r(\Delta), \tag{11}$$

where $\tilde{f}$ is a spline approximation of f. Let $\tilde{u}$ denote the spline solution of (11). We are interested in obtaining an error bound for $\mathbf{u} - \tilde{u}$. In view of Theorems 4 and 5 we only need to show that

Lemma 5. *Let $\bar{u}$ and $\tilde{u}$ be solutions of (6) and (11) respectively. Then*

$$\|\bar{u} - \tilde{u}\|_{H^1} \le \frac{1}{\beta}\|f - \tilde{f}\|_{L^2}, \tag{12}$$

and

$$\|\bar{u} - \tilde{u}\|_{L^2} \le \frac{1}{\omega^2}\|f - \tilde{f}\|_{L^2}. \tag{13}$$

Proof: Since $e := \bar{u} - \tilde{u} \in S_d^r(\Delta)$ we must have

$$\langle -\Delta^* \bar{u} + \omega^2 \bar{u}, e\rangle = \langle f, e\rangle,$$

and

$$\langle -\Delta^* \tilde{u} + \omega^2 \tilde{u}, e\rangle = \langle \tilde{f}, e\rangle.$$

Therefore by Lemma 1

$$\begin{aligned}
\beta\|e\|_{H^1}^2 &\le \langle -\Delta^* e + \omega^2 e, e\rangle = \langle -\Delta^* \bar{u} + \omega^2 \bar{u}, e\rangle - \langle -\Delta^* \tilde{u} + \omega^2 \tilde{u}, e\rangle \\
&= \langle f, e\rangle - \langle \tilde{f}, e\rangle = \langle f - \tilde{f}, e\rangle \le \|f - \tilde{f}\|_{L^2}\|e\|_{L^2} \\
&\le \|f - \tilde{f}\|_{L^2}\|e\|_{H^1}.
\end{aligned}$$

Then

$$\beta\|\bar{u} - \tilde{u}\|_{H^1} \le \|f - \tilde{f}\|_{L^2}.$$

On the other hand by (5)

$$\begin{aligned}
\|e\|_{L^2}^2 + (\omega^2 - 1)\|e\|_{L^2}^2 &\le \|e\|_{H^1}^2 + (\omega^2 - 1)\|e\|_{L^2}^2 \\
&= \langle -\Delta^* e + \omega^2 e, e\rangle \\
&= \langle f - \tilde{f}, e\rangle \le \|f - \tilde{f}\|_{L^2}\|e\|_{L^2},
\end{aligned}$$

and therefore $\omega^2\|e\|_{L^2} \le \|f - \tilde{f}\|_{L^2}$. □

The error bound results on spherical data fitting can be found in [5] and [4].

§3. A computational Method

Suppose we are given a set of measurements $f(v)$ corresponding to locations $v \in \mathcal{V}$ on the unit sphere. Construct a triangulation Δ based on the locations $\mathcal{V}$. Recall that on each triangle in Δ we can define Bernstein-Bézier spherical homogeneous polynomials of degree d [1]

$$P = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}.$$

Here

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k,$$

and

$$b_1(v)v_1 + b_2(v)v_2 + b_3(v)v_3 = v$$

are spherical barycentric coordinates of a point v on the unit sphere relative to the spherical triangle with vertices v_1, v_2 and v_3. Recall that the space of such polynomials is $\mathcal{H}^d$ [1]. Let

$$S^r_d(\Delta) := \{s : s|_\tau \in \mathcal{H}^d(\mathbb{S}^2)\} \cap C^r(\mathbb{S}^2).$$

be the spline space defined in Section 2. A spline approximation of the weak solution $\mathbf{u}$ in $S^r_d(\Delta)$ is

$$\bar{u} = \sum_{\tau\in\Delta} \sum_{i+j+k=d} u^\tau_{ijk} B^{\tau,d}_{ijk}.$$

Substituting these two representations in (11) together with $v = B^{\tau,d}_{rst}, r + s + t = d$ we obtain

$$\sum_{i+j+k=d} u^\tau_{ijk} \langle(-\Delta^* + \omega^2) B^\tau_{ijk}, B^\tau_{rst}\rangle = \langle f, B^\tau_{rst}\rangle.$$

Viewing $\langle f, B^\tau_{ijk}\rangle, \tau \in \Delta, i + j + k = d$ and $(u^\tau_{ijk}), \tau \in \Delta, i + j + k = d$ as vectors $\vec{f}$ and $\vec{u}$ correspondingly we can write

$$(D + \omega^2 B)\vec{u} = \vec{f},$$

where the matrices D and B are defined below with indices i and j corresponding to the ordering of $\vec{u}$ and $\vec{f}$

$$D_{ij} = \int_{\mathbb{S}^2} -\Delta^* B_i B_j,$$

$$B_{ij} = \int_{\mathbb{S}^2} B_i B_j.$$

Furthermore let S be the smoothness matrix (cf. [1] and [6]), i.e., $S\vec{u} = 0$ ensures that $\bar{u} \in S_d^r(\Delta)$. We need to solve

$$\begin{aligned} (D + \omega^2 B)\vec{u} &= \vec{f} \\ S\vec{u} &= 0. \end{aligned}$$

Using the Lagrange multiplier method, we convert the above equations into the following matrix equation

$$\begin{bmatrix} D + \omega^2 B & S^T \\ S & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \vec{f} \\ 0 \end{bmatrix}.$$

We use the iterative method introduced in [2] and [3] to solve the above matrix equation. The convergence is guaranteed since the matrix $D+\omega^2 B$ is positive definite with respect to the side conditions $S\vec{u} = 0$.

§4. Numerical Examples

Example 1. We first demonstrate that the spline method reproduces certain solutions of (1). Let Δ be a triangulation of the unit sphere based on the vertices $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ in $\mathbb{R}^3$. Note that if the exact solution u belongs to $S_d^r(\Delta)$, (7) and Lemma 1 imply

$$0 \le \beta \|u - \bar{u}\|_{H^1}^2 \le \langle (-\Delta^* + \omega^2)(u - \bar{u}), u - \bar{u} \rangle = 0.$$

Therefore we expect to reproduce polynomial solutions of (1) of even degree in spline spaces of even degree, and similarly for odd degrees. We test the following functions

$$\begin{aligned} u_1 &= 1 & \qquad f_1 &= \omega^2 \\ u_2 &= x^2 & f_2 &= (6 + \omega^2)x^2 - 2 \\ u_3 &= y^4 & f_3 &= (20 + \omega^2)y^4 - 12y^2 \end{aligned}$$

in $S_4^1(\Delta)$. We compute the spline approximation $\bar{u}$ for different values of ω, and evaluate the relative errors $e_i = \frac{\|u_i - \tilde{u}_i\|_\infty}{\|u_i\|_\infty}, i = 1, 2, 3$ in each case. The results presented in Table 1 demonstrate reproduction of the exact solutions. Similarly, in spline spaces of odd degree we expect to reproduce polynomial solutions u of odd degree. We test

$$\begin{aligned} u_4 &= x & \qquad f_4 &= (2 + \omega^2)x \\ u_5 &= y^3 + z^3 & f_5 &= (12 + \omega^2)(y^3 + z^3) - 6(y + z) \\ u_6 &= x^2 z^3 & f_6 &= (30 + \omega^2)x^2 z^3 - 2z(x^2 + z^2) \end{aligned}$$

in $S_5^1(\Delta)$ and list the relative errors in Table 2.

ω	e_1	e_2	e_3
.1	$3.3815e-12$	$5.3049e-12$	$3.3906e-12$
1	$8.8818e-14$	$9.1404e-14$	$8.4936e-14$
10	$2.6046e-13$	$1.0218e-13$	$4.9889e-14$

Tab. 1. Reproduction of exact solutions by spherical splines.

ω	e_4	e_5	e_6
.1	$6.2355e-13$	$3.3546e-13$	$4.7145e-13$
1	$4.9981e-13$	$4.6103e-13$	$7.2343e-13$
10	$1.7763e-12$	$1.5280e-12$	$1.1452e-12$

Tab. 2. Reproduction of exact solutions by spherical splines.

Example 2. Next we test how well the spline solution of the PDE with the right hand side function $f(x,y,z) = e^x(\omega^2 + x^2 + 2x - 1)$ can approximate the exact solution $u(x,y,z) = e^x$. The initial triangulation Δ_0 is based on 6 vertices as in the previous example. The next refined triangulation Δ_1 is obtained from Δ_0 by connecting the midpoints of edges of Δ_0. Similarly we form Δ_2 and Δ_3. The errors of the form $e_i := e(\Delta_i) = \frac{\|u-\bar{u}\|_\infty}{\|u\|_\infty}$ are computed based on a set of 46592 almost uniformly spaced points over the unit sphere. In Table 3 we list the error results in $S_5^1(\Delta)$. Corresponding convergence rates e_i/e_{i+1} are listed in Table 4.

ω	e_0	e_1	e_2	e_3
0.01	$4.2791e-1$	$5.7706e-2$	$8.8282e-3$	$1.8378e-2$
0.1	$3.1387e-1$	$1.0510e-3$	$2.4704e-4$	$3.5504e-4$
1	$2.9608e-2$	$5.7173e-4$	$7.8374e-6$	$1.8207e-6$
10	$2.4472e-2$	$5.8554e-4$	$8.9892e-6$	$4.2849e-7$
100	$2.4765e-2$	$5.8647e-4$	$9.8447e-6$	$1.6522e-7$

Tab. 3. Error values in $S_5^1(\Delta)$ for $u(x,y,z) = e^x$.

The results of similar experiments using triangulated spherical spline spaces $S_6^1(\Delta)$ and $S_8^2(\Delta)$ are recorded in Tables 5, 6, 7 and 8, respectively.

Example 3. In this example we test the exact solution

$$u^*(x,y,z) = (1-\sqrt{2-2z})_+^6(35(2-2z) + 18\sqrt{2-2z} + 3)$$

with the right hand side function

$$f(x,y,z) = 112(1-\sqrt{2-2z})_+^4(25z^2 - 9z + 4z\sqrt{2-2z} - 15)+$$

ω	e_0/e_1	e_1/e_2	e_2/e_3
0.01	7.4154	6.5366	0.4804
0.1	298.6521	4.2542	0.6958
1	51.7867	72.9489	4.3046
10	41.7951	65.1382	20.9788
100	42.2269	59.5722	59.5854

Tab. 4. Convergence in $S_5^1(\Delta)$ for $u(x,y,z) = e^x$.

ω	e_0	e_1	e_2	e_3
0.01	$2.3526e-3$	$1.8382e-3$	$5.1806e-4$	$2.7671e-3$
0.1	$2.3460e-3$	$5.7889e-5$	$9.7877e-6$	$1.2302e-4$
1	$2.4370e-3$	$4.6854e-5$	$2.4196e-6$	$4.3631e-6$
10	$2.4282e-3$	$5.1107e-5$	$7.7150e-7$	$6.3742e-7$
100	$2.3004e-3$	$5.9546e-5$	$6.9452e-7$	$1.8974e-7$

Tab. 5. Error values in $S_6^1(\Delta)$ for $u(x,y,z) = e^x$.

$$\omega^2(1-\sqrt{2-2z})_+^6(35(2-2z)+18\sqrt{2-2z}+3).$$

We compute the spline approximation to the solution u using triangulated spherical spline space $S_5^1(\Delta_i)$ for $i = 0, 1, \ldots, 4$. The maximal error values are computed based on the 46592 almost uniformly spaced points over the unit sphere used in the previous examples. The maximal errors are tabulated in Table 9. Similarly, we also use spline space $S_6^1(\Delta_i)$ for $i = 0, 1, 2, 3$ and several values of ω. The error values, computed based on the 46592 almost uniformly spaced points on the sphere are recorded in Table 10, and convergence rates are listed in Table 11. Here N denotes the number of triangles in Δ_i.

Remark 1. (Comparison of Triangulated Spherical Splines and Spherical Radial Basis Functions) The solution u^* in Example 3 was tested in [8] and thus allows us to some extent to compare the results on approximation by polynomial spherical splines and the spherical basis functions(SBF). From Tables 9 and 10 we can see that spline functions provide much better convergence than Table 1 in [Le Gia'04]. Although we have to solve much large linear systems than that associated with the SBF method, the numerical solutions in Table 1 in [8] show that the SBF method soon loses its approximation due to the condition numbers of the linear system when using 900 points. It seems that the linear systems from our method have better condition numbers than that from the SBF method. In addition, the second example in [8] shows how well the SBF method approximates the spherical polynomials. Our method can reproduce spherical polynomials

ω	e_0/e_1	e_1/e_2	e_2/e_3
0.01	1.2798	3.5483	0.1872
0.1	40.5258	5.9144	0.0796
1	52.0125	19.3642	0.5546
10	47.5112	66.2437	1.2103
100	38.6324	85.7357	3.6604

Tab. 6. Convergence in $S_6^1(\Delta)$ for $u(x,y,z) = e^x$.

ω	e_0	e_1	e_2	e_3
0.01	$1.0845e-3$	$2.2059e-2$	$1.6706e-1$	$3.0517e-2$
0.1	$8.4258e-4$	$1.7193e-4$	$2.5036e-3$	$2.1760e-3$
1	$8.5125e-4$	$8.9492e-6$	$8.4661e-5$	$1.4490e-5$
10	$8.8980e-4$	$7.9679e-6$	$1.0348e-5$	$8.4810e-6$
100	$9.0556e-4$	$7.6868e-6$	$7.3615e-7$	$1.8235e-6$

Tab. 7. Error values in $S_8^2(\Delta)$ for $u(x,y,z) = e^x$.

as in Example 1. Thus, our method is again better in this case.

Remark 2. The condition number of the matrix $D+\omega^2 B$ increases with the degree d of the spline space. It also increases as the triangulation size Δ becomes smaller and as the parameter ω decreases. We can see that even though higher convergence rates are expected for higher d the rates are often not achieved on triangulations with smaller triangles and smaller ω.

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ω	e_0/e_1	e_1/e_2	e_2/e_3
0.01	0.0492	0.1320	5.4749
0.1	4.9009	0.0687	1.1505
1	95.1197	0.1057	5.8425
10	111.6726	0.7700	1.2201
100	117.8071	10.4419	0.4037

Tab. 8. Convergence in $S_8^2(\Delta)$ for $u(x,y,z) = e^x$.

ω	$N = 8$	$N = 32$	$N = 128$	$N = 512$	$N = 2048$
.01	0.6669	$1.6310e-1$	$0.9548e-2$	$2.9058e-3$	$2.5568e-2$
.1	0.6547	$0.5743e-1$	$0.6417e-2$	$4.4364e-4$	$3.1584e-4$
1	0.6243	$0.5618e-1$	$0.6361e-2$	$3.5112e-4$	$1.9637e-5$
10	0.6125	$0.5446e-1$	$0.6185e-2$	$3.4913e-4$	$1.2325e-5$
100	0.6052	$0.5203e-1$	$0.5311e-2$	$2.8143e-4$	$1.1004e-5$

Tab. 9. Absolute error in approximation of u^* in $S_5^1(\Delta_i)$.

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ω	e_0	e_1	e_2	e_3
0.01	$5.1241e-1$	$1.2080e-2$	$6.5882e-4$	$2.5350e-3$
0.1	$8.8429e-2$	$4.1005e-3$	$1.9788e-4$	$5.2131e-5$
1	$9.2798e-2$	$4.0264e-3$	$2.0796e-4$	$9.6968e-6$
10	$9.0430e-2$	$3.8962e-3$	$2.1110e-4$	$6.3173e-6$
100	$8.8432e-2$	$3.5803e-3$	$2.2105e-4$	$6.3944e-6$

Tab. 10. Error values in $S_6^1(\Delta)$ for u^*.

ω	e_0/e_1	e_1/e_2	e_2/e_3
0.01	42.4174	18.3361	0.2599
0.1	21.5656	20.7220	3.7959
1	23.0476	19.3616	21.4459
10	23.2099	18.4564	33.4164
100	24.6996	16.1967	34.5693

Tab. 11. Convergence in $S_6^1(\Delta)$ for u^*.

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Victoria Baramidze
Department of Mathematics
Western Illinois University
Macomb, IL 61544
vbaramidze@gmail.com

and

Ming-Jun Lai
Department of Mathematics
University of Georgia
Athens, GA 30602
mjlai@math.uga.edu, http://www.math.uga.edu/∼mjlai