

NONUNIFORM VARIATIONAL INTERPOLATORY REFINEMENT

SCOTT KERSEY AND MING-JUN LAI

ABSTRACT. In this paper we investigate certain properties of the curves s_k generated by a non-uniform variational refinement scheme defined over a k -window of some bi-infinite sequence of points. In particular, we show that error between s_n and s_k for $k \ll n$ over some fixed interval decays exponentially as $k \uparrow n$, showing that s_n can be well-approximated by s_k for k much smaller than n . We use this result to approximate the variational refinement curves locally, and to approximate the global curve using a *domain decomposition* algorithm. Since the computation of the localized curves is more efficient, the new scheme results in a savings in computation for certain applications. We also generalize our non-uniform variational refinement scheme to the modeling of parametric surfaces defined over a rectangular grid.

1. INTRODUCTION

Uniform variational refinement was introduced in [5] as a method for producing smooth refinement curves by minimizing certain quadratic energy functionals. One such functional is the sum of the squares of second differences of the point set, which mimics the approximate *thin-beam* functional commonly used in variational spline theory. In [1] and [2], variational refinement was generalized to data over non-uniformly spaced knots by minimizing the sum of squares of second divided differences, which we continue in this paper. While variational refinement methods produce nice curve fits, they have the disadvantage that they are global, meaning that the value of each new point added in the refinement process depends on all other points. Moreover, the number of points grows exponentially with the level of refinement. For very large data sets this can be a computational burden, and so it is the goal of this paper to show that these variational refinement curves can be well approximated from a small subset of the original point set (a so-called a k -window in [5]). In this paper we verify this statement by showing that the decay in the error over a fixed window between k -window and n -window refinement curves for $k \ll n$ decreases exponentially as $k \uparrow n$.

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The results in this paper are perhaps not surprising. Indeed, variational refinement (minimizing divided differences) is a discrete analog of variational spline interpolation, which was shown in [3] to exhibit a similar exponential decay property. The key idea in that paper depends on the a certain L_2 orthogonality condition between local and global minimizers whereby the error between local and global interpolants over a fixed interval is “pushed” to the boundary of the ring centered at that interval. This is shown in Lemma 5 in [3]. Replacing L_2 by l_2 inner products, we adapt this result to Lemma 5.3 in this paper. But the generalization from splines to non-uniform variational refinement is not straight-forward because a separate variational problem is solved at each level of refinement. To handle this problem, we introduce refinement operators at each level of refinement, and provide bounds on the interaction between these operators at successive levels of the refinement process. To get a feel for this, the proof of main result in Theorem 5.5 illustrates how the whole process fits together.

In the multivariate setting, the exponential rate of decay of the error between local and global interpolants was shown in [4] for bivariate variational splines. This paper goes further, describing an algorithm to compute an approximation to a global variational spline surface by piecing together local spline surfaces. Moreover, the key lemma in this paper (Lemma 5.3) is based on an idea used in [3] that was adapted from [4]. Based on this we describe in the last section of this paper a domain composition for variational refinement, and we generalize curve scheme to surfaces over rectangular grids, much as was done in [3].

This paper is organized as follows:

- In Section 2 we introduce the problem of variational refinement, and state three assumptions used throughout the paper.
- In Section 3 we give bounds on the linear operators associated to our refinement scheme (in Theorem 3.1).
- In Section 4 we derive a stability estimate needed for the main results.
- The main results of this paper are in Section 5. We first prove the key lemma (Lemma 5.3) that gives the exponential decay on the error between s_n and s_k for $k < n$ in a particular inner product space norm. The key idea method used in the proof of this lemma is to some extent a discretized version of Lemma 8 in [3], which was adapted from Lemma 4.2 in [4] (based on a 2003 draft). From this, we are able to bound the norm of the difference between the global and local refinement operators at a particular level of refinement in Lemma 5.4, which leads to the main result in this paper in Theorem 5.5.

- In Section 6, we give examples that demonstrate the exponential decay in the error between localized variational refinement curves. This shows that over a fixed interval we can approximate the variational refinement curve using a small subset of the original data. Following this, we use a domain decomposition similar to that in [4] to produce an approximation to some variational refinement curves with large number of data. Finally, we adapt these results to compute approximate variational refinement surfaces over over gridded data.

2. NON-UNIFORM VARIATIONAL REFINEMENT

Let $p = (p_i^0)$ be a bi-infinite sequence of points defined over knot sequence $t^0 = (t_i^0)$ with $t_i^0 < t_{i+1}^0$ for all i . The interpolatory refinement process with n -ring support determines a sequence of piecewise linear spline curves

$$s_n^\alpha := \sum_{|i| \leq 2^\alpha n} p_{n,i}^\alpha N_i^\alpha$$

indexed by α for $\alpha = 0, 1, \dots$, determined by the points $p_{n,i}^\alpha := s_n^\alpha(t_i^\alpha)$ and knots t_i^α , such that

$$\begin{aligned} p_n^0 &:= p^0, \\ p_{n,i}^\alpha &= 0 \text{ for } |i| > 2^\alpha n, \quad (\text{supp}(s_n^\alpha) \subset (t_{-2^\alpha n-1}^\alpha, t_{2^\alpha n+1}^\alpha)) \\ p_{n,2i}^\alpha &= p_{n,i}^{\alpha-1} \text{ for } |i| \leq 2^{\alpha-1} n, \quad \alpha > 0. \quad (\text{interpolation}) \end{aligned}$$

In variational refinement, the curves are computed by minimizing some energy functional. One of the simplest and most useful is the l_2 norm of the second divided differences, which mimics the functional commonly used in variational spline theory. With S^α the linear space of all piecewise linear curves over the knot sequence t^α , we define the energy functional

$$E_n^\alpha : S^\alpha \longrightarrow \mathbb{R} : s \mapsto \sum_{|i| \leq 2^\alpha n-1} |\Delta_i^2 s|^2$$

with $\Delta_i^2 : s \mapsto [t_{i-1}^\alpha, t_i^\alpha, t_{i+1}^\alpha]s$ the second divided-difference operator centered at t_i^α . The functional E_n^α is quadratic, and so the $2^\alpha n$ new points p_{2i+1}^α for $|i| < 2^{\alpha-1} n$ are computed by solving a (banded) linear system in terms of the $2^\alpha n + 1$ (generally) non-zero interpolated points p_{2i}^α for $|i| \leq 2^\alpha n$. We then compute the refined curves s_n^α at level α by solving the variational problem

$$(2.1) \quad s_n^\alpha := \operatorname{argmin} \{E_n^\alpha(s) : s \in S^\alpha, p_{n,2i}^\alpha = p_{n,i}^{\alpha-1} \text{ for } |i| \leq 2^{\alpha-1} n, p_{n,i}^\alpha = 0 \text{ for } |i| > 2^\alpha n\},$$

and we define $s_n := s_n^\infty$.

Throughout this paper it is assumed that the knot sequence $t^\alpha = (t_i^\alpha)$ at each level α have been preassigned in some way such that they are strictly increasing and satisfy the interpolation condition $t_{2i}^\alpha = t_i^{\alpha-1}$. In general these sequences are non-uniform, from which it follows that the refinement scheme is in general non-uniform and non-stationary (the *refinement mask* changes for each i and α). In this paper the knot differences are given by $h_{i,m} := t_{i+m}^1 - t_i^1$ with $h_i = h_{i,1}$, with $h_{\max}^\alpha := \max\{h_i^\alpha : i \in \mathbb{Z}\}$ and $h_{\min}^\alpha := \min\{h_i^\alpha : i \in \mathbb{Z}\}$. While these are defined over the entire infinite knot sequences, we remark that in practice we are concerned only with the finite number of knots involved in the energy functional $E_n(s)$. At the α -th stage in the refinement process, we define the *local mesh ratio* as

$$\max_i \frac{h_i^\alpha}{h_{i\pm 1}^\alpha}$$

and the *global mesh ratio* as

$$\rho^\alpha := \frac{h_{\max}^\alpha}{h_{\min}^\alpha}.$$

The following lemma gives a connection between the local and global mesh ratios that will be used later in the paper.

Lemma 2.1. *Suppose that $\frac{1}{\rho} \leq \frac{h_{2i+1}^\alpha}{h_{2i}^\alpha} \leq \rho$. Then $\frac{1}{1+\rho} \leq \frac{h_*^\alpha}{h_i^{\alpha-1}} \leq \frac{\rho}{1+\rho}$ for $*$ = $2i$ or $2i+1$.*

Proof. For the left inequality,

$$\frac{h_i^{\alpha-1}}{h_{2i}^\alpha} = \frac{h_{2i}^\alpha + h_{2i+1}^\alpha}{h_{2i}^\alpha} = 1 + \frac{h_{2i+1}^\alpha}{h_{2i}^\alpha} \leq 1 + \rho.$$

Likewise, $h_i^{\alpha-1}/h_{2i+1}^\alpha \leq 1 + \rho$. For the right inequality,

$$\frac{h_i^{\alpha-1}}{h_{2i}^\alpha} = \frac{h_{2i}^\alpha + h_{2i+1}^\alpha}{h_{2i}^\alpha} = 1 + \frac{h_{2i+1}^\alpha}{h_{2i}^\alpha} \geq 1 + \frac{1}{\rho} = \frac{\rho+1}{\rho}.$$

Likewise, $h_i^{\alpha-1}/h_{2i+1}^\alpha \geq (1+\rho)/\rho$. □

Lastly, the main results in this paper will depend on the following assumptions:

- A1: (s_n^α) and (s_k^α) are sequences generated by solving (2.1) with $k < n$.
- A2: $\sup_\alpha \rho^\alpha \leq \rho$ for some constant $\rho \geq 1$ ($\rho = 1$ for a uniform mesh).
- A3: $\sup_\alpha \|\Delta s_n^\alpha\|_\infty \leq M$ for some constant $M < \infty$ (uniformly Lipschitz).

3. REFINEMENT OPERATORS

For $0 \leq \alpha < \beta$ and over an n -window of the point set (p_i^0) , we define the operator

$$T_n^{\alpha,\beta} : S^\alpha \rightarrow S^\beta : s \mapsto T_n^{\alpha,\alpha+1} T_n^{\alpha+1,\alpha+2} \dots T_n^{\beta-1,\beta} s$$

with

$$T_n^{\alpha-1,\alpha} : S^{\alpha-1} \rightarrow S^\alpha : s \mapsto T_n^{\alpha-1,\alpha} s$$

the solution to problem (2.1) at any particular level α . Then,

$$T_n^{0,\alpha} : S^0 \rightarrow S^\alpha : s \mapsto T_n^{\alpha-1,\alpha} T_n^{\alpha-2,\alpha-1} \dots T_n^{0,1} s$$

is the operator that computes the variational refinement curve at level α , and

$$T_n^{0,\infty} : S^0 \rightarrow C^0 : s \mapsto \left(\lim_{\alpha \rightarrow \infty} T_n^{0,\alpha} \right) s$$

is the solution to the problem of variation refinement with n -window support (continuity of the limit curves is verified below). The norms of these linear operators are defined as:

$$\|T_n^{\alpha,\beta}\| := \sup \{ \|T_n^{\alpha,\beta} s\|_{\infty,n} : s \in S^\alpha, \|s\|_{\infty,n} = 1 \}$$

with

$$\|f\|_{\infty,n} := \max\{|f(t)| : t \in [t_{-n}^0, t_n^0]\}.$$

In the next result we bound these operators. Later in the paper we will bound $\|T_n^{0,\infty} - T_k^{0,\infty}\|$ restricted to a small interval, showing that this error decays exponentially as $k \uparrow n$ (Theorem 5.5).

Theorem 3.1. *Assume A1, A2 and A3. Let $\beta := \frac{\rho}{1+\rho}$ and $\Theta := 2M\beta h_{\max}^0$. For $\alpha \geq 1$,*

$$(3.1) \quad \|T_n^{0,\alpha} - T_n^{0,\alpha-1}\| \leq \Theta \beta^\alpha,$$

$$(3.2) \quad 1 \leq \|T_n^{\alpha-1,\alpha}\| \leq \Theta \beta^\alpha + 1,$$

$$(3.3) \quad 1 \leq \|T_n^{0,\alpha}\| \leq \|T_n^{0,\infty}\| \leq \rho \Theta + 1.$$

Proof. Since $h_{\max}^\alpha/h_{\min}^\alpha \leq \rho$ for all α , it follows that $1/\rho \leq h_{2i+1}^\alpha/h_{2i}^\alpha \leq \rho$ for all i . By Lemma 2.1, $h_*^\alpha/h_i^{\alpha-1} \leq \beta$ for $*$ = $2i$ and $*$ = $2i+1$. Moreover, this implies that $h_{\max}^\alpha \leq \beta^\alpha h_{\max}^0$. Let $s \in S^0$, and $s_n^{\alpha-1} := T_n^{0,\alpha-1} s$ and $s_n^\alpha := T_n^{0,\alpha} s$. Then, since $s_n^\alpha = T_n^{\alpha-1,\alpha} s_n^{\alpha-1}$, it follows that

$s_n^\alpha(t_{2i}^\alpha) = s_n^{\alpha-1}(t_{2i}^\alpha)$ for $|i| \leq 2^{\alpha-1}n$, and

$$\begin{aligned}
|s_n^\alpha(t_{2i+1}^\alpha) - s_n^{\alpha-1}(t_{2i+1}^\alpha)| &= \left| s_n^\alpha s(t_{2i+1}^\alpha) - \left(\frac{h_{2i+1}^\alpha}{h_{2i,2}^\alpha} s_n^{\alpha-1}(t_{2i}^\alpha) + \frac{h_{2i}^\alpha}{h_{2i,2}^\alpha} s_n^{\alpha-1}(t_{2i+2}^\alpha) \right) \right| \\
&= \left| \frac{h_{2i+1}^\alpha}{h_{2i,2}^\alpha} (s_n^\alpha(t_{2i+1}^\alpha) - s_n^{\alpha-1}(t_{2i}^\alpha)) - \frac{h_{2i}^\alpha}{h_{2i,2}^\alpha} (s_n^{\alpha-1}(t_{2i+2}^\alpha) - s_n^\alpha(t_{2i+1}^\alpha)) \right| \\
&= \frac{h_{2i}^\alpha h_{2i+1}^\alpha}{h_{2i,2}^\alpha} \left| \frac{s_n^\alpha(t_{2i+1}^\alpha) - s_n^{\alpha-1}(t_{2i}^\alpha)}{h_{2i}^\alpha} - \frac{s_n^{\alpha-1}(t_{2i+2}^\alpha) - s_n^\alpha(t_{2i+1}^\alpha)}{h_{2i+1}^\alpha} \right| \\
&\leq 2 \frac{h_{2i}^\alpha h_{2i+1}^\alpha}{h_{2i,2}^\alpha} M \leq 2 \beta h_{\max}^\alpha M \\
&\leq 2 \beta h_{\max}^0 \beta^\alpha M = \Theta \beta^\alpha.
\end{aligned}$$

Since this is independent of i , it follows that

$$\|s_n^\alpha - s_n^{\alpha-1}\|_{\infty, n} \leq \Theta \beta^\alpha,$$

and so

$$\|T_n^{0,\alpha} - T_n^{0,\alpha-1}\| \leq \Theta \beta^\alpha.$$

For the second bound, let $s \in S^{\alpha-1}$ such that $\|s\|_{\infty, n} = 1$. Then

$$\|T_n^{\alpha-1,\alpha} s\|_{\infty, n} \leq \|T_n^{\alpha-1,\alpha} s - s\|_{\infty, n} + \|s\|_{\infty, n} \leq \Theta \beta^\alpha + 1,$$

and so

$$\|T_n^{\alpha-1,\alpha}\| \leq \Theta \beta^\alpha + 1.$$

Moreover, since the coefficients of the broken line $s \in S^{\alpha-1}$ are a subset of the coefficients of $T_n^{\alpha-1,\alpha} s$, it follows that $\|T_n^{\alpha-1,\alpha}\|_{\infty, n}$ is at least as large as $\|s\|_{\infty, n}$, and so $\|T_n^{\alpha-1,\alpha}\| \geq 1$. For the third bound let $s \in S^0$. Then, we have

$$\begin{aligned}
\|T_n^{0,\alpha} s\|_{\infty, n} &\leq \|T_n^{0,\alpha} s - T_n^{0,\alpha-1} s\|_{\infty, n} + \|T_n^{0,\alpha-1} s - T_n^{0,\alpha-2} s\|_{\infty, n} + \dots \\
&\quad + \|T_n^{0,2} s - T_n^{0,1} s\|_{\infty, n} + \|T_n^{0,1} s\|_{\infty, n} \\
&\leq (\Theta \beta^\alpha + \Theta \beta^{\alpha-1} + \dots + \Theta \beta^2 + (\Theta \beta + 1)) \|s\|_{\infty, n} \\
&\leq \left(\frac{\Theta}{1-\beta} - \Theta + 1 \right) \|s\|_{\infty, n} = \left(\frac{\beta}{1-\beta} + 1 \right) \|s\|_{\infty, n} = (\rho \Theta + 1) \|s\|_{\infty, n}.
\end{aligned}$$

Therefore, $\|T_n^{0,\alpha}\| \leq \rho \Theta + 1$. Since this is independent of α , it follows that $\|T_n^{0,\infty}\| \leq \rho \Theta + 1$ as well. And since $T_n^{0,\alpha} = T_n^{\alpha-1,\alpha} T_n^{0,\alpha-1}$, then by the argument used above it must be that $\|T_n^{0,\alpha}\|$ is at least as large as $\|T_n^{0,\alpha-1}\|$, and so we get monotonicity on the sequence of norms of these operators for increasing α . \square

Corollary 3.2. *Assume A1, A2 and A3. Then, the sequence s_n^α converges uniformly to a continuous function s_n^∞ .*

Proof. Since $\beta < 1$, it follows by the first inequality in Theorem 3.1 that (s_k^α) is a uniformly Cauchy sequence of continuous functions, hence converges uniformly to a continuous function. \square

Remark: Smoothness is investigated in more detail in [1].

4. STABILITY ESTIMATE

The results in this paper depend on the following *stability estimate*. For this, we introduce the notation:

$$\|\Delta^2 s\|_m^2 = \sum_{|i| \leq m} |\Delta_i^2 s|^2 \quad \text{and} \quad |c|_m^2 = \sum_{|i| \leq m} c_i^2.$$

Lemma 4.1. (*Stability*) *Suppose that $s = \sum_{|i| \leq m+p} c_i N_i$ on the knot sequence $(t_i : i \in \mathbb{Z})$ for some $p \in \mathbb{Z}$ with $m+p > 0$. Let $K_1 := \frac{1}{h_{\max}^4}$ and $K_2 := \frac{2}{h_{\min}^4}$. Assume that $c_i := 0$ if $|i| > m+p$. Then,*

$$\|\Delta^2 s\|_m^2 \leq \frac{9}{4} K_2 |c|_{m+1}^2.$$

If moreover $s(t_{2i}) = 0$ for all i , then

$$(4.1) \quad K_1 |c|_{m+1}^2 \leq \|\Delta^2 s\|_m^2 \leq K_2 |c|_{m+1}^2,$$

and if $p = 0$ as well, then

$$(4.2) \quad K_1 |c|_m^2 \leq \|\Delta^2 s\|_m^2 \leq K_2 |c|_m^2.$$

Proof. Firstly, note that if $p = 0$ we can define $c_i = 0$ for $i = \pm(m+1)$, from which it follows that $|c|_{m+1} = |c|_m$. Hence, the third result follows from the second when $p = 0$.

The first inequality follows from

$$\begin{aligned}
\|\Delta^2 s\|_m^2 &= \sum_{|i| \leq m} |\Delta_i^2 s|^2 = \sum_{|i| \leq m} \left| \frac{1}{t_{i+1} - t_{i-1}} \left(\frac{c_{i+1} - c_i}{t_{i+1} - t_i} - \frac{c_i - c_{i-1}}{t_i - t_{i-1}} \right) \right|^2 \\
&\leq \frac{1}{4h_{\min}^4} \sum_{|i| \leq m} |c_{i+1} - 2c_i + c_{i-1}|^2 \\
&\leq \frac{1}{4h_{\min}^4} \sum_{|i| \leq m} 3(c_{i+1}^2 + 4c_i^2 + c_{i-1}^2) \quad (\text{by Cauchy - Schwarz}) \\
&\leq \frac{18}{4h_{\min}^4} \sum_{|i| \leq m+1} c_i^2 = \frac{9}{2} \frac{1}{h_{\min}^4} |c|_{m+1}^2 = \frac{9}{4} K_2 |c|_{m+1}^2.
\end{aligned}$$

In the special case that $s(t_{2i}) = 0$ for all i , it follows that $|c_{i+1} - 2c_i + c_{i-1}|^2$ equals $4c_i^2$ if i is odd and is bounded by $2(c_{i+1}^2 + c_{i-1}^2)$ if i is even. Therefore, each nonzero $|c_i|^2$ appears at most 8 times, and so

$$\|\Delta^2 s\|_m^2 \leq \frac{1}{4h_{\min}^4} \sum_{|i| \leq m+1} 8|c_i|^2 = \frac{2}{h_{\min}^4} \sum_{|i| \leq m+1} |c_i|^2.$$

For the remaining inequality, let $p(t)$ be the quadratic polynomial through the three points $s(t_{2i}) = 0$, $s(t_{2i+1})$ and $s(t_{2i+2}) = 0$. Then, for $t_{2i} \leq t \leq t_{2i+2}$,

$$\begin{aligned}
|p(t)| &= \left| s(t_{2i}) + \frac{s(t_{2i+2}) - s(t_{2i})}{t_{2i+2} - t_{2i}} (t - t_{2i}) + \Delta_{2i+1}^2 s (t - t_{2i})(t - t_{2i+2}) \right| \\
&= |\Delta_{2i+1}^2 s (t - t_{2i})(t - t_{2i+2})| \\
&\leq |\Delta_{2i+1}^2 s| \left(\frac{t_{2i} + t_{2i+2}}{2} \right)^2 \\
&\leq |\Delta_{2i+1}^2 s| h_{\max}^2.
\end{aligned}$$

In particular, we have that $|c_{2i+1}| = |p(t_{2i+1})| \leq |\Delta_{2i+1}^2 s| h_{\max}^2$ and $c_{2i} = s(t_{2i}) = 0$ for all i . And so,

$$\sum_{|i| \leq m+1} |c_i|^2 = \sum_{|2i+1| \leq m+1} |c_{2i+1}|^2 \leq h_{\max}^4 \sum_{|2i+1| \leq m} |\Delta_{2i+1}^2 s|^2 \leq h_{\max}^4 \sum_{|i| \leq m} |\Delta_i^2 s|^2 = h_{\max}^4 \|\Delta^2 s\|_m^2.$$

□

5. ERROR IN THE LOCAL VARIATIONAL REFINEMENT SCHEME

To characterize solutions to (2.1) we will develop some orthogonality conditions in appropriate inner product spaces of piecewise linear splines, and on sequences. For this, we introduce some notation. Let

$$(p \cdot q)_{k,n} := \sum_{k \leq |i| \leq n} p_i q_i \quad \text{and} \quad |p|_{k,n} := \sqrt{(p \cdot p)_{k,n}}$$

with $(p \cdot q)_n := (p \cdot q)_{0,n}$ and $|p|_n := |p|_{0,n}$. On piecewise linear splines, let

$$\langle f, g \rangle_{k,n} := \sum_{k \leq |i| \leq n} f(t_i) \cdot g(t_i) \quad \text{and} \quad \|\cdot\|_{k,n} := \sqrt{\langle \cdot, \cdot \rangle_{k,n}}$$

with $\langle f, g \rangle_n := \langle f, g \rangle_{0,n}$ and $\|\cdot\|_n := \|\cdot\|_{0,n}$. Note that for $s \in S^\alpha$

$$E_n^\alpha(s) = \sum_{|i| \leq 2^{\alpha n-1}} \Delta_i^2 s \cdot \Delta_i^2 s = \langle \Delta^2 s, \Delta^2 s \rangle_{2^{\alpha n-1}} = \|\Delta^2 s\|_{2^{\alpha n-1}}^2.$$

Then, for $r = 1, \dots, 2^{\alpha n}$, let

$$H_r^\alpha := \{g \in S^\alpha : g(t_{2i}^\alpha) = 0 \text{ for all } i \text{ and } g(t_i^\alpha) = 0 \text{ for } |i| > r\}$$

be the set of variations for problem (2.1), with inner product

$$\langle \Delta^2 f, \Delta^2 g \rangle_{2^{\alpha n-1}}.$$

Note that while the inner product is not positive definite on S^α , it is positive definite on H_r^α . In particular, while $\|\Delta^2 \cdot\|_{2^{\alpha n-1}}$ is only a semi-norm on the space S^α of all piecewise linear splines, it is a norm on H_r^α .

Lemma 5.1. *Let $k < n$ and $0 < r \leq 2^{\alpha k} - 2$. Let $s \in S^{\alpha-1}$ and $s_n := T_n^{\alpha-1, \alpha} s$ and $s_k := T_k^{\alpha-1, \alpha} s$. Then, $s_n - s_k \in (H_r^\alpha)^\perp$. I.e., $\langle \Delta^2(s_n - s_k), \Delta^2 g_r \rangle_{2^{\alpha n-1}} = 0$ for all $g_r \in H_r^\alpha$.*

Proof. Let $g_r \in H_r^\alpha$. Since $H_r^\alpha \subset H_{2^{\alpha k}}^\alpha \subset H_{2^{\alpha n}}^\alpha$, it follows that $g_r \in H_{2^{\alpha k}}^\alpha$ and $g_r \in H_{2^{\alpha n}}^\alpha$. That is, g_r is a variation of (2.1) for both s_n and s_k . Therefore, it follows that

$$0 = \frac{d}{d\theta} \Big|_{\theta=0} \langle \Delta^2 s_n + \theta \Delta^2 g_r, \Delta^2 s_n + \theta \Delta^2 g_r \rangle_{2^{\alpha n-1}} = 2 \langle \Delta^2 s_n, \Delta^2 g_r \rangle_{2^{\alpha n-1}},$$

and likewise $\langle \Delta^2 s_k, \Delta^2 g_r \rangle_{2^{\alpha k-1}} = 0$. Moreover, since $g_r(t_i^\alpha) = 0$ for $i \geq 2^{\alpha k} - 1$, it follows that $\langle \Delta^2 s_k, \Delta^2 g_r \rangle_{2^{\alpha n-1}} = \langle \Delta^2 s_k, \Delta^2 g_r \rangle_{2^{\alpha k-1}} = 0$, and so $0 = \langle \Delta^2 s_k, \Delta^2 g_r \rangle_{2^{\alpha n-1}} = 0$. Therefore,

$$\langle \Delta^2 s_n - \Delta^2 s_k, \Delta^2 g_r \rangle_{2^{\alpha n-1}} = 0$$

for all $g_r \in H_r^\alpha$, and so $s_n - s_k \in (H_r^\alpha)^\perp$ with respect to the inner product $\langle \Delta^2 \cdot, \Delta^2 \cdot \rangle_{2^{\alpha n-1}}$. \square

The next lemma is a generalization to Lemma 6 in [3], to which it reduces when $l = 0$. It will be used in the Lemma 5.3, the main lemma in this section.

Lemma 5.2. *Let γ and a_0, \dots, a_m be non-negative real numbers. Suppose that*

$$\gamma(a_0 + \dots + a_{k-1}) \leq a_k$$

for $k = 1, \dots, m$. Then, for $k = l + 1, \dots, m$ and $0 \leq l < k$,

$$\gamma(1 + \gamma)^{k-1-l} (a_0 + \dots + a_l) \leq a_k.$$

Proof. Let $S_l := a_0 + \dots + a_l$. Note that $\gamma S_l \leq a_{l+1}$ establishes the result for $k = l + 1$. Now, assume true for $l + 1, \dots, k$ for some k . Then

$$\begin{aligned} a_{k+1} &\geq \gamma(S_l + a_{l+1} + \dots + a_k) \\ &\geq \gamma S_l(1 + \gamma + \gamma(1 + \gamma) + \dots + \gamma(1 + \gamma)^{k-1-l}) \\ &= \gamma S_l(1 + \gamma) (1 + \gamma + \gamma(1 + \gamma) + \dots + \gamma(1 + \gamma)^{k-2-l}) \\ &= \gamma S_l(1 + \gamma) \left(1 + \gamma \frac{(1 + \gamma)^{k-1-l} - 1}{(1 + \gamma) - 1} \right) \\ &= \gamma(S_l 1 + \gamma)(1 + (1 + \gamma)^{k-1-l} - 1) \\ &= \gamma S_l(1 + \gamma)^{k-l}. \end{aligned}$$

□

The rest of this paper depends on the following key result that shows that the error between k -windowed and n -windowed solutions on a fixed small interval is bounded by the error over a ring far away from this interval with an error that decreases exponentially as k increases towards n . Note that the interval in question is t_i^α for $|i| \leq 2^\alpha l$. But this is the same as the interval t_i^0 for $|i| \leq l$ in the original knot sequence. That is,

$$[t_{-l}^0, t_l^0] = [t_{-2l}^1, t_{2l}^1] = [t_{-4l}^2, t_{4l}^2] = \dots = [t_{-2^\alpha l}^\alpha, t_{2^\alpha l}^\alpha].$$

Hence, we are interested in the error over a *fixed* interval.

Lemma 5.3. *Let $s \in S^{\alpha-1}$, $\alpha \geq 1$. Assume A2. Assume $0 \leq l < k < n$. Then,*

$$\|(T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha})s\|_{2^\alpha l} \leq C_1 \sigma^{\alpha(k-l)} \|(T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha})s\|_{2^\alpha k-3, 2^\alpha k-1}$$

with

$$C_1 = 2 \frac{\rho^4}{\sigma^3} \quad \text{and} \quad \sigma = \sqrt{\frac{2\rho^4}{1 + 2\rho^4}}.$$

Proof. Let $v_n := T_n^{\alpha-1, \alpha} s$ and $v_k := T_k^{\alpha-1, \alpha} s$. Then $e := v_n - v_k = \sum_{|i| \leq 2^\alpha n} c_i N_i^\alpha$ with $c_{2i} = e(t_{2i}^\alpha) = 0$ for all i and $c_i = e(t_i^\alpha) = 0$ for $|i| \geq 2^\alpha n$. Hence, $e \in H_{2^\alpha n}^\alpha$. Now let

$$w_r := \sum_{|i| \leq r} c_i N_i^\alpha \quad \text{and}$$

$$u_r := e - w_r = \sum_{|i| > r} c_i N_i^\alpha$$

for any r such that $0 < r \leq 2^\alpha k - 2$. Then $w_r \in H_r^\alpha$, and so by Lemma 5.1 it follows that $\langle \Delta^2 e, \Delta^2 w_r \rangle_{2^\alpha n} = 0$. Observing that $\Delta_i^2 w_r = 0$ for $|i| > r + 1$ and $\Delta_i^2 u_r = 0$ for $|i| < r$, we have

$$\begin{aligned} \|\Delta^2 w_r\|_{2^\alpha n-1}^2 &= \langle \Delta^2 w_r, \Delta^2 w_r \rangle_{2^\alpha n-1} = \langle \Delta^2(e - u_r), \Delta^2 w_r \rangle_{2^\alpha n-1} \\ &= \langle -\Delta^2 u_r, \Delta^2 w_r \rangle_{2^\alpha n-1} = - \sum_{r+1 \leq |i| \leq r+1} \Delta_i^2 u_r \Delta_i^2 w_r \\ &\leq \|\Delta^2 u_r\|_{r, r+1} \|\Delta^2 w_r\|_{2^\alpha n-1}. \end{aligned}$$

This gives $\|\Delta^2 w_r\|_{2^\alpha n-1} \leq \|\Delta^2 u_r\|_{r, r+1}$. By the stability condition in Lemma 4.1,

$$K_1 |c_r|^2 \leq \|\Delta^2 w_r\|_r^2 \leq \|\Delta^2 w_r\|_{2^\alpha n-1}^2 \leq \|\Delta^2 u_r\|_{r, r+1}^2 \leq K_2 \sum_{r+1 \leq |i| \leq r+2} |c_i|^2,$$

which can be written

$$\gamma(a_0 + a_1 + \cdots + a_{s-1}) \leq a_s$$

with

$$\begin{aligned} a_i &:= \sum_{2i \leq j \leq 2i+1} |c_j|^2, \quad (\text{noting that } c_{2i} = 0 \text{ for all } i) \\ \gamma &:= K_1/K_2 = 1/(2\rho^4) \quad \text{and} \\ s &:= \lfloor (r+1)/2 \rfloor. \end{aligned}$$

Since $0 < r \leq 2^\alpha k - 2$, this inequality holds for $s = 0, \dots, m := 2^{\alpha-1}k - 1$, and by Lemma 5.2,

$$a_0 + \cdots + a_t \leq \frac{1}{\gamma} \hat{\sigma}^{s-1-t} a_s$$

for $s = t+1, \dots, m$ and $0 \leq t < s$, with $\hat{\sigma} := \frac{1}{1+\gamma} = \frac{K_2}{K_1+K_2} = \frac{2\rho^4}{1+2\rho^4}$. In particular, with $t = 2^\alpha l$ and $s = 2^\alpha k - 2$, it follows that

$$a_0 + \cdots + a_{2^\alpha l} \leq \frac{1}{\gamma} \hat{\sigma}^{2^\alpha k - 2^\alpha l - 3} a_{2^\alpha k - 2}.$$

Therefore,

$$\begin{aligned}
\|\Delta^2 e\|_{2^{\alpha l}}^2 &\leq \|\Delta^2 \sum_{|i| \leq 2^{\alpha l+1}} c_i N_i^\alpha\|_{2^{\alpha n-1}}^2 \\
&\leq K_2 |c|_{2^{\alpha l+2}} \quad (\text{by stability}) \\
&\leq K_2 (a_0 + \cdots + a_{2^{\alpha l+1}}) \\
&\leq \frac{1}{\gamma} \hat{\sigma}^{2^{\alpha k} - 2^{\alpha l} - 3} a_{2^{\alpha k} - 2} \\
&= \frac{K_2^2}{K_1^2} \hat{\sigma}^{2^{\alpha k} - 2^{\alpha l} - 3} \sum_{|i|=2^{\alpha k} - 2} |c_i|^2 \\
&\leq \frac{K_2^2}{K_1^2} \hat{\sigma}^{2^{\alpha k} - 2^{\alpha l} - 3} \|\Delta^2 e\|_{2^{\alpha k-3}, 2^{\alpha k-1}}^2 \quad (\text{by stability})
\end{aligned}$$

And so

$$\|\Delta^2 e\|_{2^{\alpha l}} \leq 2\rho^4 \sigma^{2^{\alpha(k-l)} - 3} \|\Delta^2 e\|_{2^{\alpha k-3}, 2^{\alpha k-1}},$$

with, from Lemma 4.1, $K_2/K_1 = 2\rho^4$ and $\sigma := \sqrt{\hat{\sigma}} = \sqrt{K_2/(K_1 + K_2)} = \sqrt{\frac{2\rho^4}{1+2\rho^4}}$. And since $0 < \sigma < 1$ and $\alpha < 2^\alpha$ when $\alpha \geq 0$, we have

$$\|\Delta^2 e\|_{2^{\alpha l}} \leq C_1 \sigma^{\sigma(k-l)} \|\Delta^2 e\|_{2^{\alpha k-3}, 2^{\alpha k-1}}$$

$$C_1 = 2\rho^4/\sigma^3. \quad \square$$

Now we will bound the operator norm of the difference $T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha}$.

Lemma 5.4. *Assume A1, A2 and A3. Let $\beta := \frac{\rho}{1+\rho}$ and $\Theta := 2M\beta h_{\max}^0$. Let $0 \leq l < k < n$. Then,*

$$\|T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha}\|_{2^{\alpha l}} \leq C_2 \sigma^{\alpha(k-l)} (\Theta \beta^\alpha + 1)$$

with $C_2 := 20\rho^6/\sigma^3$ and $\sigma = \sqrt{\frac{2\rho^4}{1+2\rho^4}}$.

Proof. Let $s \in S^{\alpha-1}$ and $e := T_n^{\alpha-1, \alpha} s - T_k^{\alpha-1, \alpha} s$. Then $e = \sum_{|i| \leq 2^{\alpha n}} c_i N_i^\alpha$ for some sequence (c_i) with $c_{2i} = 0$ for $|i| \leq 2^{\alpha-1}k$. Therefore,

$$\begin{aligned}
\Delta_{2^{j+1}}^2 e &= \frac{1}{h_{2^j, 2}^\alpha} \left(\frac{c_{2^{j+2}} - c_{2^{j+1}}}{h_{2^{j+1}}^\alpha} - \frac{c_{2^{j+1}} - c_{2^j}}{h_{2^j}^\alpha} \right) \\
&= \frac{1}{h_{2^j, 2}^\alpha} \left(\frac{-c_{2^{j+1}}}{h_{2^{j+1}}^\alpha} - \frac{c_{2^{j+1}}}{h_{2^j}^\alpha} \right) = -\frac{c_{2^{j+1}}}{h_{2^j}^\alpha h_{2^{j+1}}^\alpha}.
\end{aligned}$$

Then, with $\|e\|_{\infty, 2^{\alpha l}} = |c_{2j+1}|$ for some j with $|2j+1| \leq 2^{\alpha l}$, it follows that

$$\|e\|_{\infty, 2^{\alpha l}} = |c_{2j+1}| \leq (h_{\max}^{\alpha})^2 |\Delta_{2j+1}^2 e| \leq (h_{\max}^{\alpha})^2 \|\Delta^2 e\|_{\infty, 2^{\alpha l}}.$$

And so, with $C_1 = 2\rho^4/\sigma^3$,

$$\begin{aligned} \|(T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha})s\|_{\infty, 2^{\alpha l}} &= \|e\|_{\infty, 2^{\alpha l}} \leq (h_{\max}^{\alpha})^2 \|\Delta^2 e\|_{\infty, 2^{\alpha l}} \leq (h_{\max}^{\alpha})^2 \|\Delta^2 e\|_{2^{\alpha l}} \\ &\leq (h_{\max}^{\alpha})^2 C_1 \sigma^{\alpha(k-l)} \|\Delta^2 e\|_{2^{\alpha k-3}, 2^{\alpha k-1}} \quad (\text{by Lemma 5.3}) \\ &\leq (h_{\max}^{\alpha})^2 C_1 \sigma^{\alpha(k-l)} 5 \|\Delta^2 e\|_{\infty, 2^{\alpha k-3}, 2^{\alpha k-1}} \\ &\leq (h_{\max}^{\alpha})^2 C_1 \sigma^{\alpha(k-l)} 5 \|e\|_{\infty, 2^{\alpha k-3}, 2^{\alpha k-1}} / (h_{\min}^{\alpha})^2 \\ &\leq 5\rho^2 C_1 \sigma^{\alpha(k-l)} (\|T_n^{\alpha-1, \alpha} s\|_{\infty, 2^{\alpha k-3}, 2^{\alpha k-1}} + \|T_k^{\alpha-1, \alpha} s\|_{\infty, 2^{\alpha k-3}, 2^{\alpha k-1}}) \\ &\leq 5\rho^2 C_1 \sigma^{\alpha(k-l)} (\|T_n^{\alpha-1, \alpha} s\|_{\infty, 2^{\alpha n}} + \|T_k^{\alpha-1, \alpha} s\|_{\infty, 2^{\alpha k}}) \\ &\leq 5\rho^2 C_1 \sigma^{\alpha(k-l)} (\|T_n^{\alpha-1, \alpha}\| \|s\|_{\infty, 2^{\alpha n}} + \|T_k^{\alpha-1, \alpha}\| \|s\|_{\infty, 2^{\alpha k}}) \\ &\leq 10\rho^2 C_1 \sigma^{\alpha(k-l)} (\Theta \beta^{\alpha} + 1) \|s\|_{\infty, 2^{\alpha n}} \\ &= C_2 \sigma^{\alpha(k-l)} (\Theta \beta^{\alpha} + 1) \|s\|_{\infty, 2^{\alpha n}} \end{aligned}$$

□

We can now establish the main result of this paper concerning the exponential decay in the error between local and global variational refinement curves.

Theorem 5.5. *Assume A1, A2 and A3. Let $0 \leq l < k < n$. Then, for $1 \leq \alpha \leq \infty$,*

$$\|T_n^{0, \alpha} - T_k^{0, \alpha}\|_{2^{\alpha l}} \leq C_3 \sigma^{k-l}$$

with

$$\hat{\Theta} = \frac{2M h_{\max}^0 \rho^2}{1 + \rho}, \quad C_3 := \frac{20\rho^6 (\hat{\Theta} + 1) \exp(\hat{\Theta})}{(1 - \sigma)\sigma^3} \quad \text{and} \quad \sigma = \sqrt{\frac{2\rho^4}{1 + 2\rho^4}}.$$

Proof. By Theorem 3.1 and Lemma 5.4 we have that

$$\begin{aligned} \|T_n^{\alpha-1, \alpha}\| &\leq \Theta \beta^{\alpha} + 1, \\ \|T_n^{0, \alpha}\| &\leq \rho \Theta + 1, \\ \|T_n^{\alpha-1, \alpha} - T_k^{\alpha-1, \alpha}\|_{2^{\alpha l}} &\leq C_2 \sigma^{\alpha(k-l)} (\Theta \beta^{\alpha} + 1) \end{aligned}$$

with $C_2 := 20\rho^6/\sigma^3$. We will also need the estimate

$$\prod_{j=1}^m (\Theta \beta^j + 1) \leq \exp(\rho \Theta),$$

which can be derived in a standard way:

$$\ln \left(\prod_{j=1}^m (\Theta \beta^j + 1) \right) = \sum_{j=1}^m \ln(\Theta \beta^j + 1) \leq \sum_{j=1}^m \Theta \beta^j \leq \Theta \frac{\beta}{1-\beta} = \Theta \rho.$$

Assume from here on that all operator norms are restricted to $|i| \leq 2^\alpha l$, and let $\nu := \sigma^{k-l}$. At the first level of subdivision we have

$$\|T_n^{0,1} - T_k^{0,1}\| \leq C_2 \nu (\Theta \beta + 1),$$

and at the next level

$$\begin{aligned} \|T_n^{0,2} - T_k^{0,2}\| &= \|T_n^{1,2} T_n^{0,1} - T_k^{1,2} T_k^{0,1}\| \\ &= \|T_n^{1,2} (T_n^{0,1} - T_k^{0,1}) + (T_n^{1,2} - T_k^{1,2}) T_k^{0,1}\| \\ &\leq \|T_n^{1,2}\| \|T_n^{0,1} - T_k^{0,1}\| + \|T_n^{1,2} - T_k^{1,2}\| \|T_k^{0,1}\|. \end{aligned}$$

Proceeding recursively, we have

$$\begin{aligned} \|T_n^{0,\alpha} - T_k^{0,\alpha}\| &\leq \|T_n^{\alpha-1,\alpha} - T_k^{\alpha-1,\alpha}\| \|T_k^{0,\alpha-1}\| \\ &\quad + \|T_n^{\alpha-1,\alpha}\| \|T_n^{\alpha-2,\alpha-1} - T_k^{\alpha-2,\alpha-1}\| \|T_k^{0,\alpha-2}\| \\ &\quad + \|T_n^{\alpha-1,\alpha}\| \|T_n^{\alpha-2,\alpha-1}\| \|T_n^{\alpha-3,\alpha-2} - T_k^{\alpha-3,\alpha-2}\| \|T_k^{0,\alpha-3}\| \\ &\quad \vdots \\ &\quad + \|T_n^{\alpha-1,\alpha}\| \|T_n^{\alpha-2,\alpha-1}\| \cdots \|T_n^{1,2}\| \|T_n^{0,1} - T_k^{0,1}\|. \end{aligned}$$

From the operator bounds above, it follows that

$$\begin{aligned} \|T_n^{0,\alpha} - T_k^{0,\alpha}\| &\leq C_2 (\rho \Theta + 1) (\Theta \beta^\alpha + 1) \nu^\alpha \\ &\quad + C_2 (\rho \Theta + 1) (\Theta \beta^\alpha + 1) (\Theta \beta^{\alpha-1} + 1) \nu^{\alpha-1} \\ &\quad + C_2 (\rho \Theta + 1) (\Theta \beta^\alpha + 1) (\Theta \beta^{\alpha-1} + 1) \nu^{\alpha-2} \\ &\quad + C_2 (\rho \Theta + 1) (\Theta \beta^\alpha + 1) (\Theta \beta^{\alpha-1} + 1) (\Theta \beta^{\alpha-2} + 1) \nu^{\alpha-3} \\ &\quad \vdots \\ &\quad + C_2 (\rho \Theta + 1) (\Theta \beta^\alpha + 1) (\Theta \beta^{\alpha-1} + 1) \cdots (\Theta \beta + 1) \nu. \end{aligned}$$

And so,

$$\begin{aligned}
\|T_n^{0,\alpha} - T_k^{0,\alpha}\| &\leq C_2(\rho\Theta + 1) \exp(\rho\Theta)(\nu^\alpha + \nu^{\alpha-1} + \cdots + \nu) \\
&\leq C_2(\rho\Theta + 1) \exp(\rho\Theta) \frac{\nu}{1-\nu} \\
&= C_2(\rho\Theta + 1) \exp(\rho\Theta) \frac{\sigma^{k-l}}{1-\sigma^{k-l}} \\
&\leq C_2(\rho\Theta + 1) \exp(\rho\Theta) \frac{\sigma^{k-l}}{1-\sigma}
\end{aligned}$$

This result is independent of α , and so it follows that

$$\|T_n^{0,\alpha} - T_k^{0,\alpha}\| \leq C_2(\rho\Theta + 1) \exp(\rho\Theta) \frac{\sigma^{k-l}}{1-\sigma} = C_3 \sigma^{k-l}$$

for any α , including $\alpha = \infty$. □

By Theorem 5.5, the decay in the error between n -window and k -window variational refinement curves restricted to an l -ring

$$[t_{-l}^0, t_l^0] = [t_{-2l}^1, t_{2l}^1] = [t_{-4l}^2, t_{4l}^2] = \cdots = [t_{-2^\alpha l}^\alpha, t_{2^\alpha l}^\alpha]$$

centered at $i = 0$ is exponential as k increases to n . In particular, if we choose $l = 1$, the error is measured over the smallest nontrivial ring $[t_{-1}^0, t_1^0]$ (if $l = 0$, then the interval reduces to the point t_i^0 , which is interpolated by the refinement curves). These rings can of course be centered at any location on the knot interval.

6. EXAMPLES

Example 6.1. Variational Refinement of the Delta Function.

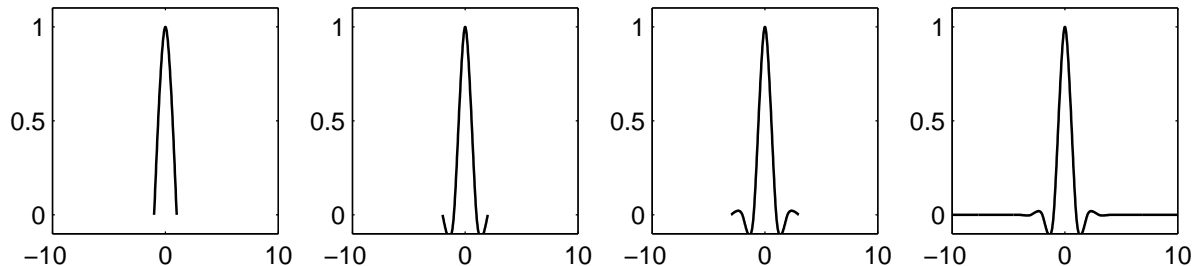
Let $p^0(i) = \delta_{i,0}$ be points on the Kroenecker delta function refined over integer knot sequence $t^0 = (-10, -9, \dots, 9, 10)$ centered around $t_0^0 = 0$ (hence, $n = 10$). In this example we will subdivide uniformly. In this example we compute s_k^α for $k = 1, \dots, n$ up to $\alpha = 5$ (five levels of refinement). At the final level $2^\alpha n = 2^5 10 = 320$, and so the final coefficient and knot sequences are $p_{-320}^5, p_{-319}^5, \dots, p_{320}^5$ and $t_{-320}^5, t_{-319}^5, \dots, t_{320}^5$, respectively.

To assess the convergence of the localized schemes, we evaluate the errors

$$e_{k,n}^5 := \|s_n^5 - s_k^5\|_{\infty, 2^5 l} = \max\{|s_n^5(t_i^5) - s_k^5(t_i^5)| : |i| \leq 2^5 l\}$$

for $l = 1$ and $k = 1, 2, \dots, n = 10$. Equivalently, since $t_i = -1$ for $i = -2^5 l$ and $t_i = 1$ for $i = 2^5 l$, the maximum error is being evaluated over the interval $[-1, 1]$. In Figure 6.1,

the refinement curves s_k^5 at level $\alpha = 5$ are plotted for $k = 1, 2, 3$ and 10 , and a table with the errors $e_{k,n}^5$ for $k = 1, 2, \dots, n$ with $n = 10$, illustrating the exponential rate of decay verified in this paper. The log of the error is given as well, which is approximately linear with respect to the iteration number k . From this, we estimate the rate of decay in the error as $e_{k,n}^5 = 3.4 (.0296)^k$ as $k \uparrow n$.



	k=1	2	3	4	5	6	7	8	8	10
$e_{k,n}^5$	9.5e-2	3.0e-3	8.9e-5	2.6e-6	7.7e-8	2.3e-9	6.7e-11	2.0e-12	5.6e-14	0
$\log(e_{k,n}^5)$	-2.35	-5.80	-9.33	-1.28	-1.64	-1.99	-2.34	-2.69	-3.05	

FIGURE 6.1. Plot of s_k^5 for $k = 1, 2, 3, 10$, with errors $e_{k,n}^5$ for $k = 1, 2, \dots, 10$.

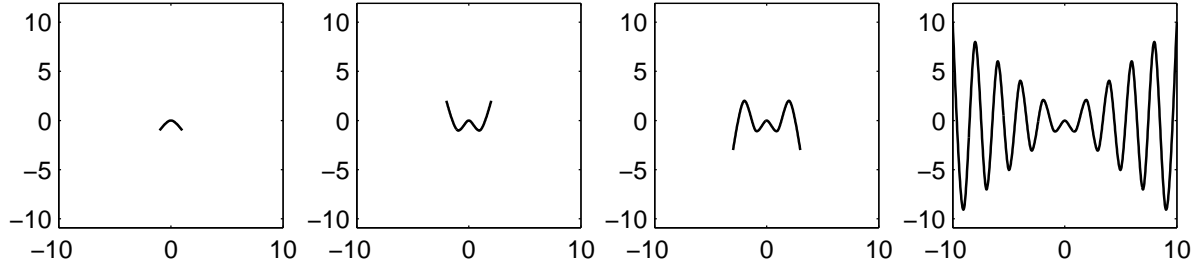
Example 6.2. Another Curve Example.

Let p^0 be the sequence of points on the curve $f(t) = t \cos(3\pi t)$ evaluated at the knot sequence $t^0 = (-10, -9, \dots, 10)$. As in Example 6.1, we will subdivide uniformly. In Figure 6.2, the refinement curves s_k^5 at level $\alpha = 5$ are plotted for $k = 1, 2, 3$ and 10 , and a table with the errors $e_{k,n}^5$ for $k = 1, 2, \dots, n$ with $n = 10$, illustrating the exponential rate of decay verified in this paper. The rate of decay in the error in this example is approximately $e_{k,n}^5 = 3.5 (.2188)^k$ as $k \uparrow n$, which is slower than the rate for the delta function, as might be expected. It is interesting that the constant 3.5 is approximately the same as above.

Example 6.3. Domain Decomposition.

The domain decomposition algorithm given in [4] can be adapted to localized variational refinement in the following way:

- (1) Input a sequence of data $(p_i^0 : -n \leq i \leq n)$ and knots sequences $t^\alpha = (t_i^\alpha)$ for $\alpha = 0, 1, \dots$
- (2) Choose $k \ll n$ and interval size (*ring*) $1 \leq l < k$ over which the coefficients of the localized k -window approximations are taken.



	k=1	2	3	4	5	6	7	8	8	10
$e_{k,n}^5$	4.4e-1	1.5e-1	3.7e-2	8.6e-2	1.8e-3	3.8e-4	7.5e-5	1.4e-5	2.3e-6	0
$\log(e_{k,n}^5)$	-0.819	-1.91	-3.29	-4.76	-6.30	-7.88	-9.50	-11.1	-13.0	

FIGURE 6.2. Plot of s_k^5 for $k = 1, 2, 3, 10$, with errors $e_{k,n}^5$ for $k = 1, 2, \dots, 10$.

- (3) For $k \ll n$, compute the k -window non-uniform variational refinement curves centered over interval rings of radius l .
- (4) After several levels of subdivision, construct the global interpolant by choosing coefficients over each l -ring as those of the corresponding k -window refinement curve centered about the l -ring.

Example 6.4. Surface refinement for gridded data.

There are already sophisticated methods for producing surfaces based on subdivision over rectangular and more general grids. Our goal here is to follow up with the curve scheme studied in this paper to construct a simple variational refinement scheme over a rectangular grid of points that depends only on the univariate functional minimized in this paper. The advantage of using univariate schemes to construct a surface is that it is computationally less expensive to compute several curve fits than to do a surface variational subdivision scheme; the main disadvantage is that we are not minimizing a surface functional to fair the surface which may not lead to good results in all cases. Our algorithm proceeds as follows:

- (1) Let $P = S(u_i, v_i)$ for knots u_1, \dots, u_m and v_1, \dots, v_m .
- (2) for $k=1:n$ iterations
 - (a) For $i=1:m$, refine all curves in the v -direction. Redefine n as $2n-1$.
 - (b) For $j=1:n$, refine all curves in the u -direction. Redefine m as $2m-1$.
- (3) end
- (4) Plot the surface.

To demonstrate the effectiveness of our surface scheme we applied it to the modeling of an aircraft wing. The results are shown in Figures 6.3 and 6.4 – the same data displayed in wire frame in Figure 6.3 and flat shaded in Figure 6.4. The original data is given in the top left of each figure, and the refined surface at three levels is displayed in the next three images of each figure. This original parametric surface data $(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j))$ depends on knot values u_i and v_j , which is non-uniformly spaced. To maintain a uniform global mesh ratio we bisect the knot sequences at each level of refinement. I.e.,

$$(u_{2i}^\alpha, v_{2j}^\alpha) = (u_i^{\alpha-1}, v_j^{\alpha-1})$$

$$(u_{2i+1}^\alpha, v_{2j+1}^\alpha) = \left(\frac{u_i^{\alpha-1} + u_{i+1}^{\alpha-1}}{2}, \frac{v_i^{\alpha-1} + v_{i+1}^{\alpha-1}}{2} \right).$$

Visually, we conclude that our surface refinement scheme does a very nice job of producing a smooth surface model of the aircraft wing.

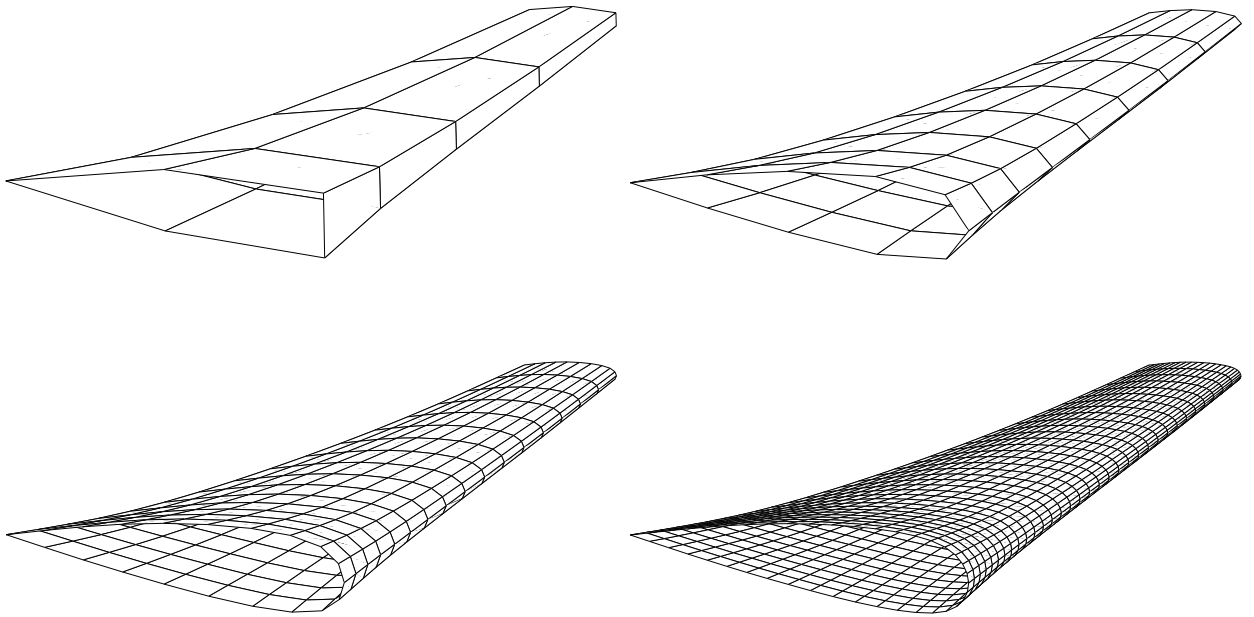


FIGURE 6.3. Wing data wireframe: Refinement levels: 0, 1, 2, 3.

REFERENCES

1. S. Kersey, *Sufficient conditions for smooth non-uniform variational refinement curves*, Approximation Theory XI, Eds. C. Chui, M. Neamtu and L. Schumaker, Nashboro Press (Brentwood) (2005), 185–196.
2. S. Kersey, *Constrained variational refinement*, JCAM **223** (2009), 983–996.
3. S. Kersey and M.J. Lai, *Convergence of local variational spline interpolation*, JMAA (2007).

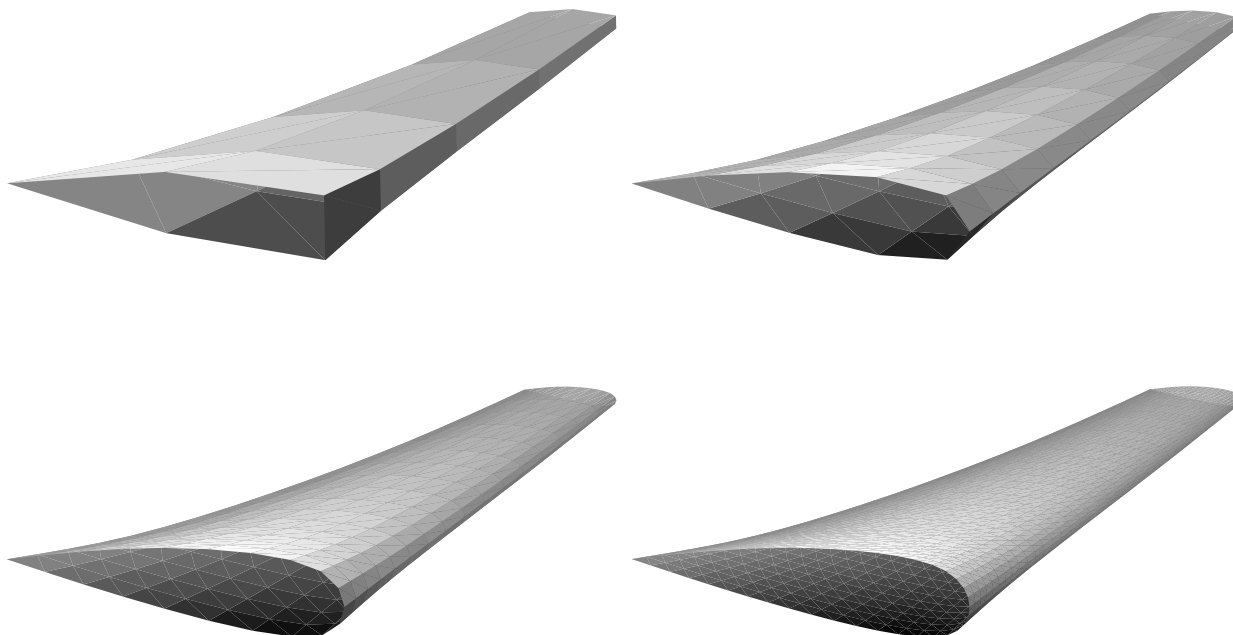


FIGURE 6.4. Wing data shaded: Refinement levels: 0, 1, 2, 3.

4. M.J. Lai and L.L. Schumaker, *Domain decomposition method for scattered data fitting*, SJNA **47** (2009), 911–928.
5. Kobbelt, L., A variational approach to subdivision, *Computer Aided Geometric Design* **13** (1996), 743–761.

Current address: Dr. Scott Kersey, Dept. of Math. Sci., Georgia Southern Univ., Statesboro, GA 30460-8093, and Dr. Ming-Jun Lai, Dept. of Math., Univ. of Georgia, Athens, GA 30602

E-mail address: skersey@georgiasouthern.edu, mjlai@math.uga.edu