A New Estimate of Restricted Isometry Constants for Sparse Solutions

Ming-Jun Lai *and Louis Y. Liu†

January 12, 2011

Abstract

We show that as long as the restricted isometry constant $\delta_{2k} < 1/2$, there exist a value $q_0 \in (0, 1]$ such that for any $q < q_0$, each minimizer of the nonconvex ℓ_q minimization for the sparse solution of any under-determined linear system is the sparse solution.

1 Introduction

Let us start with the one of basic problems in compressed sensing: seek the minimizer $\mathbf{x}^* \in \mathbb{R}^n$ solving

$$\min\{\|\widetilde{\mathbf{x}}\|_0 : \quad \Phi\widetilde{\mathbf{x}} = \mathbf{b}\}, \tag{1}$$

where $\|\widetilde{\mathbf{x}}\|_0$ stands for the number of nonzero entries of vector $\widetilde{\mathbf{x}}$, Φ is a matrix of size $m \times n$ with $m << n$. That is, the purpose of the research is to find the sparse solution satisfying the under-determined linear system $\Phi\mathbf{x} = \mathbf{b}$ with $\|\mathbf{x}\|_0$ as small as possible. A key concept to describe the solution of (1) is the restricted isometry constants of a matrix Φ introduced in [5].

Definition 1 *For each integer $k = 1, 2, \cdots$, let δ_k be the smallest number such that*

$$(1 - \delta_k)\|\mathbf{x}\|_2^2 \leq \|\Phi\mathbf{x}\|_2^2 \leq (1 + \delta_k)\|\mathbf{x}\|_2^2 \tag{2}$$

*mjlai@math.uga.edu. This author is partly supported by the National Science Foundation under grant DMS-0713807. Department of Mathematics, The University of Georgia, Athens, GA 30602

†yliu@marlboro.edu, Department of Mathematics, Marlboro College, Marlboro, Vermont 05344.

holds for all k-sparse vectors $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_0 \le k$, where $\|\mathbf{x}\|_2$ the standard ℓ_2 norm for vector $\mathbf{x}$. δ_k is called restricted isometry constant.

One of the standard approaches to find the minimizer $\mathbf{x}^*$ is to seek the minimizer $\mathbf{x}^1 \in \mathbb{R}^n$ solving

$$\min\{\|\widetilde{\mathbf{x}}\|_1 : \quad \Phi\widetilde{\mathbf{x}} = \mathbf{b}, \widetilde{\mathbf{x}} \in \mathbb{R}^n\}, \tag{3}$$

where $\|\widetilde{\mathbf{x}}\|_1$ is the standard ℓ_1 norm of vector $\widetilde{\mathbf{x}}$.

Suppose that $\|\mathbf{x}^*\|_0 = k$. Let $T_0 \subset \{1, 2, \cdots, n\}$ be the subset of indices for the k largest entries of $\mathbf{x}^*$. For any vector $\mathbf{x}$, let $\mathbf{x}_{T_0}$ denote the vector whose entries agree with that of $\mathbf{x}$ at the indices in T_0 and zeros for other entries. Many researchers have established the following result in various literature:

Theorem 1 (Noiseless Recovery) *For appropriate $\delta_{2k} > 0$, the solution $\mathbf{x}^1$ of the minimization problem (3) satisfies*

$$\|\mathbf{x} - \mathbf{x}^1\|_2 \le C_0 k^{-1/2} \|\mathbf{x} - \mathbf{x}_{T_0}\|_1, \tag{4}$$

for any $\mathbf{x}$ with $\Phi\mathbf{x} = \mathbf{b}$, where C_0 is a positive constant dependent on δ_{2k}. In particular, if $\mathbf{x}$ is k-sparse, the recovery is exact.

It is known from Candès, 2008[4] that the above result holds when $\delta_{2k} < \sqrt{2} - 1$. This condition is improved in Foucart and Lai, 2009[11] to be $\delta_{2k} < 2/(3 + \sqrt{2}) \approx 0.4531$. Subsequently, this condition is further improved in Cai, Wang, and Xu, 2010[2] for special k (multiple of 4), $\delta_{2k} < 2/(2 + \sqrt{5}) \approx 0.4721$ as well as in Foucart, 2010[10] to be

$$\delta_{2k} < \frac{3}{4 + \sqrt{6}} \approx 0.4652 \text{ and for large } k, \delta_{2k} < \frac{4}{6 + \sqrt{6}} \approx 0.4734.$$

Recently, Li and Mo proposed another approach in [15] and showed that the inequality (4) holds as long as

$$\delta_{2k} < 0.4931.$$

The problem (3) was extended in [12] by seeking a minimizer $\mathbf{x}^q \in \mathbb{R}^n$ for a number $q \in (0, 1)$ which solves

$$\min\{\|\widetilde{\mathbf{x}}\|_q^q : \quad \Phi\widetilde{\mathbf{x}} = \mathbf{b}, \widetilde{\mathbf{x}} \in \mathbb{R}^n\}, \tag{5}$$

where $\|\widetilde{\mathbf{x}}\|_q$ is the standard ℓ_q quasi-norm of vector $\widetilde{\mathbf{x}}$. See also [7], [11], [6], [14] for study of the nonconvex ℓ_q minimization problem (5). In Foucart and Lai, 2009[11], the following result was established.

Theorem 2 *Suppose that $\delta_{2k} < 2(3-\sqrt{2})/7 \approx 0.4531$. Then for any $q \in (0,1]$,*

$$\|\mathbf{x}^q - \mathbf{x}\|_q \le C_0 \|\mathbf{x} - \mathbf{x}_{T_0}\|_q, \tag{6}$$

for any $\mathbf{x}$ with $\Phi\mathbf{x} = \mathbf{b}$, where C_0 is a positive constant dependent on δ_{2k}. In particular, if $\mathbf{x} = \mathbf{x}^*$ is k-sparse, the recovery is exact.

To improve the result in Theorem 2, our main result in this paper is

Theorem 3 *Suppose that $\delta_{2k} < 1/2$. There exists a number $q_0 \in (0,1]$ such that for any $q < q_0$, each minimizer $\mathbf{x}^q$ of the ℓ_q minimization (5) is the sparse solution of (1). Furthermore, there exists a positive constant C_q such that for any $\mathbf{x} \in \mathbb{R}^n$ with $\Phi\mathbf{x} = \mathbf{b}$ such that*

$$\|\mathbf{x} - \mathbf{x}^q\|_q \le C_q \|\mathbf{x} - \mathbf{x}_{T_0}\|_q,$$

where C_q is dependent on q and δ_{2k} and T_0 is the index set of the k nonzero entries of the sparse solution $\mathbf{x}^*$.

Under the assumption that the ℓ_q minimization (5) can be computed, the sensing matrix Φ requires a more relaxed condition on the restricted isometry constant to be able to find the sparse solution than the conditions listed above for Theorem 1.

For simplicity, we only discuss the sparse solution for noiseless recovery in this paper. We leave the discussion on noisy recovery to the interested reader. After we establish an elementary inequality in Preliminary section §2, we prove our main result in §3. Finally we give a few remarks in §4.

2 Preliminary Results

Let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be a vector in $\mathbb{R}^n$ and we use $\|\mathbf{x}\|_p$ be the standard norm for vector $\mathbf{x}$ for any $p \ge 1$ and $\|\mathbf{x}\|_q$ be the standard quasi-norm when $q < 1$. Recall that we have the following standard inequality

$$\|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2 \text{ or } \|\mathbf{x}\|_2 \ge \frac{\|\mathbf{x}\|_1}{\sqrt{n}}. \tag{7}$$

by the well-known Cauchy-Schwarz inequality. A converse of the above inequality is

$$\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$$

which can be seen directly after dividing $\|\mathbf{x}\|_\infty = \max\{|x_i|, i = 1, \cdots, n\}$ both sides. Recently, Cai, Wang and Xu proved the following interesting inequality in [3].

Lemma 1 (Cai, Wang and Xu'10) *For any* $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_2 \leq \frac{\|\mathbf{x}\|_1}{\sqrt{n}} + \frac{\sqrt{n}}{4}\left(\max_{1\leq i\leq n}|x_i| - \min_{1\leq i\leq n}|x_i|\right). \tag{8}$$

We now extend the inequality to the setting of quasi-norm $\|\mathbf{x}\|_q$ with $q \in (0,1)$. It is easy to see that for $0 < q < 1$,

$$\|\mathbf{x}\|_2 \geq \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}} \tag{9}$$

by using Hölder's equality. The following converse inequality

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_q, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

is often used in the literature. Motivated by the new inequality in (8), we would like to see the converse of the inequality (9).

Lemma 2 *Fix* $0 < q < 1$. *For any* $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_2 \leq \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}} + \sqrt{n}\left(\max_{1\leq i\leq n}|x_i| - \min_{1\leq i\leq n}|x_i|\right). \tag{10}$$

Proof. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and not all x_i are equal. Let

$$f(\mathbf{x}) = \|\mathbf{x}\|_2 - \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}}.$$

Let us fix x_1 and find an upper bound for $f(\mathbf{x})$. Note that

$$\frac{\partial f}{\partial x_i} = \frac{x_i}{\|\mathbf{x}\|_2} - \frac{\|\mathbf{x}\|_q^{1-q} x_i^{q-1}}{n^{1/q-1/2}}$$

is an increasing function as a function of x_i. Indeed, it is easy to see that both functions

$$\frac{\|\mathbf{x}\|_2}{x_i} = \sqrt{\sum_{j=1}^n (\frac{x_j}{x_i})^2}$$

and

$$\|\mathbf{x}\|_q^{1-q} x_i^{q-1} = \left(\sum_{j=1}^n (\frac{x_j}{x_i})^q\right)^{(1-q)/q}$$

of x_i are decreasing. Thus, $\frac{\partial f}{\partial x_i}$ is an increasing function of x_i. It follows that $f(\mathbf{x})$ is convex as a function of x_i for each $i = 2, \cdots, n-1$. The maximum achieves at either $x_i = x_{i-1}$ or $x_i = x_{i+1}$. It follows that when f achieves its maximum, $\mathbf{x}$ must be of the form that $x_1 = x_2 = \cdots = x_k$ and $x_{k+1} = \cdots = x_n$ for some $1 \leq k < n$. Thus,

$$f(\mathbf{x}) = \sqrt{k(x_1^2 - x_n^2) + nx_n^2} - \frac{(k(x_1^q - x_n^q) + nx_n^q)^{1/q}}{n^{1/q-1/2}}.$$

It is easy to see that

$$f(\mathbf{x}) \leq \sqrt{n(x_1^2 - x_n^2) + nx_n^2} - \frac{(nx_n^q)^{1/q}}{n^{1/q-1/2}} = \sqrt{n}(x_1 - x_n)$$

■

To find a better upper bound for some $q < 1$, see Remark 4.4. One can see from Remark 4.4 that it is not an easy task to find out which k to maximize the function

$$g(k) = \sqrt{k(x_1^2 - x_n^2) + nx_n^2} - \frac{(k(x_1^q - x_n^q) + nx_n^q)^{1/q}}{n^{1/q-1/2}}$$

(cf. Remark 4.4). Anyway, the result in Lemma 2 is good enough for our application in the next section.

3 Main Results and Proofs

To describe our results, we need more notation. We use Null(Φ) to denote the null space of Φ and $S(\mathbf{x})$ to denote the support of $\mathbf{x} \in \mathbb{R}^n$, i.e., $S = \{i, x_i \neq 0\}$ for $\mathbf{x} = (x_1, \cdots, x_n)^T$. Recall that $\mathbf{x}^*$ is a sparse solution, i.e., $\Phi\mathbf{x}^* = \mathbf{b}$ with $S(\mathbf{x}^*) \subset T_0$ with cardinality of T_0 less or equal to k. Let $\mathbf{x}^q$ be the solution of the minimization problem (5). Recall from [12] that $\mathbf{x}^q$ is the unique spare solution $\mathbf{x}^*$ if and only if

$$\|h_{T_0}\|_q < \|h_{T_0^c}\|_q \tag{11}$$

for all nonzero vector h in the null space of Φ. It is called the null space property. Indeed, we have

$$\begin{aligned}\|\mathbf{x}^*\|_q^q = \|\mathbf{x}^*_{T_0}\|_q^q &\leq \|\mathbf{x}^*_{T_0} + h_{T_0}\|_q^q + \|h_{T_0}\|_q^q \\ &< \|\mathbf{x}^*_{T_0} + h_{T_0}\|_q^q + \|h_{T_0^c}\|_q^q = \|\mathbf{x}^* + h\|_q^q\end{aligned}$$

by (11) for any nonzero vector h in the null space of Φ. Thus, $\mathbf{x}^*$ is the solution of (5). Another way to show the sufficiency is to let $\mathbf{x}^q$ be the solution of (5) and let $h = \mathbf{x}^* - \mathbf{x}^q$ which is in the null space of Φ. If $h \neq 0$, we have

$$\begin{aligned} \|\mathbf{x}^*_{T_0}\|_q^q = \|\mathbf{x}^*\|_q^q \geq \|\mathbf{x}^q\|_q^q = \|\mathbf{x}^q_{T_0}\|_q^q + \|\mathbf{x}^q_{T_0^c}\|_q^q \\ \geq \|\mathbf{x}^*_{T_0}\|_q^q - \|h_{T_0}\|_q^q + \|h_{T_0^c}\|_q^q. \end{aligned}$$

It follows that $\|h_{T_0^c}\|_q^q \leq \|h_{T_0}\|_q^q < \|h_{T_0^c}\|_q^q$ which is a contradiction, where we have used (11). Thus, $\mathbf{x}^q$ is the sparse solution. The necessity of the null space property (11) can be seen as follows: suppose that there is a nonzero vector $h \in \text{null}(\Phi)$ such that $\|h_{T_0^c}\|_q \leq \|h_{T_0}\|_q$. Let $\mathbf{x}^* = h_{T_0}$ and $\mathbf{b} = \Phi\mathbf{x}^*$. If $\|h_{T_0^c}\|_q < \|h_{T_0}\|_q$, then $-h_{T_0^c}$ satisfies $\Phi(-h_{T_0^c}) = \mathbf{b}$ and the minimization (5) should find a solution $\mathbf{x}^q$ which is not h_{T_0}, the sparse solution of this vector $\mathbf{b}$ which is a contradiction to the assumption that $\mathbf{x}^q$ is the unique sparse solution $\mathbf{x}^*$. Similarly, if $\|h_{T_0^c}\|_q = \|h_{T_0}\|_q$, the minimization (5) may find two solutions h_{T_0} and $-h_{T_0^c}$ which is a contradiction.

In fact, one can find the smallest constant $\rho < 1$ such that

$$\|h_{T_0}\|_q \leq \rho\|h_{T_0^c}\|_q, \quad \forall h \in \text{ null } (\Phi).$$

Indeed, it is easy to see that the following equality holds

$$\sup_{\substack{h\in\text{Null}(\Phi)\\ h\neq 0}} \frac{\sum_{i\in T_0}|h_i|^q}{\sum_{i\notin T_0}|h_i|^q} = \max_{\substack{h\in\text{Null}(\Phi)\\ \|h\|_2=1}} \frac{\sum_{i\in T_0}|h_i|^q}{\sum_{i\notin T_0}|h_i|^q}$$

which is denoted by ρ. In general, for $h = \mathbf{x} - \mathbf{x}^q$, let

$$\|h_{T_0}\|_q = \tau(h,q)\|h_{T_0^c}\|_q. \tag{12}$$

The purpose of the study is to show how to make $\tau(h,q) < 1$ for all nonzero vector h in the null space of Φ.

For any nonzero vector h in the null space of Φ, we rewrite h as a sum of vectors $h_{T_0}, h_{T_1}, h_{T_2}, \cdots$, each of sparsity at most k. Here, T_0 corresponds to the locations of the k largest entries of $\mathbf{x}^*$; T_1 to the locations of the k largest entries of $h_{T_0^c}$; T_2 to the locations of the next k largest entries of $h_{T_0^c}$, and so on, where T_0^c stands for the complement index set of T_0 in $\{1,2,\cdots,n\}$. Without loss of generality, we may assume that $h = (h_{T_0}, h_{T_1}, h_{T_2}, \cdots)^T$ with the cardinality of T_i being equal to k for all $i = 0,1,2,\cdots$. Let us introduce another ratio $t := t(h,q) \in [0,1]$ be a number such that $\|h_{T_1}\|_q^q = t\sum_{i\geq 1}\|h_{T_i}\|_q^q$. First of all, we have

Lemma 3 *For $q \in (0,1)$, we have*

$$\sum_{i\geq 2} \|h_{T_i}\|_2^2 \leq \frac{1}{k^{(2-q)/q}}(1-t)t^{(2-q)/q} \left(\sum_{i\geq 1} \|h_{T_i}\|_q^q\right)^{2/q}. \qquad (13)$$

Proof. It is easy to see that

$$\begin{aligned}
\sum_{i\geq 2} \|h_{T_i}\|_2^2 &\leq |h_{2k+1}|^{2-q} \sum_{i\geq 2} \|h_{T_i}\|_q^q \leq \left(\frac{\|h_{T_1}\|_q^q}{k}\right)^{(2-q)/q} \sum_{i\geq 2} \|h_{T_i}\|_q^q \\
&\leq \left(\frac{\|h_{T_1}\|_q^q}{k}\right)^{(2-q)/q} \frac{1-t}{t} \|h_{T_1}\|_q^q \\
&\leq \frac{1}{k^{(2-q)/q}} \frac{1-t}{t} t^{2/q} \left(\sum_{i\geq 1} \|h_{T_i}\|_q^q\right)^{2/q}.
\end{aligned}$$

The result in (13) follows. ■

Next we have

Lemma 4 *For $q \in (0,1)$, we have*

$$\sum_{i\geq 2} \|h_{T_i}\|_2 \leq \frac{1}{k^{1/q-1/2}} \left(\sum_{i\geq 1} \|h_{T_i}\|_q^q\right)^{1/q}. \qquad (14)$$

Proof. By Theorem 2, we have

$$k^{1/q-1/2}\|h_{T_i}\|_2 \leq \|h_{T_i}\|_q + k^{1/q}(|h_{ik+1}| - |h_{ik+k}|)$$

for $i \geq 2$. It follows

$$\begin{aligned}
k^{1/q-1/2} \sum_{i\geq 2} \|h_{T_i}\|_2 &\leq \sum_{i\geq 2} \|h_{T_i}\|_q + k^{1/q}\|h_{T_1}\|_q / k^{1/q} \\
&\leq \sum_{i\geq 1} \|h_{T_i}\|_q \leq \left(\sum_{i\geq 1} \|h_{T_i}\|_q^q\right)^{1/q}
\end{aligned}$$

since $q \leq 1$. ■

Furthermore, we have

Lemma 5 *For $q \in (0,1)$, we have*

$$\|\Phi(h_{T_0} + h_{T_1})\|_2^2 \geq \frac{1-\delta_{2k}}{k^{2/q-1}}(\tau(h,q)^2 + t^{2/q}) \left(\sum_{i\geq 1} \|h_{T_i}\|_q^q\right)^{2/q}. \qquad (15)$$

Proof. By the definition of δ_{2k} and using (9), we have

$$\begin{aligned}\|\Phi(h_{T_0}+h_{T_1})\|_2^2 &\geq (1-\delta_{2k})\|h_{T_0}+h_{T_1}\|_2^2 = (1-\delta_{2k})(\|h_{T_0}\|_2^2+\|h_{T_1}\|_2^2)\\ &\geq (1-\delta_{2k})(\|h_{T_0}\|_q^2+\|h_{T_1}\|_q^2)/k^{2/q-1}\\ &= \frac{1-\delta_{2k}}{k^{2/q-1}}(\tau(h,q)^2+t^{2/q})\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}.\end{aligned}$$

■

It is easy to see that $\Phi(h_{T_0}+h_{T_1}) = \Phi h - \Phi(\sum_{j\geq 2} h_{T_j}) = -\Phi(\sum_{i\geq 2} h_{T_i})$. We have the following estimate

Lemma 6 *For $q \in (0,1)$, we have*

$$\|\Phi(h_{T_0}+h_{T_1})\|_2^2 = \|\Phi(\sum_{j\geq 2} h_{T_j})\|_2^2 \leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{(2-q)/q}} + \frac{\delta_{2k}}{k^{2/q-1}}\right)\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}. \tag{16}$$

Proof. A straightforward calculation shows

$$\begin{aligned}\|\Phi(\sum_{j\geq 2} h_{T_j})\|_2^2 &= \sum_{i,j\geq 2}\langle\Phi(h_{T_i}),\Phi(h_{T_j})\rangle\\ &= \sum_{j\geq 2}\langle\Phi(h_{T_j}),\Phi(h_{T_j})\rangle + 2\sum_{2\leq i<j}\langle\Phi(h_{T_i}),\Phi(h_{T_j})\rangle\\ &\leq (1+\delta_k)\sum_{i\geq 2}\|h_{T_i}\|_2^2 + 2\delta_{2k}\sum_{i>j\geq 2}\|h_{T_i}\|_2\|h_{T_j}\|_2\\ &\leq \sum_{i\geq 2}\|h_{T_i}\|_2^2 + \delta_{2k}(\sum_{i\geq 2}\|h_{T_i}\|_2)^2\\ &\leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{(2-q)/q}} + \frac{\delta_{2k}}{k^{2/q-1}}\right)\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}.\end{aligned}$$

■

By using (15) and (16), we have

$$(1-\delta_{2k})(\tau(h,q)^2+t^{2/q}) \leq (1-t)t^{(2-q)/q}+\delta_{2k}$$

or

$$\tau(h,q)^2 \leq (\delta_{2k}+t^{(2-q)/q}-(2-\delta_{2k})t^{2/q})/(1-\delta_{2k}). \tag{17}$$

Let us study the maximum of the right-hand side as a function of $t \in [0, 1]$. Letting $s = (2-q)/(2-\delta_{2k})$ with $s \le 2$, it is easy to see that the maximum happens at $t = s/2$ and

$$\tau(h,q)^2 \le (\delta_{2k} + \frac{2}{s}\left(\frac{s}{2}\right)^{2/q} - (2-\delta_{2k})\left(\frac{s}{2}\right)^{2/q})/(1-\delta_{2k}) = \frac{\delta_{2k}s + \left(\frac{s}{2}\right)^{2/q} q}{s(1-\delta_{2k})}.$$

If the term on the right-hand of the inequality is less than 1, then we will have $\tau(h,q) < 1$ and hence, $\mathbf{x}^q$ is the sparse solution of (1). To see the range of value of δ_{2k}, we continue the following simple analysis:

$$\delta_{2k}s + q\left(\frac{s}{2}\right)^{2/q} < s - \delta_{2k}s$$

or

$$2\delta_{2k} + \left(\frac{s}{2}\right)^{2/q}\frac{q}{s} < 1.$$

Further simplification yields

$$\delta_{2k} + q\left(\frac{2-q}{2(2-\delta_{2k})}\right)^{2/q}\frac{2-\delta_{2k}}{2(2-q)} < 1/2. \tag{18}$$

Since the second term on the left-hand side goes to zero as $q \to 0_+$ as $\delta_{2k} < 1$, $q \le 1$ and

$$\left(\frac{2-q}{2(2-\delta_{2k})}\right)^{2/q}\frac{2-\delta_{2k}}{2(2-q)} \le \left(\frac{2-q}{2}\right)^{2/q} \approx \frac{1}{e},$$

we can establish the results in Theorem 3.

Proof. of Theorem 3. Based on the proofs of Lemmas 5 and 6, we have

$$\|h_{T_0}\|_q^2 \le \rho_q^2\left(\sum_{i\ge1}\|h_{T_i}\|_q^q\right)^{2/q}.$$

where ρ_q^2 is

$$\rho_q^2 := \frac{\delta_{2k}s + \left(\frac{s}{2}\right)^{2/q} q}{s(1-\delta_{2k})}.$$

That is,

$$\|h_{T_0}\|_q \le \rho_q\left(\sum_{i\ge1}\|h_{T_i}\|_q^q\right)^{1/q}. \tag{19}$$

Since $\delta_{2k} < 1$, there exists a q_0 such that (18) holds and hence, we will have $\rho_q < 1$ for any $q < q_0$.

As $\mathbf{x}^q$ is a minimizer of (5) and for any $\mathbf{x}$ which is a solution of the under-determined linear equations $\Phi\mathbf{x} = \mathbf{b}$, we let $h = \mathbf{x}^q - \mathbf{x}$ and

$$\begin{aligned}\|\mathbf{x}_{T_0}\|_q^q + \|\mathbf{x}_{T_0^c}\|_q^q &= \|\mathbf{x}\|_q^q \geq \|\mathbf{x}^q\|_q^q = \|\mathbf{x}+h\|_q^q = \sum_{i\in T_0} |x_i + h_i|^q + \sum_{i\in T_0^c} |x_i + h_i|^q \\ &\geq \|\mathbf{x}_{T_0}\|_q^q - \|h_{T_0}\|_q^q + \|h_{T_0^c}\|_q^q - \|\mathbf{x}_{T_0^c}\|_q^q.\end{aligned}$$

Thus, we have

$$\|h_{T_0^c}\|_q^q \leq \|h_{T_0}\|_q^q + 2\|\mathbf{x}_{T_0^c}\|_q^q. \tag{20}$$

Together with (19), we conclude

$$\sum_{i\geq 1} \|h_{T_i}\|_q^q \leq \rho_q^q \sum_{i\geq 1} \|h_{T_i}\|_q^q + 2\|\mathbf{x}_{T_0^c}\|_q^q.$$

That is,

$$\sum_{i\geq 1} \|h_{T_i}\|_q^q \leq \frac{2}{1-\rho_q^q} \|\mathbf{x}_{T_0^c}\|_q^q.$$

By (19), we have

$$\|h\|_q^q = \|h_{T_0}\|_q^q + \sum_{i\geq 1} \|h_{T_i}\|_q^q \leq (\rho_q^q + 1) \sum_{i\geq 1} \|h_{T_i}\|_q^q \leq \frac{2(1+\rho_q^q)}{1-\rho_q^q} \|\mathbf{x}_{T_0^c}\|_q^q.$$

These complete the proof. ■

4 Remarks

We have a few remarks in order.

Remark 4.1 *Clearly, the results in Theorem 3 can be extended to the noisy recovery setting as in [4] and [11]. We leave the discussion to the interested reader.*

Remark 4.2 *The results in Theorem 3 can also be extended to dealing with sparse solution for multiple measurement vectors as discussed in [13]. We omit the details.*

Remark 4.3 *Recently the block sparse solution of compressed sensing problems was introduced and studied in [8], [1], which have many practical applications, such as DNA microarrays [17], multiband signal [16], and magnetoencephalography (MEG) [9]. In recovering the sparse solution* $\mathbf{x}$ *from* $\Phi\mathbf{x} = \mathbf{b}$*, the entries of* $\mathbf{x}$ *are grouped into blocks. That is,* $\mathbf{x} = (\mathbf{x}_{t_1}, \mathbf{x}_{t_2}, \cdots, \mathbf{x}_{t_\ell})$ *with* $\mathbf{x}_{t_i}$ *being a block of entries for each* i*. One looks for the fewest number of nonzero blocks* $\mathbf{x}_{t_i}$ *such that* $\Phi\mathbf{x} = \mathbf{b}$*. Letting*

$$|||\mathbf{x}|||_{2,q} = \left(\sum_{i=1}^{\ell} \|\mathbf{x}_{t_i}\|_2^q\right)^{1/q}$$

be a mixed norm with $\|\mathbf{x}_{t_i}\|_2$ *is the standard* ℓ_2 *norm of vector* $\mathbf{x}_{t_i}$*, one finds the block sparse solution*

$$\min\{|||\mathbf{x}|||_{2,q}, \quad \Phi\mathbf{x} = \mathbf{b}\}.$$

(Cf. [8] for $q = 1$*.) The concept of restricted isometry constant was extended in this mixed norm minimization when* $q = 1$ *in [8]. Our study in §3 can be generalized to the setting. We leave the details to the interested reader.*

Remark 4.4 *In order to find a better upper bound for Lemma 2, we need to find out which* k *maximizes* $f(\mathbf{x})$*. Let us treat the right-hand side of the equation in the end of the proof of Theorem 2 as a function* $g(k)$*. Note that* $g(n) = 0$ *and* $g(0) = 0$*. The maximum of* g *must happen inside* k *between* 1 *and* $n - 1$*. The derivative of* g *is*

$$g'(k) = \frac{x_1^2 - x_n^2}{2\sqrt{k(x_1^2 - x_n^2) + nx_n^2}} - \frac{(x_1^q - x_n^q)}{qn^{1/q-1/2}}(k(x_1^q - x_n^q) + nx_n^q)^{1/q-1}.$$

The critical point satisfies

$$\frac{qn^{1/q-1/2}(x_1^2 - x_n^2)}{2(x_1^q - x_n^q)} = \sqrt{k(x_1^2 - x_n^2) + nx_n^2}(k(x_1^q - x_n^q) + nx_n^q)^{1/q-1}$$

That is,

$$\sqrt{k(x_1^2 - x_n^2) + nx_n^2} = \frac{qn^{1/q-1/2}(x_1^2 - x_n^2)}{2(x_1^q - x_n^q)}(k(x_1^q - x_n^q) + nx_n^q)^{1-1/q}. \quad (21)$$

The critical point of k *is not easy to find except for* $q = 1$*. Let us try a particular* $q = \frac{2}{3}$*. In this case, we have*

Lemma 7 *For any* $\mathbf{x} \in \mathbb{R}^n$*, one has*

$$\|\mathbf{x}\|_2 - \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}} \leq \frac{2}{3}\sqrt{\frac{n}{3}} \left(\max_{1\leq i\leq n} |x_i|^q - \min_{1\leq i\leq n} |x_i|^q \right)^{1/q} \tag{22}$$

for $q = \frac{2}{3}$*. In particular, one has*

$$\|\mathbf{x}\|_2 - \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}} \leq \frac{2}{3}\sqrt{\frac{n}{3}} \left(\max_{1\leq i\leq n} |x_i| - \min_{1\leq i\leq n} |x_i| \right). \tag{23}$$

Proof. *It is easy to see that*

$$g(k) = \sqrt{nx_n^2 + k\left(x_1^2 - x_n^2\right)} - \frac{\left(k\left(x_1^{2/3} - x_n^{2/3}\right) + nx_n^{2/3}\right)^{3/2}}{n}$$

achieves its maximum at

$$\begin{aligned} k_0 \quad = \quad & \frac{n\sqrt{4x_1^4+12x_1^{10/3}x_n^{2/3}+33x_1^{8/3}x_n^{4/3}+46x_1^2x_n^2+33x_1^{4/3}x_n^{8/3}+12x_1^{2/3}x_n^{10/3}+4x_n^4}}{6\left(x_1^2-x_n^2\right)} \\ & -\frac{3n\left(x_1^{4/3}x_n^{2/3}+x_1^{2/3}x_n^{4/3}+2x_n^2\right)}{6\left(x_1^2-x_n^2\right)} \end{aligned}$$

by the standard calculation. Let

$$p(s,t) := 4s^6 + 12s^5t + 33s^4t^2 + 46s^3t^3 + 33s^2t^4 + 12st^5 + 4t^6.$$

Setting $s := x_1^{2/3}$ *and* $t := x_n^{2/3}$*, we see that the maximum of* $g(k)$ *is*

$$g(k_0) = \frac{\sqrt{n}}{6\sqrt{6}} \left(6\sqrt{\sqrt{p(s,t)} - 3st(s+t)} - \left(\frac{\sqrt{p(s,t)} + 3st(s+t)}{s^2+st+t^2} \right)^{3/2} \right).$$

Let

$$F(s,t) := 6\sqrt{\sqrt{p(s,t)} - 3st(s+t)} - \left(\frac{\sqrt{p(s,t)} + 3st(s+t)}{s^2+st+t^2} \right)^{3/2}$$

so that $g(k_0) = \frac{\sqrt{n}}{6\sqrt{6}}F(s,t)$*. To find an upper bound of* $F(s,t)$*, we may consider* $F(1,y)$ *with* $y = t/s$ *for a fixed* s*. It is easy to plot* $F(1,y)$ *and* $4\sqrt{2}\,(1-y)^{3/2}$ *in Fig. 1 and their difference. Hence, the inequality (22) follows. Furthermore, by the quasi-triangle inequality for* $q = 2/3$*,*

$$\left(\max_{1\leq i\leq n} |x_i|^q - \min_{1\leq i\leq n} |x_i|^q \right)^{1/q} \leq \max_{1\leq i\leq n} |x_i| - \min_{1\leq i\leq n} |x_i|$$

one obtains the inequality in (23). ■

The analysis above just shows that a better estimate for Lemma 2 is hard to find.

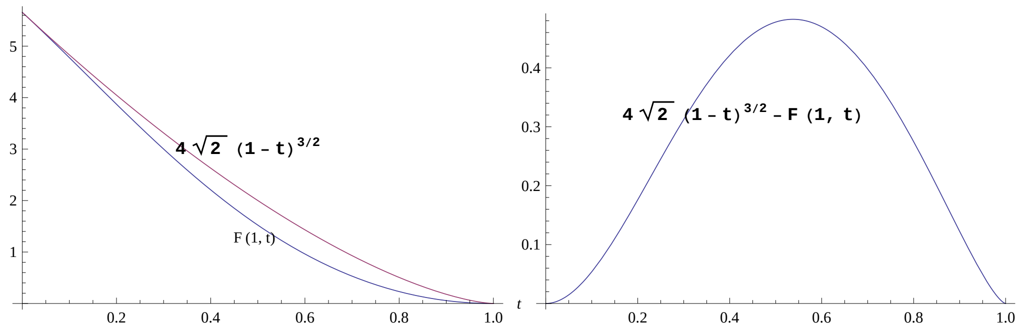

Figure 1: The graphs of $F(1,t)$ and $4\sqrt{2}\,(1-t)^{3/2}$ (left) and the graph of their difference (right)

References

[1] R. G. Baraniuk, V. Cevher, M. F. Duarte, C. Hegde, Model-based compressive sensing, IEEE Trans. Inform. Theory 56 (2010). 1982 – 2001.

[2] T. Cai, L. Wang, G. Xu, Shifting inequality and recovery of sparse signals, IEEE Trans. Signal Process., 58(2010), pp. 1300-1308.

[3] T. Cai, L. Wang, and G. Xu, New Bounds for Restricted Isometry Constants, Information Theory, IEEE Transactions, 2010, pp.4388–4394.

[4] E. Candès, The restricted isometry property and its implications for compressed sensing, C. R. Acad. Sci. Ser. I 346 (2008) 589-592.

[5] E. Candès and T. Tao, Decoding by linear programing, IEEE Trans. Inform. Theory 51 (2005) 42034215.

[6] M. Davies and R. Gribonval, Restricted isometry constants where ℓ_q sparse recovery can fail for $0 < p < 1$, IEEE Trans. Inform. Theory, in press.

[7] R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, IEEE Signal Process. Lett. 14 (2007) 707–710.

[8] Y. C. Eldar, M. Mishali, Robust recovery of signals from a structured union of subspaces, IEEE Trans. Inform. Theory 55 (2009) 5302–5316.

[9] Y. C. Eldar, P. Kuppinger, H. Bolcskei, Block-sparse signals uncertainty relations and efficient recovery, IEEE Trans. Signal Process. 58 (2010) 3042-3054.

[10] S. Foucart, A note on guaranteed sparse recovery via ℓ_1-minimization, Appl. Comput. Harmon. Anal. 29 (2010), 97–103.

[11] S. Foucart, M.J. Lai, Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < q < 1$, Appl. Comput. Harmon. Anal. 26 (2009) 395-407.

[12] R. Gribonval and M. Nielsen, Sparse decompositions in unions of bases. IEEE Trans. Info. Theory, vol. 49, no. 12, pp. 3320–3325, Dec 2003.

[13] M. J. Lai and Louis Y. Liu, The null space property for sparse recovery from multiple measurement vectors, to appear in Applied Computational Harmonic Analysis, 2010.

[14] M. J. Lai and J. Wang, An unconstrained lq minimization for sparse solution of under determined linear systems, to appear in SIAM J. Optimization, 2010.

[15] S. Li and Q. Mo, New bounds on the restricted isometry constant δ_{2k}, submitted, 2010.

[16] M. Mishali, Y. C. Eldar, Blind multi-band signal reconstruction: Compressed sensing for analog signals, IEEE Trans. Signal Process. 57 (2009) 993–1009.

[17] F. Parvaresh, H. Vikalo, S. Misra, B. Hassibi, Recovering sparse signals using sparse measurement matrices in compressed DNA microarrays, IEEE J. Sel. Top. Signal Process. 2 (2008) 275–285.