# The Method of Virtual Components in the Multivariate Setting

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**Abstract**

We describe the so-called method of virtual components for tight wavelet framelets to increase their approximation order and vanishing moments in the multivariate setting. Two examples of the virtual components for tight wavelet frames based on bivariate box splines on three or four direction mesh are given. As a byproduct, a new construction of tight wavelet frames based on box splines under the quincunx dilation matrix is presented.

**Keywords:** Tight wavelet frames, Approximation Order, Vanishing moments, Virtual Components

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# 1 Introduction

Finding systems admitting sparse representations of objects of some functional classes and methods for constructing such representations is an important problem attracting the attention of many applied and pure mathematicians. From naive point of view the sparseness means compression. At the same time, this naive fact has rigorous justification from the point of view of information theory. The problem of sparse representations admits many different approaches. Those approaches can be partitioned into linear and nonlinear methods. The non-linear methods of representation consist in finding the most sparse representation for individual functions, whereas the linear methods are independent on a function and result in a linear operator providing optimization of some goal function for the class. Among those goal functions for wavelets (or wavelet frames) representations providing the higher order of vanishing moments or the higher frame approximation order are most popular. Unfortunately, natural constructions of wavelet frames based on polynomial filter-banks (like Unitary Extension Principle) fail to provide those properties even if the wavelet frame has a potential ability to represent higher order polynomials only with shifts of a scaling function (cf. [Ron and Shen'99]).

In [Lai and Petukhov'07], we introduced a concept of virtual components for tight wavelet frames. The purpose of the virtual components is to increase their approximation order and vanishing moments of tight wavelet frames. More precisely, using a non-uniqueness of decomposition in wavelet frames, we parameterize all such decompositions and select parameters (virtual components) providing the maximum order of polynomials annihilated with an analysis operator and reduction the leakage of the low-pass part of a signal into the high-pass parts. In that paper, we mainly discussed the construction of virtual components in the univariate setting. We now continue the investigation the same problems in the multivariate setting.

While we concentrate our attention on linear methods, we have to emphasize that virtual components are a natural tool for non-linear optimization. We consider such an opportunity as a topic for future research.

To construct virtual components, we first need to do a matrix extension and then compute virtual components. In this paper, we mainly address the existence of the matrix extension in the multivariate setting. Our proof of the existence is dependent on the Quillen-Suslin theorem. There is an algorithmic approach of the Quillen-Suslin Theorem. In general, the matrix extension is not an easy step. However, if a compactly supported multivariate tight wavelet frame which is constructed based on the method in [Lai and Stöckler'06], then a matrix extension can be easily obtained. Once we have a matrix extension, we can construct the virtual components to increase the order of polynomial reproduction and reduce the leakage of the low-pass contents into the high-pass parts of a signal. We shall provide with two examples in the bivariate setting to illustrate the construction of virtual components. That is, we use tight wavelet frames based on box splines on three or four direction mesh to construct examples. In each case, we are able to find the unitary matrix extension. Then we find the solution for virtual components.

The paper is organized as follows. We first explain the concept of the virtual components in the multivariate setting in §2 and then we discuss how to construct virtual components in §3. In §4 and §5 we use bivariate box spline tight wavelet frames to illustrate how to compute the virtual components. One byproduct of this study is our new construction of tight wavelet frames based on box splines on a four direction mesh. One significant advantage of our new construction is that the number of tight wavelet frames is less or equal to 6 no matter how smooth the box spline functions are.

# 2 Formation of Problems

To describe the method of virtual components in the multivariate setting, we need some notations and definitions. We consider an integer dilation matrix $D$ of size $d \times d$, where $d \geq 2$ is the dimensionality of the Euclidean space, with the dilation factor $|D|$ greater than 1. For the convenience, we will use the notation $|D|$ for $|\det D|$. The linear transform $D$ partition $\mathbf{Z}^d$ into $|D|$ non-overlapping classes. two vectors $k, l \in \mathbf{R}^d$ belong to the same class if $k - l = Dn$ for some $n \in \mathbf{Z}^d$. Any set of $|\det D|$ vectors including exactly one representative of each class is called a set of digits associated with the matrix $D$. In what follows, we need a set of digits $\delta = \{d_i\}$ and $\delta^* = \{d_i^*\}$ associted with the matrices $D$ and its complex conjugate and transpose $D^*$.

Let $\phi \in L_2(\mathbf{R}^d)$ be a compactly supported refinable function, i.e.,

$$\hat{\phi}(\omega) = P(D^{*-1}\omega)\hat{\phi}(D^{*-1}\omega),$$

where $P(\omega)$ is a trigonometric polynomial in $e^{i\omega}$. Let $V_j = \text{span}\{\phi_{j,m}(\cdot), m \in \mathbf{Z}^d\}$ be the linear span of integer translates of $\phi(D^j\cdot)$ for $j \in \mathbf{Z}$, where

$$\phi_{j,m}(x) = |D|^{1/2}\phi(D^j x - m)$$

is a dilation and translate of $\phi$. Since $\phi$ is refinable, $V_j \subset V_{j+1}$. In this paper we shall first assume that we can construct a set of tight framelets associated with $\phi$, i.e., tight wavelet frame generators $\psi^{(i)}$ defined in terms of Fourier transform by

$$\hat{\psi}^{(i)}(\omega) = Q_i(D^{*-1}\omega)\hat{\phi}(D^{*-1}\omega), \quad i = 1, \ldots, r,$$

with $r$ necessarily $\geq |D|$. Here $Q_i$ are Laurent polynomials in $e^{i\omega}$. (See, e.g., [Lai and Stöckler'06] for a method of constructing tight wavelet framelets in the multivariate setting.) Furthermore let $W_{j,k}$ be the linear span of $\psi_k$ similar to $V_j$. We know that

$$V_{j+1} = V_j \dot{+} W_{j,1} \dot{+} \cdots \dot{+} W_{j,r} \tag{1}$$

Suppose that $P$ and $Q_i$'s satisfy the following unitary extension principle (cf. [Ron and Shen'97a] and [Ron and Shen'97b]):

$$\mathcal{P}\mathcal{P}^* + \mathcal{Q}\mathcal{Q}^* = I_{|D|}, \tag{2}$$

where $I_{|D|}$ denotes the identity matrix of size $|D| \times |D|$, $\mathcal{P} = (P(\omega + 2\pi D^{*-1}d_n^*)$, $n = 1, 2, \ldots, |D|$, be a vector of size $|D| \times 1$ and $\mathcal{Q} = (Q_i(\omega + 2\pi D^{*-1}d_n^*)$, $i = 1, \cdots, r$, $n = 1, 2, \ldots, |D|$ , be a matrix of size $|D| \times r$. In other words, letting

$$\mathcal{M} = [\,\mathcal{P} \quad \mathcal{Q}\,]$$

be a matrix of size $|D| \times (r+1)$, we can rewrite (2) in the following matrix format:

$$\mathcal{M}\mathcal{M}^* = I_{|D|}. \tag{3}$$

As we know that (3) is fundamental in sequence decomposition and reconstruction. Let us explain it a little bit in detail. Let $\{s_j, j \in \mathbf{Z}^d\}$ be a digital sequence and $S(\omega)$ be the discrete Fourier transform of the signal. That is,

$$S(\omega) = \sum_{m \in \mathbf{Z}^d} s_m e^{-im\omega}.$$

Let $f_j = \sum\limits_{m \in \mathbf{Z}^d} s_m^{(j)} \phi_{j,m}$ be a projection of the digital signal $S$ in $V_j$. Its Fourier transform $\widehat{f_j}$ of $f_j$ is

$$\widehat{f_j}(\omega) = F_j(D^{*-j}\omega)\widehat{\phi}(D^{*-j}\omega)$$

with $F_j(\omega) = |D|^{-j/2} \sum_{m \in \mathbf{Z}^d} s_m^{(j)} e^{-im\omega}$. Similarly, for $g_{j,k} = \sum\limits_{m \in \mathbf{Z}^d} t_m^{(j,k)} \psi_{j,k,m} \in W_{j,k}$, the Fourier transform of $g_{j,k}$ is

$$\widehat{g_{j,k}}(\omega) = G_{j,k}(D^{*-j}\omega)\widehat{\psi^{(k)}}(D^{*-j}\omega),$$

where $G_{j,k} = |D|^{-j/2} \sum_{m \in \mathbf{Z}^d} t_m^{(j,k)} e^{-im\omega}$.

The decomposition relationship (1) implies that for $f_{j+1} \in V_{j+1}$, there exist $f_j$ and $g_{j,k}, k = 1, \cdots, r$ such that

$$\begin{aligned} & F_{j+1}(D^{*-j-1}\omega)\widehat{\phi}(D^{*-j-1}\omega) \\ = \; & F_j(D^{*-j}\omega)\widehat{\phi}(D^{*-j}\omega) + \sum_{k=1}^{r} G_{j,k}(D^{*-j}\omega)\widehat{\psi_k}(D^{*-j}\omega). \end{aligned}$$

Using the dilation relation and the definition of framelets, we have

$$\begin{aligned} F_{j+1}(D^{*-j}\omega)\widehat{\phi}(D^{*-j}\omega) \; = \; & F_j(\omega)P(D^{*-1}\omega)\widehat{\phi}(D^{*-1}\omega) + \\ & \sum_{k=1}^{r} G_{j,k}(\omega)Q_k(D^{*-1}\omega)\widehat{\phi}(D^{*-1}\omega). \end{aligned}$$

That is,

$$F_{j+1}(\omega + 2\pi D^{*-1}d_k^*) = F_j(\omega)P(\omega + 2\pi D^{*-1}d_k^*) + \sum_{k=1}^{r} G_{j,k}(\omega)Q_k(\omega + 2\pi D^{*-1}d_k^*)$$

for $k = 1, 2, \ldots, |D|$. In terms of matrix form and without explicit reference of level $j$, we have

$$X(\omega) = \mathcal{M}Y(\omega), \tag{4}$$

where

$$\begin{aligned} Y(\omega) &= [F_j(\omega), G_{j,1}(\omega), \cdots, G_{j,r}(\omega)]^T, \\ X(\omega) &= \left[F_{j+1}(\omega + 2\pi D^{*-1}d_k^*), k = 1, 2, \ldots, |D|\right]^T, \end{aligned} \tag{5}$$

are two vectors of size $(r+1)\times 1$ and $|D|\times 1$, respectively. By the fundamental condition (3), we obtain the following

$$Y(\omega) = \mathcal{M}^* X(\omega). \tag{6}$$

That is, given a sequence in $V_{j+1}$, i.e, $F_{j+1}$, we can find the decomposition components $F_j, G_{j,k}, k = 1, \cdots, r$, i.e, $Y(\omega)$ by using (6). On the other hand, for given $Y(\omega)$, we can recovery $F_{j+1}$ using (4).

Let us introduce the matrix $\Phi = \|D\|^{-1/2}\{e^{-i(D^{-1}j,r)}\}_{j\in\delta, r\in\delta^*}$ and the diagonal matrix $\Delta(\omega)$ with the diagonal entries $\{e^{-i(d_k,\omega)}\}_{1\le k\le |D|}$. Both matrices are unitary. Then the matrix $\mu = \Delta\Phi\mathcal{M}$ is a polyphase form of the matrix $\mathcal{M}$ and $\xi = \Delta\Phi X$ is a polyphase form of the vectors $X$. The polyphase forms of equations (4) and (6) are as follows

$$\xi = \mu Y, \tag{7}$$

$$Y = \mu^* \xi. \tag{8}$$

While original entries of $X$ and $\mathcal{M}$ are Laurent polynomial and formal Laurent series in $e^{i\omega}$, entries of $\xi$ and $\mu$ are polynomials and Laurent series in $e^{iD^*\omega}$. Moreover, while the matrix $\mathcal{M}$ is defined by the first row and $X$ is defined by the first component, the entries of $\xi$ and $\mu$ does not have this dependence and can be chosen with more freedom. For this reasons, formulas (2.4') usually are preferable for both theoretical analysis and applications.

Note that in terms of signal processing terminology, $F_j$ is associated with low-pass part of the sequence in $F_{j+1}$ while $G_{j,k}, k = 1, \cdots, r$ are associated with high-pass part of the signal. We would like to know if we have $F_j = F_{j+1}$ in the above computation (6) when $F_{j+1}$ happens to be the low-pass part of the signal. This relates to the question if the low-pass part of $F_{j+1}$ leaks to $W_{j,k}$ or not. For example, let $\phi$ be a box spline function and let $F_{j+1}$ be associated with a box spline function in $V_j$ written in term of the refined level $V_{j+1}$. That is, suppose that $b = \sum_{m\in\mathbf{Z}^d} c_m\phi_{j,m}$ which may be rewritten in $b = \sum_{m\in\mathbf{Z}^d} \tilde{c}_m\phi_{j+1,m}$ by using the refinement relation of the box spline. Letting $F_{j+1} = \sum_{m\in\mathbf{Z}^d} \tilde{c}_m e^{-im\omega}$, the computation (6) yields $F_j$. Our question is if $F_j = \sum_{m\in\mathbf{Z}^d} c_m e^{-im\omega}$ or not.

Next let $\mathcal{V}_j$ be the extension of $V_j$ in the distribution sense so that $\mathcal{V}_j$ contains all polynomials of total degree $n$ assuming that the refinable function $\phi$ are able to reproduce polynomials of total degree $n$. Our second question is when $F_{j+1}$ is associated with a polynomial in $\mathcal{V}_{j+1}$, if $F_j$ is associated with the same polynomial. Do we have $G_{j,k} = 0$ for all $k = 1, \cdots, r$ in this case? This is related to the vanishing moment order of the tight framelets. Thus, our questions may be formulated as follows: Suppose that $f_{j+1} \in \mathcal{V}_{j+1}$ is a polynomial. How can we have $G_{j,k} = 0$ for all $k = 1, \cdots, r$?

We shall use the method of virtual components (MVC) to give the above two questions an answer. The MVC is a computational method which shows how to modify $\mathcal{M}$ and $\mathcal{M}^*$ to avoid the leakage of the low-pass part of signals to the high pass parts and to increase the vanishing moment order.

# 3 The Method of Virtual Components

The method of virtual components (MVC) starts with extending $\mathcal{M}$ and $\mathcal{M}^*$ to square matrices $\mathcal{M}_e$ and $\widetilde{\mathcal{M}}_e$ of size $(r+1)\times(r+1)$, respectively such that

$$\mathcal{M}_e\widetilde{\mathcal{M}}_e = I_{r+1}. \tag{9}$$

Here we say $\mathcal{M}_e$ is an extension of $\mathcal{M}$ if $\mathcal{M}_e$ consists of matrix block $\mathcal{M}$ and another block $\mathcal{N}$ in the following form:

$$\mathcal{M}_e = \begin{pmatrix} \mathcal{M} \\ \mathcal{N} \end{pmatrix}.$$

Similarly, $\widetilde{\mathcal{M}}_e = (\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is an extension of $\widetilde{\mathcal{M}}$. We notice that we are interested only in extensions whose entries are Laurent polynomials in $e^{iD^*\omega}$. This is ensured by the following (cf. [Logar and Sturmfels'92] for an algorithmic proof). Let $\mathcal{R}$ be the ring consisting of all Laurent polynomials in $z \in (\mathbf{C}\backslash\{0\})^d$.

**Theorem 3.1** *(Quillen-Suslin). Let $n < r$. Suppose that a matrix $A$ of size $n \times r$ with Laurent polynomial entries in $z = e^{i\eta}$ is unimodular. That is, the maximal minors of $A$ generate the unit ideal ove the ring $\mathcal{R}$ of all Laurent polynomials in $z$. Then there exists a matrix $U$ of size $r \times r$ unimodular over $\mathcal{R}$ such that*

$$AU = [I_{n\times n}, 0_{r\times r-n}].$$

Indeed, we use the above Quillen-Suslin Theorem to prove the following (11). There are many approaches available in the literature to find matrix extension (or matrix completion) in (11). For example, see [Ji, Riemenschneider and Shen'99] and [Chen, Micchelli, and Xu'07] and the references therein. In this paper, we provide another new proof which is based on [He'98].

**Theorem 3.2** *. Let $\mathcal{M} := [m_{jk}]_{1\le j\le n, 1\le k\le r}$ and $\widetilde{\mathcal{M}} := [\tilde{m}_{jk}]_{1\le j\le r, 1\le k\le n}$ be two rectangular matrices with trigonometric polynomials in $e^{i\omega}$ in their entries, where $n < r$. Suppose that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ satisfy*

$$\mathcal{M}\widetilde{\mathcal{M}} = I_n, \tag{10}$$

*where $I_n$ denotes the identity of size $n \times n$. Then there exist two extension matrices $\mathcal{M}_e$ and $\widetilde{\mathcal{M}}_e$ of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, respectively, such that*

$$\mathcal{M}_e\widetilde{\mathcal{M}}_e = I_r. \tag{11}$$

**Proof:** We use induction. When $n = 1$, $\mathcal{M} = [m_{11}(\omega), \cdots, m_{1r}(\omega)]$ and $\widetilde{\mathcal{M}} = [\tilde{m}_{11}(\omega), \cdots, \tilde{m}_{1r}(\omega)]^T$. Then (10) is

$$m_{11}(\omega)\tilde{m}_{11}(\omega) + \cdots + m_{1r}(\omega)\tilde{m}_{1r}(\omega) = 1, \quad \forall \omega \in [0, 2\pi]^d.$$

It follows that $m_{11}(\omega), \cdots, m_{1r}(\omega)$ have no common zeros for $z = e^{i\omega} \in (\mathbf{C}\backslash\{0\})^d$. By using the Quillen-Suslin Theorem, i.e., the algorithm in [Logar and Sturmfels'92, Theorem 2.1], we can find $U(\omega)$ which is invertible in $\mathcal{R}$ such that

$$[m_{11}(\omega), \cdots, m_{1r}(\omega)]U = [1, 0, \cdots, 0]_{1\times r}.$$

Let $A(\omega) = U^{-1}(\omega)$. Then the above equation implies that $[m_{11}(\omega), \cdots, m_{1r}(\omega)] = [1, 0, \cdots, 0]_{1,m} A(\omega)$ or the first row of $A(\omega)$ is $\mathcal{M}$. We now look at

$$A(\omega) \begin{pmatrix} \tilde{m}_{11}(\omega) \\ \vdots \\ \tilde{m}_{1r}(\omega) \end{pmatrix} = \begin{pmatrix} 1 \\ h_1 \\ \vdots \\ h_{m-1} \end{pmatrix},$$

where $h_1, \cdots h_{m-1}$ are polynomials in $z$. Multiplying the both sides of the equation above by matrix

$$L(\omega) := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -h_1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -h_{m-1} & 0 & \cdots & 1 \end{pmatrix}$$

we have

$$L(\omega) A(\omega) \begin{pmatrix} \tilde{m}_{11}(\omega) \\ \vdots \\ \tilde{m}_{1r}(\omega) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Notice that the first row of $L(\omega)A(\omega)$ is still $[m_{11}(\omega), \cdots, m_{1r}(\omega)]$ and $L(\omega)A(\omega))$ is also invertible in $\mathcal{R}$. Thus, we have

$$\begin{pmatrix} \tilde{m}_{11}(\omega) \\ \vdots \\ \tilde{m}_{1r}(\omega) \end{pmatrix} = (L(\omega)A(\omega))^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

That is, $\begin{pmatrix} \tilde{m}_{11}(\omega) \\ \vdots \\ \tilde{m}_{1r}(\omega) \end{pmatrix}$ is the first column of $(L(zA(\omega))^{-1}$. Let $\mathcal{M}_e(\omega) = L(\omega)A(\omega)$ and $\widetilde{\mathcal{M}}_e(\omega) = (L(\omega)A(\omega))^{-1}$. Then we have the matrix extension and

$$\mathcal{M}_e(\omega)\widetilde{\mathcal{M}}_e(\omega) = I_r.$$

Assume now that we can find two desired extension matrices for $n = \ell < r$. For $n = \ell + 1 < r$, suppose $\mathcal{M}_{\ell+1}(\omega)$ and $\widetilde{\mathcal{M}}_{\ell+1}(\omega)$ are $(\ell+1) \times r$ and $r \times (\ell+1)$ polynomial matrices satisfying

$$\mathcal{M}_{\ell+1}(\omega)\widetilde{\mathcal{M}}_{\ell+1}(\omega) = I_{\ell+1}.$$

Let $\mathcal{M}_\ell(\omega)$ be the first $\ell$ rows of $\mathcal{M}_{\ell+1}(\omega)$ and $\widetilde{\mathcal{M}}_\ell(\omega)$ be the first $\ell$ columns of $\widetilde{\mathcal{M}}_{\ell+1}(\omega)$. We also have

$$\mathcal{M}_\ell(\omega)\widetilde{\mathcal{M}}_\ell(\omega) = I_\ell.$$

By the induction assumption, we can find one $(r-\ell)\times r$ and one $r\times(r-\ell)$ polynomial matrices $\mathcal{N}_\ell(\omega)$ and $\widetilde{\mathcal{N}}_\ell(\omega)$ such that

$$\begin{pmatrix} \mathcal{M}_\ell(\omega) \\ \mathcal{N}_\ell(\omega) \end{pmatrix} (\, \widetilde{\mathcal{M}}_\ell(\omega)\ \widetilde{\mathcal{N}}_\ell(\omega)\,) = I_r. \tag{12}$$

Now we look at

$$\mathcal{M}_{\ell+1}(\omega)\,(\, \widetilde{\mathcal{M}}_\ell(\omega)\ \widetilde{\mathcal{N}}_\ell(\omega)\,) = \begin{pmatrix} I_\ell & 0 \\ 0 & h \end{pmatrix},$$

where $h=(h_1,h_2,\cdots,h_{r-\ell})$ is a vector with polynomial entries. Notice that the matrix $\mathcal{M}_{\ell+1}(\omega)$ is of full rank and $\det([\widetilde{\mathcal{M}}_\ell(\omega),\widetilde{\mathcal{N}}_\ell(\omega)])\neq 0$ for all $z\in(\mathbf{C}\backslash\{0\})^d$ by (3.4). It implies that $(r-\ell)$ polynomials $h_1,h_2,\cdots,h_{r-\ell}$ do not have any common zeros in $(\mathbf{C}\backslash\{0\})^d$. Otherwise, the matrix on the right-hand side would not be of full rank. By the algorithm in [Logar and Sturmfelt'92, Theorem 2.1], we can find a polynomial matrix $T(\omega)$ which is invertible in $\mathcal{R}$ such that

$$[h_1,\cdots,h_{r-\ell}]T(\omega) = [1,0,\cdots,0]_{1\times(r-\ell)}.$$

Multiplying both sides of (3.4) from the right by $\begin{pmatrix} I_\ell & 0 \\ 0 & T(\omega) \end{pmatrix}$, we have

$$\mathcal{M}_{\ell+1}(\omega)(\widetilde{\mathcal{M}}_\ell(\omega)\ \widetilde{\mathcal{N}}_\ell(\omega)) \begin{pmatrix} I_\ell & 0 \\ 0 & T(\omega) \end{pmatrix} = [I_{\ell+1}\ 0_{(\ell+1)\times(r-\ell-1)}]_{(\ell+1)\times r}.$$

That is, $\mathcal{M}_{\ell+1}(\omega)[\widetilde{\mathcal{M}}_\ell(\omega),\widetilde{\mathcal{N}}_\ell(\omega)T(\omega)] = [I_{\ell+1}\ 0_{(\ell+1)\times(r-\ell-1)}]_{(\ell+1)\times r}$. Thus, if we let $U = [\widetilde{\mathcal{M}}_\ell(\omega),\widetilde{\mathcal{N}}_\ell(\omega)T(\omega)]$, then $U$ is invertible and $U^{-1}$ is also a polynomial matrix. It is easy to see that $\mathcal{M}_{\ell+1}(\omega) = [I_{\ell+1},0_{(\ell+1)\times(r-\ell-1)}]U^{-1}$, i.e., $\mathcal{M}_{\ell+1}(\omega)$ is the first $(\ell+1)$ rows of matrix $U^{-1}$. Denote by $U^{-1} = \begin{bmatrix} \mathcal{M}_{\ell+1}(\omega) \\ \mathcal{N}_{\ell+1}(\omega) \end{bmatrix}$. A simple computation shows

$$\begin{pmatrix} \mathcal{M}_{\ell+1}(\omega) \\ \mathcal{N}_{\ell+1}(\omega) \end{pmatrix} \widetilde{\mathcal{M}}_\ell(\omega) = \begin{pmatrix} I_\ell \\ 0 \end{pmatrix}.$$

We have, by the assumption,

$$\begin{pmatrix} \mathcal{M}_{\ell+1}(\omega) \\ \mathcal{N}_{\ell+1}(\omega) \end{pmatrix} \widetilde{\mathcal{M}}_{\ell+1}(\omega) = \begin{pmatrix} I_\ell & 0 \\ 0 & 1 \\ 0 & g \end{pmatrix}, \tag{13}$$

where $g=[g_1(\omega),\cdots,g_{r-\ell-1}(\omega)]^T$ with polynomial entries in $e^{i\omega}$. Let

$$W = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -g_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_{r-\ell-1} & 0 & \cdots & 1 \end{pmatrix}_{(r-\ell)\times(r-\ell)}.$$

Multiply both sides of (3.5) from the left by $\begin{pmatrix} I_\ell & 0 \\ 0 & W \end{pmatrix}$, we get

$$\begin{pmatrix} \mathcal{M}_{\ell+1}(\omega) \\ W\mathcal{N}_{\ell+1}(\omega) \end{pmatrix} \widetilde{\mathcal{M}}_{\ell+1}(\omega) = \begin{pmatrix} I_{\ell+1} \\ 0 \end{pmatrix}.$$

Let $V = \begin{pmatrix} \mathcal{M}_{\ell+1}(\omega) \\ W\mathcal{N}_{\ell+1}(\omega) \end{pmatrix}$. Note that $V$ is invertible in $\mathcal{R}$ and $V^{-1}$ is also a polynomial matrix. We can see

$$\widetilde{\mathcal{M}}_{\ell+1}(\omega) = V^{-1} \begin{pmatrix} I_{\ell+1} \\ 0 \end{pmatrix}.$$

That is, $\widetilde{\mathcal{M}}_{\ell+1}(\omega)$ is the first $\ell+1$ rows of $(V^{-1})$. We thus choose the matrix extensions to be

$$\mathcal{M}_e = \begin{pmatrix} \mathcal{M}_{\ell+1}(\omega) \\ W\mathcal{N}_{\ell+1}(\omega) \end{pmatrix} \text{ and } \widetilde{\mathcal{M}}_e = \left(\mathcal{M}_e^{-1}\right).$$

Then the extension matrices satisfy the desired properties. This completes the proof. ■

Since the proof of Quillen-Suslin's Theorem is not constructive, the above matrix extension is not a constructive proof. However, for those tight wavelet frames which were constructed based on the method in [Lai and Stöckler'06], we can give a constructive proof of the matrix extension. Indeed, let us recall the method for construction of compactly supported multivariate tight wavelet frames from [Lai and Stöckler'06]. Let $P(\omega)$ be a mask associated with a refinable function $\phi$ corresponding to the dilation matrix $D$ and suppose that $\sum_{d^*\in\delta^*} |P(\omega + 2\pi D^{*-1}d^*)|^2 \leq 1$. Let us further assume that there exist polynomials $\tilde{p}_i, i = 1, \cdots, n$ such that

$$\sum_{d^*\in\delta^*} |P(\omega + 2\pi D^{*-1}d^*)|^2 + \sum_{j=1}^{n} |\tilde{p}_j(D^*\omega)|^2 = 1.$$

A computational method of $\tilde{p}_j$ can be found in [Geronimo and Lai'06] for certain Laurant polynomials $P$. As shown in [Lai and Stöckler'06], the Laurant polynomial associated with any bivariate box spline function on three and four direction mesh, such $\tilde{p}_j$ can be constructed for a special form of the dilation matrix $D$.

Let $P_k(\omega), 1 \leq k \leq |D|$, be the polyphase components of $P$, i.e.,

$$P(\omega) = |D|^{-1/2}| \sum_{d_\ell^*\in\delta^*} e^{id_\ell^*\cdot\omega} P_\ell(D^*\omega).$$

Let $\widetilde{P}(\omega) = [P_\ell(\omega), 1 \leq \ell \leq |D|, \tilde{p}_j(\omega), j = 1, \cdots, n]$, be a vector of size $(2^d + n) \times 1$ and let

$$\widetilde{Q} = I_{|D|+n} - \widetilde{P}(\omega)\widetilde{P}(\omega)^*,$$

where $I_{|D|+n}$ is the identity matrix of size $|D|+n$. Then it is easy to verify that $\widetilde{Q}\widetilde{Q}^* = I_{|D|+n} - \widetilde{P}(\omega)\widetilde{Q}(\omega)^*$. Let $\mathcal{U} = \Phi^*\Delta^*$, then

$$\left[P(\omega + 2\pi D^{-1*}d_\ell^*), 1 \le \ell \le |D|\right]^T = \mathcal{U}\left[P_\ell(D^*\omega), 1 \le \ell \le |D|\right]^T.$$

Let $\widehat{Q}$ be the first $|D|$ rows of $\widetilde{Q}$. From the above we have

$$\widehat{Q}(\omega)\widehat{Q}(\omega)^* = I_{|D|} - \widehat{P}(\omega)\widehat{P}(\omega)^*$$

where $\widehat{P}(\omega) = [P_\ell(D^*\omega), 1 \le \ell \le |D|]$ is of size $|D| \times 1$. We let

$$\mathcal{Q}(\omega) = \mathcal{U}\widehat{Q}(D^*\omega).$$

Then letting $[Q_1, \cdots, Q_{|D|+n}]$ be the first row of the above matrix $\mathcal{Q}$ and $\widehat{\psi}_j(\omega) = Q_j(D^{*-1}\omega)\widehat{\psi}(D^{*-1}\omega)$, we know that these $\psi_j$ are tight wavelet frames by using the unitary extension principle (UEP). In this situation, $\mathcal{M} = [\mathcal{P}, \mathcal{Q}]$ which is of size $|D| \times (|D|+n+1)$. As discussed in the above, we have $\mathcal{M}\mathcal{M}^* = I_{|D|}$.

We are now ready to construct a matrix extension of $\mathcal{M}$. We first extend $\mathcal{M}$ to have a matrix

$$\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix}$$

of size $(|D|+n) \times (|D|+n+1)$. It can be done easily by letting

$$\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{U} & 0 \\ 0 & I_n \end{bmatrix} [\widetilde{P}(D^*\omega), \widetilde{Q}(D^*\omega)].$$

Indeed, using the above discussion we have

$$\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix} \begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix}^* = I_{|D|+n}.$$

We now find polynomial entries $\mathcal{N}_2$ of size $1 \times |D|+n+1$ such that the following determinant is equal to 1:

$$\det \begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \\ \mathcal{N}_2 \end{bmatrix} = 1.$$

This is guaranteed by using the Hilbert Nullstellensatz. Indeed, as we saw from the above that

$$\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix}$$

is of full rank. Hence, the cofactors of the matrix

$$\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \\ \mathcal{N}_2 \end{bmatrix}$$

associated with each entry of $\mathcal{N}_2$ have no common zeros in $(\mathcal{C}\backslash\{0\})^d$. Then the above matrix is invertible in the ring of polynomials. Hence, there exists $\widetilde{\mathcal{M}}_e$ such that $\mathcal{M}_e\widetilde{\mathcal{M}}_e = I_{|D|+n+1}$. Thus, we have obtained the following

**Theorem 3.3** *Suppose that the tight wavelet framelets are constructed by using the method in [Lai and Stöckler'06]. Let* $\mathcal{M}$ *as defined above which satisfies*

$$\mathcal{M}\mathcal{M}^* = I_{|D|+n},$$

*where* $I_{|D|+n}$ *denotes the identity of size* $(|D|+n)\times(|D|+n)$*. Then there exist two extension matrices* $\mathcal{M}_e$ *and* $\widetilde{\mathcal{M}}_e$ *of* $\mathcal{M}$*, such that*

$$\mathcal{M}_e\widetilde{\mathcal{M}}_e = I_{|D|+n+1}.$$

For simplicity, we may assume that $\mathcal{M}_e$ is an unitary extension in the remaining part of this section unless specified explicitly. That is, $\widetilde{\mathcal{M}}_e = \mathcal{M}_e^*$. For convenience, we write $\mathcal{M}_e = \begin{bmatrix}\mathcal{M}\\ \mathcal{N}\end{bmatrix}$ with

$$\mathcal{N} = (n_{ij}(D^*\omega))_{\substack{1\le i\le r+1-|D|\\ 1\le j\le |D|}}.$$

That is, $\mathcal{N}$ is the extension component of $\mathcal{M}$.

We let $\widetilde{X} = \alpha X$ be a virtual component, where $\alpha$ is a matrix of size $(r+1-|D|)\times|D|$. That is, for $\alpha = [\alpha_i(\omega + 2\pi D^{*-1}d_\ell^*), 1\le \ell\le |D|]_{i=1,\cdots,r+1-|D|}$,

$$\widetilde{X} = \left[\sum_{1\le\ell\le|D|} \alpha_i(\omega+2\pi D^{*-1}d_\ell^*)F_{j+1}(\omega+2\pi D^{*-1}d_\ell^*)\right]_{i=1,\cdots,r+1-|D|}.$$

Similarly let $X_e = \begin{bmatrix}X\\ \widetilde{X}\end{bmatrix}$. Then we have

$$Y_e = \mathcal{M}_e^* X_e \text{ and } X_e = \mathcal{M}_e Y_e. \tag{14}$$

We shall show how to choose $\alpha$ such that $X$ be can be recovered from the first component from $Y_e$ when the signal $f_{j+1}\in\mathcal{V}_{j+1}$ is a polynomial. Furthermore, we shall show how to choose $\alpha$ such that the other components from $Y_e$, especially, the components $G_{j,k} = 0$ for $k = 1,\cdots,r$.

**Lemma 3.1** *Suppose that* $P(\omega+2\pi D^{*-1}d_\ell^*), 1\le \ell|D|$ *have no common zero in* $(\mathcal{C}\backslash\{0\})^d$*. Then there exists* $\alpha = (\alpha_1,\cdots,\alpha_{r+1-|D|})$ *with polynomial entries* $\alpha_i$ *such that*

$$\sum_{1\le\ell\le|D|}\alpha_i(\omega+2\pi D^{*-1}d_\ell^*)P(\omega+2\pi D^{*-1}d_\ell^*) = n_{i,1}(D^*\omega) \tag{15}$$

*for all* $i = 1,\cdots,r+1-|D|$.

**Proof:** The existence of the solution $\alpha_i$ is guaranteed by the Hilbert Nullstallensatz since $P(\omega + 2\pi D^{*-1} d^*_\ell), 1 \le \ell \le |D|$, viewing as polynomials in $z = e^{i\omega}$, have no common zero for $z \in (\mathbf{C}\backslash\{0\})^d$. ■

Suppose that $P(\omega+2\pi D^{*-1}d^*_\ell), 1 \le \ell \le |D|$ have a common zero in $(\mathcal{C}\backslash\{0\})^d$ which is not a zero for $n_{i,1}$ for some $i$. Then we use some polynomials $q_i(D^*\omega), i = 1, \cdots, r+1-|D|$ in $z = e^{i\omega}$ to generate the same zero of the same order on the right-hand side so that we can find $\alpha_i$ such that

$$\sum_{1\le\ell\le|D|} \alpha_i(\omega + 2\pi D^{*-1}d^*_\ell)P(\omega + 2\pi D^{*-1}d^*_\ell) = n_{i,1}(D^*\omega) + q_i(D^*\omega) \tag{16}$$

for all $i = 1, \cdots, r + 1 - |D|$. See an example in Section 5.

For convenience, let us use the expression in (16). We note that in (14) the first component of the vector $Y_e = [y_1, \cdots, y_{r+1}]^T$ can be explicitly written as follows:

$$\begin{aligned} y_1(\omega) &= \sum_{1\le\ell\le|D|} P(\omega + 2\pi D^{*-1}d^*_\ell)^* F_{j+1}(\omega + 2\pi D^{*-1}d^*_\ell) \\ &\quad + \sum_{i=1}^{r+1-|D|} n_{i,1}(D^*\omega)^* \sum_{1\le\ell\le|D|} \alpha_i(\omega + 2\pi D^{*-1}d^*_\ell)F_{j+1}(\omega + 2\pi D^{*-1}d^*_\ell). \end{aligned}$$

That is, we have

$$\begin{aligned} y_1(\omega) &= \sum_{1\le\ell\le|D|} F_{j+1}(\omega + 2\pi D^{*-1}d^*_\ell)\times \\ &\quad \left[P(\omega + 2\pi D^{*-1}d^*_\ell)^* + \sum_{i=1}^{r+1-|D|} n_{i,1}(D^*\omega)^*\alpha_i(\omega + 2\pi D^{*-1}d^*_\ell)\right]. \end{aligned}$$

Let us assume that $f_{j+1} \in V_{j+1}$ is also in $V_j$. That is,

$$\widehat{f_{j+1}} = F_{j+1}(D^{*-1}\omega)\hat{\phi}(D^{*-1}\omega) = \tilde{F}_j(\omega)\hat{\phi}(\omega).$$

In other words, $F_{j+1}(\omega) = \tilde{F}_j(D^*\omega)P(\omega)$ using the dilation relation of $\phi$. Thus, by using (15), we have

$$\begin{aligned} y_1(\omega) &= \sum_{1\le\ell\le|D|} \tilde{F}_j(D^*\omega)P(\omega + 2\pi D^{*-1}d^*_\ell)\times \\ &\quad \left[P(\omega + 2\pi D^{*-1}d^*_\ell)^* + \sum_{i=1}^{r+1-|D|} n_{i,1}(D^*\omega)^*\alpha_i(\omega + 2\pi D^{*-1}d^*_\ell)\right] \\ &= \tilde{F}_j(D^*\omega)\left[\sum_{1\le\ell\le|D|} |P(\omega + 2\pi D^{*-1}d^*_\ell)|^2 + \right. \\ &\quad \left. \sum_{i=1}^{r+1-|D|} (|n_{i,1}(D^*\omega)|^2 + q_i(D^*\omega)n_{i,1}(D^*\omega)^*)\right] \end{aligned}$$

$$= \tilde{F}_j(D^*\omega)\left[1 + \sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^*\right].$$

When $P(\omega + 2\pi D^{*-1}d_\ell^*), 1 \le \ell \le |D|$ have no common zero and $q_j, j = 1, \cdots, r + 1 - |D|$ are equal to zero, we clearly have $y_1(\omega) = \tilde{F}_j(D^*\omega)$. That is, $y_1$ contains the low-pass part of the signal and hence $F_{j+1}(\omega) = P(\omega)\tilde{F}_j(D^*\omega) = P(\omega)y_1(\omega)$. Otherwise, we will have

$$y_1(\omega) = \tilde{F}_j(D^*\omega) + \sum_{i=1}^{r+1-|D|} q_i(D^*\omega)\tilde{F}_j(D^*\omega)n_{i,1}(D^*\omega)^*.$$

In addition to the property of $q_i$, we further require $q_i$ to contain factors $\prod_{j=1}^{d}(1 - e^{i\omega_j})^n$ for an appropriate $n \ge 1$ for $i = 1, \cdots, r + 1 - |D|$. If $f_{j+1}$ is a polynomial of degree $n$ which is in $V_{j+1}$ in the distribution sense, then $\prod_{j=1}^{d}(1 - e^{i\omega_j})^n\tilde{F}_j(D^*\omega) = 0$ and hence,

$$y_1(\omega) = \tilde{F}_j(D^*\omega). \tag{17}$$

Similar to the computation of $y_1(\omega)$, we can compute $y_k(\omega)$ which is zero for $k = 2, \cdots, r + 1$. Hence,

$$X = \mathcal{M}\begin{bmatrix} y_1(\omega) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

That is, $X$ can be recovered from the low-pass part $y_1(\omega)$ directly. The high-pass parts $y_2(\omega), \cdots, y_{r+1}(\omega)$ are all equal to zero. We summarize the discussions above in the following

**Theorem 3.4** *Assume that polynomials $P(\omega + 2\pi D^{*-1}d_\ell^*), 1 \le \ell \le |D|$ in $z = e^{i\omega}$ have no common zero in $(\mathbf{C}\backslash\{0\})^d$. Solve (15) for $\alpha_i$. Then any sequence $f_{j+1}$ in $V_{j+1}$ which is also in $V_j$ can be recovered from the low-pass $y_1$ of the decomposition $Y_e$ of $f_{j+1}$ in $V_j$, $W_{j,k}$, $k = 1, \cdots, r$. The decompositions $G_{j,k} = 0$ for all $k = 1, \cdots, r$.*

and

**Theorem 3.5** *If polynomials $P(\omega + 2\pi D^{*-1}d_\ell^*), 1 \le \ell \le |D|$ in $z = e^{i\omega}$ have common zeros in $(\mathbf{C}\backslash\{0\})^d$, we choose a vector of Laurent polynomials $q_k, k = 1, \cdots, r + 1 - |D|$ such that (3.8) holds. In addition choose $q_k$ containing a factor $\prod_{j=1}^{d}(1 - e^{i\omega_j})^n$ for $k = 1, \cdots, r + 1 - |D|$. Then when $f_{j+1}$ is a polynomial of degree $\le n$ in $\mathcal{V}_{j+1}$ can be recovered from the low-pass $y_1$ of the decomposition $Y_e$ of $f_{j+1}$ in $V_j$, $W_{j,k}$, $k = 1, \cdots, r$. Furthermore, the decompositions $G_{j,k} = 0$ for all $k = 1, \cdots, r$.*

If $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are associated with bi-framelets (cf. [Daubechies, Han, Ron, and Shen'03] and [Daubechies and Han'04]), the construction (15) may be modified accordingly. That is, we need to find $\alpha$ be a solution which solves the following

$$\sum_{1\le \ell\le |D|} \alpha_i(\omega + 2\pi D^{*-1}d_\ell^*)\widetilde{P}(\omega+2\pi D^{*-1}d^*_\ell) = n_{i,1}(D^*\omega) + q_i(D^*\omega). \tag{18}$$

We leave the detail to the interested reader.

Finally we remark that the requirement the factor $\prod_{j=1}^d (1-e^{i\omega_j})^d$ in each $q_j$ may be too strong. We can weaken it. Suppose that $f_{j+1}\in \mathcal{V}_{j+1}$ is a polynomial of total degree $n$ under the assumption that the integer translates of $\phi$ reproduce polynomials of degree $\le n$. Then $F_{j+1}(\omega) = \tilde{F}_j(D^*\omega)P(\omega)$ as mentioned above. Based on the discussion above, we have

$$y_1(\omega) = \tilde{F}_j(D^*\omega)(1+\sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^*).$$

Multiplying both sides of the equation above by $P(\omega)$, we have

$$\begin{aligned} P(\omega)y_1(\omega) &= \tilde{F}_j(D^*\omega)P(\omega)(1+\sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^*) \\ &= F_{j+1}(\omega) + \sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^* F_{j+1}(\omega). \end{aligned}$$

Note that $y_1(\omega)$ is of $\pi$ periodic function from (3.9). That is, $y_1(\omega+2\pi D^{*-1}d^*_\ell) = y_1(\omega)$. If the second term in the above equation is zero, i.e.,

$$\sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^*F_{j+1}(\omega) = 0, \tag{19}$$

we will have $X = \mathcal{M}\begin{bmatrix} y_1(\omega) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Let us consider a box spline function $\phi$ with direction set $\Lambda$ which contains direction vector $\lambda\in \mathbf{Z}^d\backslash\{0\}$. When $f_{j+1}\in\mathcal{V}_{j+1}$ is a polynomial of degree $\le n$,

$$f_{j+1} = \sum_{m\in\mathbf{Z}^d} s_m^{(j+1)}\phi_{j+1,m}$$

with coefficient $s_m^{(j+1)}$ being a polynomial in $m$ (cf. [Chui and Lai'87]). It is known, e.g., in [de Boor, Hölig, and Riemenschneider'93] that

$$\bigcup_{n=0}^{\infty} \mathbf{P}_d \cap \mathcal{V}_{j+1} = \bigcap_{\substack{\Lambda'\in\Lambda \\ \mathrm{span}_{(\Lambda\backslash\Lambda')\neq \mathbf{R}^d}}} \ker(D_{\Lambda'})$$

where $D_{\Lambda'} = \prod_{\lambda\in\Lambda'} D_\lambda$, and $D_\lambda$ is the directional derivative along direction $\lambda$, $\ker(D_{\Lambda'}) = \{f, D_{\Lambda'}f = 0\}$. Then

$$\prod_{\lambda\in\Lambda'}(1-e^{i\lambda\cdot\omega})$$

will annilate the discrete Fourier transform of all polynomials of total degree $\le n$ if $\Lambda' \subset \Lambda$ and $\mathrm{span}(\Lambda\backslash\Lambda') \neq \mathbf{R}^d$, where

$$n \le \min\{\#(\Lambda'), \Lambda' \subset \Lambda, \qquad \mathrm{span}(\Lambda\backslash\Lambda') \neq \mathbf{R}^d\}.$$

In this case, if

$$\sum_{i=1}^{r+1-|D|} q_i(D^*\omega)n_{i,1}(D^*\omega)^* = s(D^*\omega)\prod_{\lambda\in\Lambda'}(1-e^{i\lambda\cdot D^*\omega})$$

we will have (3.9).

# 4 An Example of Virtual Components

In this section we use bivariate box spline tight wavelet frames to explain how to construct virtual components. Let

$$\widehat{\phi}_{k,l,m}(\eta,\theta) = \left(\frac{1-e^{-i\eta}}{i\eta}\right)^k \left(\frac{1-e^{-i\theta}}{i\theta}\right)^l \left(\frac{1-e^{-i(\eta+\theta)}}{i(\eta+\theta)}\right)^m$$

be the Fourier transform of a box spline on the three direction mesh (see, [de Boor, Höllig, and Riemenschneider'03]). We shall use the tight wavelet frames constructed in [Lai and Stöckler'06]. For simplicity, we use box spline $\phi_{111}$. Let $z = e^{-i\eta}$ and $w = e^{-i\theta}$ and

$$P(\eta,\theta) = (1+z)(1+w)(1+zw)/8$$

be the mask associated with box spline $\phi_{111}$ under the standard dilation matrix $2I_2$, where $I_2$ is the identity matrix of size $2\times 2$. That is,

$$\widehat{\phi}(\eta,\theta) = P(\eta/2,\theta/2)\widehat{\phi}(\eta/2,\theta/2).$$

Let $p_1(z,w) = (1+zw)/4, p_2(z,w) = (1+w)/4, p_3(z,w) = (1+x)/4, p_4(z,w) = 1/2$ be the polyphase components of $P(\eta,\theta)$. Let $\tilde{p}_1(z,w) = \sqrt{6}(1-w)/8$ and $\tilde{p}_2(z,w) = (2-z-zw)\sqrt{2}/8$. It is easy to see that letting

$$\widetilde{P}(z,w) = [p_1(z,w), p_2(z,w), p_3(z,w), p_4(z,w), \tilde{p}_1(z,w), \tilde{p}_2(z,w)],$$

$\widetilde{P}(z,w)\widetilde{P}(z,w)^T = 1$. Following the ideas of the construction of multivariate tight wavelet frames in [Lai and Stöckler'06], we first let

$$\widetilde{Q}(z,w) = I_6 - \widetilde{P}(z,w)^T\widetilde{P}(1/z,1/w),$$

with $I_6$ being the identity matrix of size $6 \times 6$ and let $\mathcal{R}(z,w)$ be the top four rows of $\widetilde{Q}(z,w)$. Since we can easily verify that $\widetilde{Q}(z,w)\widetilde{Q}(1/z,1/w)^T = I_6$, we have $\mathcal{R}(z,w)\mathcal{R}(1/z,1/w)^T = I_4$, where $I_4$ is the identity matrix of size $4 \times 4$. Letting

$$[Q_1(\eta,\theta), Q_2(\eta,\theta), \cdots, Q_6(\eta,\theta)] = \frac{1}{2}\,[1 \quad z \quad w \quad zw\,]\,\mathcal{R}(z^2,w^2),$$

we know that $\widehat{\psi}_i(\eta,\theta) := Q_i(\eta/2,\theta/2)\widehat{\phi}(\eta/2,\theta/2), i = 1,\cdots,6$ define 6 tight wavelet framelets by the unitary extension principal (UEP).

We now show how to construct virtual components. Let

$$\mathcal{U} = \frac{1}{2}\begin{bmatrix} 1 & z & w & zw \\ 1 & -z & w & -zw \\ 1 & z & -w & -zw \\ 1 & -z & -w & zw \end{bmatrix}$$

and

$$\mathcal{P} = \mathcal{U}\begin{bmatrix} p_1(z^2,w^2) \\ p_2(z^2,w^2) \\ p_3(z^2,w^2) \\ p_4(z^2,w2) \end{bmatrix}, \qquad \mathcal{Q} = \mathcal{U}\mathcal{R}(z^2,w^2).$$

With $\mathcal{M} = [\mathcal{P},\mathcal{Q}]$, we need to extend $\mathcal{M}$ to square unitary matrix $\mathcal{M}_e$ of size $7 \times 7$. As we saw above, the formula of $\widetilde{Q}(z,w)$ implies that

$$\begin{bmatrix} \mathcal{U} & 0 \\ 0 & I_2 \end{bmatrix}\widetilde{Q}(z^2,w^2) = \begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix}$$

is a partial extension of $\mathcal{M}$. That is, $\begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1 \end{bmatrix}$ contains 6 rows of unitary vectors with 7 entries. By the inspection, we can find the last row of unitary vector

$$\begin{aligned} \mathcal{N}_2(z,w) \quad = \quad & [0, (1+1/(zw))/4, (1+1/w)/4, (1+1/z)/4, 1/2, \\ & \sqrt{3}(1-1/w)/(4\sqrt{2}), (2-1/z-1/(zw))/(4\sqrt{2})]. \end{aligned}$$

That is, we have

$$\mathcal{M}_e = \begin{bmatrix} \mathcal{M} \\ \mathcal{N}_1(z^2,w^2) \\ \mathcal{N}_2(z^2,w^2) \end{bmatrix}.$$

Next we find $\beta_{i,1},\cdots,\beta_{i,4}$ such that

$$\sum_{j=1}^{4}\beta_{i,j}p_j(z,w) = \tilde{p}_i(z,w), \quad i = 1,2.$$

It is easy to find these solutions. They are

$$\begin{aligned}[\beta_{11}, \cdots, \beta_{14}] &= [0, -1/2, 0, \sqrt{6}] \\ [\beta_{21}, \cdots, \beta_{24}] &= [-1, 0, -1/\sqrt{2}, 2].\end{aligned}$$

The corresponding $\alpha_{ij}$ satisfying Lemma 3.2 can be easily converted from these $\beta_{ij}$.

The above discussion provides a detailed description of virtual components for the tight wavelet frame based on box spline $B_{111}$ on three direction mesh.

# 5 Tight Wavelet Frames using the Quincunx Dilation Matrix

In this section we use the tight wavelet frames under the quincunx dilation

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

to illustrate how to construct the virtual components. First of all we explain a new method for constructing tight wavelet frames using bivariate box splines on four direction mesh using the quincunx dilation matrix. In this case, the assumption in Theorem 3.4 is not satisfied. We have to use Theorem 3.5. Thus we explain how to construct virtual components in this situation.

Let

$$\widehat{\phi}_{k,l,m,n}(\eta, \theta) = \left(\frac{1 - e^{-i\eta}}{i\eta}\right)^k \left(\frac{1 - e^{-i\theta}}{i\theta}\right)^l \left(\frac{1 - e^{-i(\eta+\theta)}}{i(\eta + \theta)}\right)^m \left(\frac{1 - e^{-i(\eta-\theta)}}{i(\eta - \theta)}\right)^n$$

be the Fourier transform of a box spline on the four direction mesh. It is known that box spline function $\phi_{k,l,m,n}(x, y)$ is refinable under the dilation matrix $S$(cf., e.g., [Ron and Shen'99]) when $k = m$ and $l = n$. That is, let

$$\widehat{\phi}_{m,n} := \widehat{\phi}_{m,n,m,n}.$$

Then

$$\widehat{\phi}_{m,n}(S[\eta, \theta]^T) = P(\eta, \theta)\widehat{\phi}_{m,n}(\eta, \theta)$$

with

$$P(\eta, \theta) = \left(\frac{1 + e^{-i\eta}}{2}\right)^m \left(\frac{1 + e^{-i\theta}}{2}\right)^n.$$

A construction of tight wavelet frames using these box spline functions were given in [Ron and Shen'99] and the number of the tight wavelet framelets associated with $\phi_{m,n}$

is $m+n+1$. As box spline function $\phi_{m,n}$ is smoother, $m$ and $n$ get larger and hence, the number of tight wavelet framlets gets bigger. In the following we present another method of construction such that the number of tight wavelet framelets is bounded independent of $m$ and $n$. The ideas of our construction are similar to the ones in [Lai and Stöckler'06]. We begin with the following

**Lemma 5.1** *There exists at most* 2 *Laurent polynomials* $\tilde{p}_1$ *and* $\tilde{p}_2$ *such that*

$$\begin{aligned}&|P(\eta,\theta)|^2+|P(\eta+\pi,\theta)|^2+|P(\eta,\theta+\pi)|^2+|P(\eta+\pi,\theta+\pi)|^2\\&+\sum_{j=1}^{2}|\tilde{p}_j(2\eta,2\theta)|^2=1.\end{aligned}$$

**Proof:** Since $|P(\eta,\theta)|^2=\cos^{2m}(\eta/2)\cos^{2n}(\theta/2)$, we first consider

$$1-|P(2\eta,2\theta)|^2-|P(2\eta+\pi,2\theta)|^2-|P(2\eta,2\theta+\pi)|^2-|P(2\eta+\pi,2\theta+\pi)|^2$$

which is

$$\begin{aligned}&1-\cos^{2m}(\eta)(\cos^{2n}(\theta)+\sin^{2n}(\theta)-\sin^{2m}(\eta)(\cos^{2n}(\theta)+\sin^{2n}(\theta))\\=\;&1-\cos^{2m}(\eta)-\sin^{2m}(\eta)\\&+(\cos^{2m}(\eta)+\sin^{2m}(\eta))(1-\cos^{2n}(\theta)-\sin^{2n}(\theta)).\end{aligned}$$

Note that $1-\cos^{2n}(\theta)-\sin^{2n}(\theta)\geq 0$ and it can be rewritten as a Laurent polynomial in $z^{i4\theta}$. Bying use the Fejér-Riesz lemma, we find a Laurent polynomial $q_1(\theta)$ in $z=e^{i\theta}$ such that

$$1-\cos^{2n}(\theta)-\sin^{2n}(\theta)=|q_1(4\theta)|^2$$

Similarly, we can find Laurent polynomials $q_2(\eta)$ and $q_3(\eta)$ in $w=e^{i\eta}$ such that

$$\cos^{2m}(\eta)+\sin^{2m}(\eta)=|q_2(4\eta)|^2 \text{ and } 1-\cos^{2m}(\eta)-\sin^{2m}(\eta)=|q_3(4\eta)|^2.$$

It follows that

$$\begin{aligned}&1-|P(2\eta,2\theta)|^2-|P(2\eta+\pi,2\theta)|^2-|P(2\eta,2\theta+\pi)|^2-|P(2\eta+\pi,2\theta+\pi)|^2\\=\;&|q_3(4\eta)|^2+|q_2(4\eta)|^2|q_1(4\theta)|^2.\end{aligned}$$

If we choose

$$\tilde{p}_1(\eta,\theta)=q_3(\eta,\theta) \text{ and } \tilde{p}_2(\eta,\theta)=q_2(\eta)q_1(\theta),$$

then the equality in (20) holds. This completes the proof. ■

Note that when $m = n = 1$ we have $q_1 \equiv 0$ and $q_3 \equiv 0$. Thus, (20) holds with $\tilde{p}_1 = \tilde{p}_2 \equiv 0$. Next we write $P(\eta, \theta)$ in the polyphase form:

$$P(\eta,\theta) = P_1(2\eta, 2\theta) + e^{i\eta}P_2(2\eta, 2\theta) + e^{i\theta}P_3(2\eta, 2\theta) + e^{i(\eta+\theta)}P_4(2\eta, 2\theta),$$

for four Laurent trigonometric polynomia $P_1, \cdots, P_4$ in $z = e^{\theta}$ and $w = e^{i\eta}$. It is easy to see that

$$\begin{aligned} & |P(\eta,\theta)|^2 + |P(\eta+\pi,\theta)|^2 + |P(\eta,\theta+\pi)|^2 + |P(\eta+\pi,\theta+\pi)|^2 \\ = \; & 4(|P_1(2\eta,2\theta)|^2 + |P_2(2\eta,2\theta)| + |P_3(2\eta,2\theta)|^2 + |P_4(2\eta,2\theta)|^2). \end{aligned}$$

Then letting

$$\mathcal{P} = [2P_1(\eta,\theta), 2P_2(\eta,\theta), 2P_3(\eta,\theta), 2P_4(\eta,\theta), \tilde{p}_1(\eta,\theta), \tilde{p}_2(\eta,\theta)]^T$$

be a column vector of size $6 \times 1$ we define

$$\tilde{Q}(\eta,\theta) = I_{6\times 6} - \overline{\mathcal{P}}\mathcal{P}^T$$

where $I_{6\times6}$ is the $6 \times 6$ identity matrix.

Then it is easy to see $\overline{\tilde{Q}(\eta,\theta)}^T \tilde{Q}(\eta,\theta) = I_{6\times6} - \overline{\mathcal{P}}\mathcal{P}^T$ by Lemma 5.1. That is,

$$\overline{\mathcal{P}}\mathcal{P}^T + \overline{\tilde{Q}(\eta,\theta)}^T \tilde{Q}(\eta,\theta) = I_{6\times6}.$$

Let us look at the principle $4 \times 4$ block of the above matrix equation. We first use $2(\eta, \theta)$ instead of $(\eta, \theta)$ and then multiply from the left and right by

$$U = \frac{1}{2}(e^{im\cdot(\omega+\ell)})_{\substack{m\in\{0,1\}^2 \\ \ell\in\{0,1\}^2\pi}} \text{ and } U^*$$

respectively to get

$$\begin{bmatrix} \overline{P}(\eta,\theta) \\ \overline{P}(\eta+\pi,\theta) \\ \overline{P}(\eta,\theta+\pi) \\ \overline{P}(\eta+\pi,\theta+\pi) \end{bmatrix} [\, P(\eta,\theta) \quad P(\eta+\pi,\theta) \quad P(\eta,\theta+\pi) \quad P(\eta+\pi,\theta+\pi) \,]$$
$$+\overline{\tilde{Q}}^T \tilde{Q} = I_{4\times4},$$

where $I_{4\times4}$ is the identity matrix of size $4 \times 4$ and $\tilde{Q}$ are $4 \times 6$ matrix with Laurent polynomial entries defined as follows. Writing $\tilde{Q}(\eta,\theta) = [\tilde{q}_{ij}(\eta,\theta)]_{1\le i,j\le 6}$, we have

$$\tilde{Q} := \begin{bmatrix} \overline{\tilde{q}_{1,1}(2\eta,2\theta)} & \overline{\tilde{q}_{1,2}(2\eta,2\theta)} & \cdots & \overline{\tilde{q}_{1,4}(\eta,\theta)} \\ \vdots & \ddots & & \vdots \\ \vdots & \ddots & & \vdots \\ \overline{\tilde{q}_{6,1}(2\eta,2\theta)} & \overline{\tilde{q}_{6,2}(2\eta,2\theta)} & \cdots & \overline{\tilde{q}_{6,4}(2\eta,2\theta)} \end{bmatrix} U^*.$$

Among the above $4\times 4$ matrix equation, we only consider a submatrix of size $2\times 2$ which consists of the first and fourth rows and columns. That is,

$$I_{2\times 2} = \begin{bmatrix} \overline{P(\eta,\theta)} \\ \overline{P(\eta+\pi,\theta+\pi)} \end{bmatrix} [P(\eta,\theta), P(\eta+\pi,\theta+\pi)] + \begin{bmatrix} q_{11}(\eta,\theta) & q_{14}(\eta,\theta) \\ \vdots & \vdots \\ \vdots & \vdots \\ q_{61}(\eta,\theta) & q_{64}(\eta,\theta) \end{bmatrix}^* \begin{bmatrix} q_{11}(\eta,\theta) & q_{14}(\eta,\theta) \\ \vdots & \vdots \\ \vdots & \vdots \\ q_{61}(\eta,\theta) & q_{64}(\eta,\theta) \end{bmatrix}$$

where for $j = 1, 2, 3, 4, 5, 6$,

$$q_{j1}(\eta,\theta) = \tilde{q}_{j1}(2\eta, 2\theta) + e^{i\eta}\tilde{q}_{j2}(2\eta, 2\theta) + e^{i\theta}\tilde{q}_{j3}(2\eta, 2\theta) + e^{i(\eta+\theta)}\tilde{q}_{j4}(2\eta, 2\theta)$$

and $q_{j4}(\eta,\theta) = q_{j1}(\eta+\pi,\theta+\pi)$. Thus, by using the UEP in [Ron and Shen'97] we have

**Theorem 5.1** *Let $Q_j(\eta,\theta) = q_{j1}(\eta,\theta)$ and define $\psi^{(j)}(x,y)$ in terms of Fourier transform by*

$$\widehat{\psi^{(j)}(S[\eta,\theta]^T)} = Q_j(\eta,\theta)\widehat{\phi}(\eta,\theta) \tag{20}$$

*Then $\psi^{(j)}, j = 1,\cdots,6$ are tight wavelet framelets. That is,*

$$\psi^{(j)}(S^\ell[x,y]^T - [k_1,k_2]^T), j = 1,\cdots,6, \ell, k_1, k_2 \in \mathbf{Z},$$

*form a tight frame for $L_2(\mathbf{R}^2)$.*

When $m = n = 1$, we define

$$\tilde{Q}(\eta,\theta) = I_{4\times 4} - \overline{\mathcal{P}}\mathcal{P}^T.$$

Then the computation similar to the above yields four wavelet framelets. Let

$$\begin{aligned} Q_1(\eta,\theta) &= \frac{1}{8}(3 - e^{i\eta} - e^{i\theta} - e^{i(\eta+\theta)}) \\ Q_2(\eta,\theta) &= \frac{1}{8}(-1 + 3e^{i\eta} - e^{i\theta} - e^{i(\eta+\theta)}) \\ Q_3(\eta,\theta) &= \frac{1}{8}(-1 - e^{i\eta} + 3e^{i\theta} - e^{i(\eta+\theta)}) \\ Q_4(\eta,\theta) &= \frac{1}{8}(-1 - e^{i\eta} - e^{i\theta} + 3e^{i(\eta+\theta)}) \end{aligned}$$

and define $\psi^{(j)}(x,y)$ in terms of its Fourier transform by (20). Then they generate a tight wavelet frame for $L_2(\mathbf{R}^2)$.

We now turn attention to the construction of virtual components. First we recall, in [Ron and Shen'99], three tight wavelet framelets associated with box spline $\phi_{1111}$ were given. The filters associated with the tight wavelet framelets are

$$\begin{aligned} Q_1(\eta,\theta) &= (1-e^{i\eta}+e^{i\theta}-e^{i(\eta+\theta)})/4 \\ Q_2(\eta,\theta) &= (1+e^{i\eta}-e^{i\theta}-e^{i(\eta+\theta)})/4 \\ Q_3(\eta,\theta) &= (1-e^{i\eta}-e^{i\theta}+e^{i(\eta+\theta)})/4. \end{aligned}$$

Let

$$\mathcal{M}=\begin{bmatrix} P(\eta,\theta) & Q_1(\eta,\theta) & Q_2(\eta,\theta) & Q_3(\eta,\theta)] \\ P(\eta+\pi,\theta+\pi) & Q_1(\eta+\pi,\theta+\pi) & Q_2(\eta+\pi,\theta+\pi) & Q_3(\eta+\pi,\theta+\pi)] \end{bmatrix}.$$

Let us rewrite $\mathcal{M}$ in another polyphase format. Let $z=e^{i\eta}, w=e^{i\theta}$ and $u=zw$ and $v=w/z$. Then

$$\mathcal{M}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} 1+u & 1-u & 1-u & 1+u \\ 1+v & -(1-v) & 1-v & -(1+v) \end{bmatrix}.$$

The extension $\mathcal{N}$ can now be found easily by inspection:

$$\mathcal{N}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} 1-1/u & -(1+1/u) & -(1+1/u) & 1-1/u \\ 1-1/v & 1+1/v & -(1+1/v) & -(1-1/v) \end{bmatrix}.$$

Thus, $\mathcal{M}_e=\begin{bmatrix} \mathcal{M} \\ \mathcal{N} \end{bmatrix}$ is an unitary matrix. Let

$$X=\begin{bmatrix} F_{j+1}(\eta,\theta) \\ F_{j+1}(\eta+\pi,\theta+\pi) \end{bmatrix}$$

be a z-transform of polynomial signal and

$$\widetilde{X}=\begin{bmatrix} \alpha_{11}F_{j+1}(\eta,\theta)+\alpha_{12}F_{j+1}(\eta+\pi,\theta+\pi) \\ \alpha_{21}F_{j+1}(\eta,\theta)+\alpha_{22}F_{j+1}(\eta+\pi,\theta+\pi) \end{bmatrix}=\alpha X$$

be virtual components, where $\alpha$ is a $2\times 2$ matrix with entries $\alpha_{11},\alpha_{12},\alpha_{21}$ and $\alpha_{22}$ being polynomials of $e^{i\eta}$ and $e^{i\theta}$. We need to determine $\alpha$. In order to that the polynomial signal $X$ can be recovered from the first component of

$$Y_e=\mathcal{M}_e^*X_e, \text{ with } X_e=\begin{bmatrix} X \\ \alpha X \end{bmatrix},$$

we need to solve $\alpha_{11},\cdots,\alpha_{22}$ such that

$$\begin{bmatrix} \alpha_{11}P(\eta,\theta)+\alpha_{12}P(\eta+\pi,\theta+\pi) \\ \alpha_{11}P(\eta,\theta)+\alpha_{12}P(\eta+\pi,\theta+\pi) \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}\left(\frac{1}{2\sqrt{2}}\begin{bmatrix} 1-1/u \\ 1-1/v \end{bmatrix}+\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right). \quad (21)$$

Note that when $\eta = 0, \theta = \pi$ or $\eta = \pi, \theta = 0$, we have $P(\eta,\theta) = 0 = P(\eta+\pi, \theta+\pi)$. That is, $P(\eta,\theta)$ and $P(\eta+\pi,\theta+\pi)$ have two common zeros. Let us use $q_1$ and $q_2$ to make sure that the right-hand side is also zero at these locations. We have many choices, e.g.,

$$q_1(\eta,\theta) = \frac{-(1-u)(1-v)^2}{8\sqrt{2}uv} \text{ and } q_2(\eta,\theta) = \frac{(1-u)^2(1-v)}{8\sqrt{2}uv}.$$

Then (21) has a simple solution. Letting $\beta$ be 2x2 matrix with entries $\beta_{11} = 0, b_{12} = (u-1)(1+v)/(uv)$, $\beta_{21} = 0, \beta_{22} = (v-1)(1+u)/(uv)$, we have

$$\begin{bmatrix} 0 & (u-1)(1+v)/(uv) \\ 0 & (v-1)(1+u)/(uv) \end{bmatrix} \frac{1}{2\sqrt{2}} \begin{bmatrix} 1+u \\ 1+v \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1-1/u \\ 1-1/v \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

It follows that

$$\alpha = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \beta \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}^*.$$

Using these virtual components, we have

$$Y_e = \mathcal{M}_e^* X_e = \mathcal{M}^* X + \mathcal{N}^* \alpha X = (\mathcal{M}^* + \mathcal{N}^* \alpha) X.$$

If $f_{j+1} \in V_{j+1}$ is also in $V_j$, then

$$\widehat{f_{j+1}} = F_{j+1}(\eta,\theta)\widehat{\phi_{11}}(\eta,\theta) = \tilde{F}_j(S[\eta,\theta]^T)\widehat{\phi_{11}}(S[\eta,\theta]^T).$$

By the dilation relation we have $F_{j+1}(\eta,\theta) = \tilde{F}_j(S[\eta,\theta]^T)P(\eta,\theta)$. From the above, the first component $y_1$ from $Y_e$ can be seen to be

$$\begin{aligned} y_1 &= \left( [\overline{P(\eta,\theta)}, \overline{P(\eta+\pi,\theta+\pi)}] + [\frac{1-u}{2\sqrt{2}}, \frac{1-v}{2\sqrt{2}}] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1/z & -1/z \end{bmatrix} \alpha \right) \\ &\qquad \begin{bmatrix} F_{j+1}(\eta,\theta) \\ F_{j+1}(\eta+\pi,\theta+\pi) \end{bmatrix} \\ &= \tilde{F}_j(S[\eta,\theta]^T) \left( |P(\eta,\theta)|^2 + |P(\eta+\pi,\theta+\pi)|^2 + [\frac{1-u}{2\sqrt{2}}, \frac{1-v}{2\sqrt{2}}] \beta \begin{bmatrix} \frac{1+u}{2\sqrt{2}} \\ \frac{1-v}{2\sqrt{2}} \end{bmatrix} \right) \\ &= \tilde{F}_j(S[\eta,\theta]^T) \Big( |P(\eta,\theta)|^2 + |P(\eta+\pi,\theta+\pi)|^2 + \\ &\qquad + \frac{|1-u|^2}{8} + \frac{|1-v|^2}{8} + \frac{1-u}{2\sqrt{2}} q_1 + \frac{1-v}{2\sqrt{2}} q_2) \Big) \\ &= \tilde{F}_j(S[\eta,\theta]^T) \left( 1 + \frac{(1-u)^2(1-v)^2}{8uv} \right). \end{aligned}$$

When $X$ is the z-transform a quadratic polynomial signal, so is the $\tilde{F}_j$ and then $\tilde{F}_j(S[\eta,\theta]^T)\frac{(1-u)^2(1-v)^2}{8uv} = 0$ for all linear polynomials. This shows if we use the new scaling function and new tight framelets with mask

$$\widehat{M} = \mathcal{M} + \alpha^* \mathcal{N} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \frac{\sqrt{2}}{16}$$

$$\begin{bmatrix} 4+4u+u/v-uv-1/v+v & (1-u)(6+1/v+v) \\ (4+4v-1/u+u+v/u+uv & (1-v)(6+u+1/u) \end{bmatrix} \times$$
$$\begin{matrix} (u-1)(1-v)^2/v & 4+4u-u/v+uv+1/v-v \\ (1-v)(1-u)^2/u & 4+4v+1/u-u-v/u+uv \end{bmatrix},$$

then we will be able to reproduce all linear polynomial signals.

Our second choice is

$$q_1(\eta,\theta) = -(1-u)^3/(8\sqrt{(}2)u^2), \text{ and } q_2(\eta,\theta) = -(1-v)^3/(8\sqrt{(}2)v^2).$$

Then (21) has a simple solution. Letting $\beta_{11} = (u^2-1)/(4u^2), \beta_{22} = (v^2-1)/(4v^2)$, we have

$$\begin{bmatrix} \beta_{11} & 0 \\ 0 & \beta_{22} \end{bmatrix} \frac{1}{2\sqrt{2}} \begin{bmatrix} 1+u \\ 1+v \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1-1/u \\ 1-1/v \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Thus our virtual components is

$$\alpha = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} \beta_{11} & 0 \\ 0 & \beta_{22} \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}^*.$$

and hence

$$\widehat{M} = \mathcal{M} + \alpha^* \mathcal{N} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \frac{\sqrt{2}}{16}$$
$$\begin{bmatrix} (-5u-5u^2+u^3+1)/u & (u-1)^3/u & (u-1)^2/u & (5u+5u^2-u^3-1)/u \\ (5v+5v^2-1-v^3)/v & -(v-1)^3/v & (v-1)^3/v & (-5v-5v^2+1+v^3)/v \end{bmatrix}$$

is another mask of scaling function and tight wavelet framelets which enables us to reproduce certain quadratic polynomials.

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