

# Convergence of a Central Difference Discretization of ROF Model for Image Denoising

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## Abstract

We study a central difference discretization of Rudin-Osher-Fetami model for image de-noising. We show that the discrete solution  $u_k$  converges to the continuous solution  $u$  in  $L_2$  norm and a rate of convergence is given.

## 1 Introduction

One of the most influential variational models for image denoising is the total variation-based model proposed by Rudin, Osher and Fatemin in [16]. This model studies the following constrained minimization problem:

$$\begin{aligned} \arg \min_{\mathbf{u}} |u|_{\text{BV}} & \quad (1) \\ \text{with } \int_{\Omega} u &= \int_{\Omega} g \quad \text{and} \quad \int_{\Omega} |u - g|^2 = \sigma^2, \end{aligned}$$

where  $g$  is the input data,  $\sigma$  is the standard deviation of the noise,  $\Omega$  is the unit square  $[0, 1]^2$ , and  $|u|_{\text{BV}}$  is the total variation (TV) of  $u$  defined in (2) below. Letting  $\phi \in [C_0^1(\Omega)]^2$  be a compactly supported continuous

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function vector mapping  $\Omega$  to  $\mathbb{R}^2$ , the variation of a function  $u \in L^1(\Omega)$  is then defined by

$$|u|_{\text{BV}} := \int_{\Omega} |Du| := \sup_{\phi \in [C_0^1(\Omega)]^2, |\phi| \leq 1 \text{ point-wise}} \int_{\Omega} u \nabla \cdot \phi. \quad (2)$$

For more details on functions of bounded variation, we refer the reader to [11].

The existence and uniqueness of the minimizer of (1) have been studied. See, e.g., [1]. Chambolle and Lions [4] proved that the constrained problem (1) is equivalent to the following unconstrained problem:

$$\arg \min_{\mathbf{u}} |u|_{\text{BV}} + \frac{1}{2\lambda} \int_{\Omega} |u - g|^2. \quad (3)$$

They also proved more general results of existence and uniqueness of (1). For convenience, we denote by

$$E(u) = |u|_{\text{BV}} + \frac{1}{2\lambda} \int_{\Omega} |u - g|^2 \quad (4)$$

the ROF energy functional. It is a common practice to minimize the functional  $E(u)$  or minimize  $E(u)$  together with other constrained terms for image denoising, deblurring, dejittering, inpainting, segmentation, colorization, and etc. (cf., e.g., [4], [17], [15], [13], [12]).

When implementing these minimizations, the most commonly used discretization of the ROF model is based on the discrete energy functional

$$E_h(u) = \sum_{i,j=0}^{k-1} \mu_{i,j} |(\nabla u)_{i,j}| + \frac{1}{2\lambda} \sum_{i,j=0}^{k-1} \mu_{i,j} (u_{i,j} - g_{i,j})^2, \quad (5)$$

where  $u$  is defined by a 2-dimensional matrix of size  $k \times k$ ,  $\mu_{i,j}$  is related to the scale  $k$ . A simple choice of  $\mu_{i,j}$  is  $\mu_{i,j} = 1/k^2$  and

$$(\nabla u)_{i,j} = ((\nabla_x u)_{i,j}, (\nabla_y u)_{i,j}),$$

with

$$(\nabla_x u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}, \quad (\nabla_y u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h},$$

where  $h = 1/k$ . On the boundary,  $u$  is assumed to satisfy the discrete Neumann boundary conditions:

$$u_{-1,j} = u_{0,j}, \quad u_{k,j} = u_{k-1,j}, \quad u_{i,-1} = u_{i,0}, \quad u_{i,k} = u_{i,k-1}. \quad (6)$$

The discrete function  $g_{i,j}$  in (5) is the given input noised image. Many efficient algorithms have been developed to find the numerical minimizer of (5). See the algorithms in [6], [2], [3] and [9].

There are other choices for the discrete gradient operator  $\nabla u$  (cf. [3], [5], [19], and [18]). For example, in [19], the following approximation of the first term, that is, the total variation (TV) term in  $E(u)$  is considered.

$$\begin{aligned} |u^h|_{\text{BV}} = \sum_{i,j=0}^{k-1} \frac{h^2}{4} \left\{ \left( \left( \frac{u_{i+1,j}^h - u_{i,j}^h}{h} \right)^2 + \left( \frac{u_{i,j+1}^h - u_{i,j}^h}{h} \right)^2 \right)^{1/2} + \right. \\ \left( \left( \frac{u_{i+1,j}^h - u_{i,j}^h}{h} \right)^2 + \left( \frac{u_{i,j}^h - u_{i,j-1}^h}{h} \right)^2 \right)^{1/2} + \\ \left( \left( \frac{u_{i,j}^h - u_{i-1,j}^h}{h} \right)^2 + \left( \frac{u_{i,j+1}^h - u_{i,j}^h}{h} \right)^2 \right)^{1/2} + \\ \left. \left( \left( \frac{u_{i,j}^h - u_{i-1,j}^h}{h} \right)^2 + \left( \frac{u_{i,j}^h - u_{i,j-1}^h}{h} \right)^2 \right)^{1/2} \right\} \quad (7) \end{aligned}$$

For another example, the authors in [5] studied the approximation of the first term in  $E(u)$  using the “upwind” variation scheme.

These approximations lead to different forms of  $E_h$ . It is not hard to show that the solution of (5)  $\Gamma$ -converges to  $E$  (for the definition of  $\Gamma$ -convergence, we refer the reader to [7]). It therefore follows that the sequence  $\{u^h\}$  of minimizers of  $E_h$  converges to  $u$  which is the minimizer of  $E$  in  $L^1(\Omega)$  and  $E_h(u^h)$  converges to  $E(u)$  as  $h$  tends to zero (cf. [7]).

It is interesting to know the rate of convergence and the convergence in other norm, e.g., in  $L^2$  norm. The researchers in [19] and in [18] proved that if the discrete energy  $E_h$  is equipped with a discrete total variation as defined in (7) or in the “upwind” scheme discussed in [5], and if the discrete input data  $g^h$  is the projection of the continuous input data  $g$  by taking average of  $g$  on each pixel, then one can bound the error between the discrete minimizer  $u^h$  and the continuous minimizer  $u$  in  $L^2$  norm by the Lipschitz norm of  $g$  provided that  $g$  is in some Lipschitz space  $\text{Lip}(\alpha, L^2)$ , i.e.,  $\alpha$  derivatives of  $g$  in  $L^2(\Omega)$  for  $0 < \alpha \leq 1$ .

In this paper we continue the studies in [18] and [19] and study another approximation of the TV term in  $E(u)$  by using central differences. That is, we shall consider the central-difference discrete ROF model whose energy

functional is defined as follows

$$E_h(u^h) = J_c(u^h) + \frac{1}{2\lambda} \sum_{i,j=0}^k \mu_{i,j} (u_{i,j}^h - g_{i,j}^h)^2, \quad (8)$$

where the TV term  $J_c$  is defined by

$$J_c(u^h) := \sum_{i,j=0}^k \left( \left| \frac{u_{i+1,j}^h - u_{i-1,j}^h}{2h} \right|^2 + \left| \frac{u_{i,j+1}^h - u_{i,j-1}^h}{2h} \right|^2 \right)^{1/2} \mu_{i,j}, \quad (9)$$

with  $u_{i,j}^h$  satisfying the discrete Neumann boundary conditions (6). Here the weights  $\mu_{i,j}$  are given by

$$\mu_{i,j} = \begin{cases} h^2/4 & (i,j) \in \{(0,0), (0,k), (k,0), (k,k)\} \\ h^2/2 & i=0, k; 0 < j < k \text{ or } j=0, k; 0 < i < k \\ h^2 & 0 < i, j < k. \end{cases} \quad (10)$$

The discrete input data  $g^h$  is defined by

$$g_{i,j}^h = \frac{1}{|\Omega_{i,j}|} \int_{\Omega_{i,j}} g$$

with  $\Omega_{i,j} = h(\Omega + (i - 1/2, j - 1/2)) \cap \Omega$ . We note that  $|\Omega_{i,j}| = \mu_{i,j}$ .

It is clear that this version (9) is simpler than (7) or the ‘‘upwind’’ scheme. It turns out that the computation is consequently simpler too. However, a problem of this central-difference model is that it does not deal well with a special non-smooth data: a chessboard image. Nevertheless, this pathological image never happen in real-life image processing. Even if we have this digital image for a fixed  $h$ , we will not have a chessboard image for all  $h > 0$ . Thus this central difference discrete ROF model makes sense for numerical calculation of the ROF model for image de-noising.

We use techniques similar to [19] to analyze this discrete ROD model and obtained a convergence rate lower than the rates in [19] under the same assumption on the input data  $g$ . Our results improve the result announced in a conference paper [14]. In fact, the results in [14] are a special case of the following theorems with  $\alpha = 1$ .

**Theorem 1** Suppose that  $g \in \text{Lip}(\alpha, L^2(\Omega))$ . Let  $u$  be the minimizer of  $E$  in (4) and  $u^h$  be the minimizer of  $E_h$  in (8) equipped with the central difference TV operator. Then

$$|E(u) - E_h(u^h)| \leq C \left(1 + \frac{1}{\lambda}\right) (\|g\|_{\text{Lip}(\alpha, L^2(\Omega))} + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2) h^{\alpha^2/(\alpha+1)}.$$

Next we need more notation to state our next result. Let  $\|\cdot\|$  stand for the standard  $L^2$  norm for the space  $L^2(\Omega)$  of all square integrable functions over  $\Omega$ . For the discrete minimizer  $u^h$ , let  $I_h u^h$  be the piecewise constant injection of  $u^h$  into  $L^2(\Omega)$  space. A precise definition of  $I_h u^h$  will be given in (13) in the next section. Then

**Theorem 2** Under the same assumptions in Theorem 1, we have

$$\|I_h u^h - u\|^2 \leq C(\lambda + 1)(\|g\|_{\text{Lip}(\alpha, L^2(\Omega))} + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2) h^{\alpha^2/(\alpha+1)}.$$

The convergence here is more precise than that of the  $\Gamma$  convergence of  $u^h$  to  $u$  as discussed above.

The paper is organized as follows. We first explain some notations and definitions together with some preliminary results which are similar to those in [19]. We need these preliminary results in the next section. In Section §3, we present some basic lemmas which are necessary for the proofs of our main results in Theorems 1 and 2. Then we prove Theorems 1 and 2 in §4. We end this paper with some remarks in §5.

## 2 Preliminaries

### 2.1 Notations and Definitions

A continuous image  $u$  is defined as a  $L^2$  function on  $\Omega = [0, 1] \times [0, 1]$ . We always assume that the denoised image is in the space  $\text{BV}(\Omega)$  of all functions of bounded variation.

In the discrete setting, we consider the discrete domain  $\Omega^h$  to be the set of all pairs  $(i, j) \in Z^2$  with  $0 \leq i, j \leq k$ . A discrete image  $u^h$  is defined as a function on  $\Omega^h$ . We define the discrete  $L^p(\Omega^h)$  norms

$$\|u^h\|_{L^p(\Omega^h)} := \left( \sum_{(i,j) \in \Omega^h} |u_{(i,j)}^h|^p \mu_{i,j} \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p \leq \infty,$$

where  $\mu_{i,j}$  is the weight at each index  $(i, j)$  given in the introduction section. In particular, the  $L^2$  term of  $h^h - g^h$  is defined by

$$\|u^h - g^h\|_{L^2(\Omega^h)}^2 = \sum_{i,j=0}^k |u_{i,j}^h - g_{i,j}^h|^2 \mu_{i,j}.$$

In analogue of Sobolev norm, we define the discrete Sobolev norm as follows. The first order forward finite differences of  $u^h$  at point  $i = (i_1, i_2)$  are

$$\Delta_x^+ u_{i,j}^h = \frac{u_{i+1,j}^h - u_{i,j}^h}{h}; \quad \Delta_y^+ u_{i,j}^h = \frac{u_{i,j+1}^h - u_{i,j}^h}{h},$$

where  $h = 1/k$  is the step size. We can also define backward finite difference as

$$\Delta_x^- u_{i,j}^h = \frac{u_{i,j}^h - u_{i-1,j}^h}{h}; \quad \Delta_y^- u_{i,j}^h = \frac{u_{i,j}^h - u_{i,j-1}^h}{h}.$$

Thus, the second order finite difference along  $x$  direction is

$$\Delta_{xx} u_{i,j}^h = \frac{\Delta_x^+ u_{i,j}^h - \Delta_x^- u_{i,j}^h}{h}.$$

Also  $\Delta_{yy} u_{i,j}^h$  can be similarly defined.

We define  $\|\nabla u^h\|_{L^1(\Omega^h)}$ ,  $\|\Delta_{xx} u^h\|_{L^1(\Omega^h)}$ ,  $\|\Delta_{yy} u^h\|_{L^1(\Omega^h)}$  as

$$\|\nabla u^h\|_{L^1(\Omega^h)} := \sum_{i,j=0}^k (|\Delta_x^+ u_{i,j}^h| + |\Delta_y^+ u_{i,j}^h|) \mu_{i,j}; \quad (11)$$

$$\|\Delta_{xx} u^h\|_{L^1(\Omega^h)} := \sum_{i,j=0}^k |\Delta_{xx} u_{i,j}^h| \mu_{i,j}, \quad \|\Delta_{yy} u^h\|_{L^1(\Omega^h)} := \sum_{i,j=0}^k |\Delta_{yy} u_{i,j}^h| \mu_{i,j}. \quad (12)$$

Finally we need notation of (first-order)  $L^p(\Omega)$  modulus of smoothness that is defined by

$$\omega(u, t)_{L^p(\Omega)} = \sup_{\tau \in \mathbb{R}^2, |\tau| < t} \left( \int_{\mathbf{x}, \mathbf{x}+\tau \in \Omega} |u(\mathbf{x} + \tau) - u(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

Similarly we need a discrete modulus of smoothness. The discrete  $L^p$  modulus of smoothness is

$$\omega(u^h, m)_{L^p(\Omega^h)} := \sup_{\ell \in \mathbb{Z}^2, |\ell| \leq m} \left( \sum_{(i,j), (i,j)+\ell \in \Omega^h} |u_{(i,j)+\ell}^h - u_{i,j}^h|^p \mu_{i,j} \right)^{\frac{1}{p}}.$$

with  $|\ell| = |\ell_1| + |\ell_2|$  for  $\ell \in \mathbb{Z}^2$ .

## 2.2 Extension of functions

To discuss the modulus of smoothness of function  $v$  and discrete function  $v^h$ , we need to extend  $v \in L^p(\Omega)$  and  $v^h$  to all of  $\mathbb{R}^2$  and  $\mathbb{Z}^2$ , respectively. Since we only consider the variation inside the  $\Omega$  and  $\Omega^h$ . The extension shall not introduce extra variation on the boundary.

For  $v \in L^p(\Omega)$ , the extension is done by reflection along the boundary. First,

$$\text{Ext } v(x, y) = v(x, y), \quad (x, y) \in \Omega.$$

We then reflect horizontally across the line  $x = 1$ ,

$$\text{Ext } v(x, y) = \text{Ext } v(2 - x, y), \quad 1 \leq x \leq 2, \quad 0 \leq y \leq 1,$$

and reflect again vertically across the line  $y = 1$ ,

$$\text{Ext } v(x, y) = \text{Ext } v(x, 2 - y), \quad 0 \leq x \leq 2, \quad 1 \leq y \leq 2.$$

Having defined  $\text{Ext } v$  on  $2\Omega$ , we then extend  $\text{Ext } v$  periodically with period  $(2, 2)$  on all of  $\mathbb{R}^2$ .

The discrete extension is different from [19]. We use an odd extension here. First we extend  $v^h$  to

$$2\Omega^h := \{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq 2k\}$$

as follows:

$$\text{Ext}_h v_{i,j}^h = v_{i,j}^h, \quad (i, j) \in \Omega^h;$$

then we reflect horizontally

$$\text{Ext}_h v_{(i,j)}^h = \text{Ext}_h v_{(2k-i,j)}^h, \quad k+1 \leq i \leq 2k, \quad 0 \leq j \leq k,$$

and then vertically

$$\text{Ext}_h v_{(i,j)}^h = \text{Ext}_h v_{(i,2k-j)}^h, \quad 0 \leq i \leq 2k, \quad k+1 \leq j \leq 2k.$$

Now that  $\text{Ext}_h v^h$  is defined on  $2\Omega^h$ , we extend it periodically with period  $(2k, 2k)$  to all of  $\mathbb{Z}^2$ .

## 2.3 Injectors, Projectors and Smoothing Operators

We need to inject functions in the discrete space  $L^2(\Omega^h)$  into  $L^2(\Omega)$ . The first one is the piecewise constant injector to inject discrete function  $u^h$  into  $L^p(\Omega)$ : Let  $S_{i,j} = h(\Omega + (i - 1/2, j - 1/2))$ ,  $S_{i,j}$  be a square with center

$(ih, jh)$  and size  $h$ . We define  $\Omega_{i,j} = S_{i,j} \cap \Omega$  for  $0 \leq i, j \leq k$ . Note that  $\Omega_{i,j}$  is a half square or quarter square for  $(i, j)$  which are on the boundary or on four corners of  $\Omega^h$  respectively. Then

$$(I_h u^h)(x) = u_{i,j}^h \quad \text{for } x \in \Omega_{i,j}, \quad (13)$$

Next let  $L_h$  be a linear injector mapping functions in the discrete space  $L^2(\Omega^h)$  into a space of continuous, piecewise linear functions.

$$L_h u^h = \sum_{(i,j) \in \Omega^h} u_{i,j}^h \phi_{i,j}^h, \quad (14)$$

where

$$\phi_{i,j}^h(x, y) := \phi_{i,j}^h(x, y) := \phi(x/h - i, y/h - j), \quad (15)$$

is a translation of dilated tent function  $\phi$  on  $\mathbb{R}^2$  which is supported on the hexagon with boundary vertices  $\{(1, 0), (-1, 1), (0, 1), (-1, 0), (1, -1), (0, -1)\}$  and is zero for all integers in  $\mathbb{R}^2$  except for  $(0, 0)$  where  $\phi(0, 0) = 1$ .

We also consider the piecewise constant projector of  $u \in L^1(\Omega)$  onto the space of discrete functions, defined by

$$(P_h u)_{i,j} = \frac{1}{|\Omega_{i,j}|} \int_{\Omega_{i,j}} u(x, y) dx dy, \quad (i, j) \in \Omega^h,$$

where  $|\Omega_{i,j}| = \mu_{i,j}$  is the measure of  $\Omega_{i,j}$ .

We will need continuous and discrete smoothing operators. Assume that  $\eta(\mathbf{x})$  is a fixed non-negative, rotationally symmetric, mollifier with support on the unit disk which is  $C^\infty$  and has integral 1 with  $\mathbf{x} = (x, y)$ . For  $\epsilon > 0$  we define the scaled function

$$\eta_\epsilon(\mathbf{x}) := \frac{1}{\epsilon^2} \eta\left(\frac{\mathbf{x}}{\epsilon}\right), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2;$$

Let  $\mathcal{S}_\epsilon$  be a smoothing operator defined by

$$\mathcal{S}_\epsilon u = (\eta_\epsilon * \text{Ext } u) = \int_{\mathbb{R}^2} \eta_\epsilon(\mathbf{x} - \mathbf{y}) \text{Ext } u(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in 2\Omega.$$

The discrete smoothing operator  $S_L$  is defined by

$$(S_L u^h)_{i,j} = \frac{1}{(2L+1)^2} \sum_{i_1, j_1 = -L}^L \text{Ext}_h u_{(i,j)+(i_1, j_1)}^h \quad \text{for } (i, j) \in \Omega^h$$

### 3 Basic Lemmas

We now give basic lemmas that will be used to prove the main results in the next section.

We first give properties of injectors, projectors and smoothing operators. Their proofs can be found (with minor modifications) in [19].

The following lemmas bound the errors introduced by injectors and projectors which can be verified easily.

**Lemma 3.1** *Let  $v \in L^2(\Omega)$  and  $v^h$  being a discrete function. Then there exists a constant  $C$  such that the following properties hold:*

$$\|P_h v\|_{L^2(\Omega^h)} \leq \|v\|, \quad (16)$$

$$\omega(P_h v, m)_{L^2(\Omega^h)} \leq C\omega(v, mh), \quad (17)$$

$$\|v^h\|_{L^2(\Omega^h)} = \|I_h v^h\|, \quad (18)$$

$$\omega(I_h v^h, mh) \leq C\omega(v^h, m)_{L^2(\Omega^h)}, \quad (19)$$

$$\|v - I_h P_h v\| \leq C\omega(v, h), \quad (20)$$

and

$$\|L_h v^h - I_h v^h\| \leq C\omega(v^h, 1)_{L^2(\Omega^h)}. \quad (21)$$

We next give some properties of the smoothing operators which will be used in the proofs for our main results. They can be verified straightforwardly.

**Lemma 3.2** *For any integer  $M > 0$ ,*

$$\omega(S_L v^h, M)_{L^2(\Omega^h)} \leq C\omega(v^h, M)_{L^2(\Omega^h)}, \quad (22)$$

$$\|S_L v^h - v^h\|_{L^2(\Omega^h)} \leq C\omega(v^h, L)_{L^2(\Omega^h)}, \quad (23)$$

$$J_c(S_L v^h) \leq J_c(v^h), \quad (24)$$

and

$$\|\Delta_{xx}S_L v^h\|_{L^1(\Omega^h)} + \|\Delta_{yy}S_L v^h\|_{L^1(\Omega^h)} \leq \frac{C}{Lh} \|\nabla v^h\|_{L^1(\Omega^h)}. \quad (25)$$

Furthermore, for all  $t > 0$ ,

$$\omega(\mathcal{S}_\epsilon v, t)_{L^2(\Omega)} \leq C\omega(v, t)_{L^2(\Omega)}, \quad (26)$$

$$\|\mathcal{S}_\epsilon v - v\| \leq C\omega(v, \epsilon)_{L^2(\Omega)}, \quad (27)$$

$$|\mathcal{S}_\epsilon v|_{\text{BV}} \leq |v|_{\text{BV}}, \quad (28)$$

and

$$\|D_x^2 \mathcal{S}_\epsilon v\|_{L^1(\Omega)} + \|D_y^2 \mathcal{S}_\epsilon v\|_{L^1(\Omega)} \leq \frac{C}{\epsilon} |v|_{\text{BV}}. \quad (29)$$

The first inequality in Lemma 3.2 shows that the smoothing operator does not decrease the smoothness. Inequality (23) shows the error between  $u^h$  and smoothed  $u^h$  can be bounded by its discrete modulus of continuity. Inequality (24) shows smoothing does not increase the discrete total variation. Inequality (25) shows the second order difference of the smoothed function can be bounded by its first order finite difference.

Inequalities (26), (27), (28), and (29) are continuous analogues of (22), (23), (24), and (25), respectively.

The following lemma shows the consistency of the central-difference discrete variation. They can be proved by a standard inequality:

$$\sqrt{x^2 + y^2} \leq \sqrt{(x - u)^2 + (y - v)^2} + \sqrt{u^2 + v^2}$$

and the fact that the central difference approximates the first order derivatives with error terms  $O(h)$  when a function is not very smooth.

**Lemma 3.3** (TV Consistency) *There exists a  $C > 0$  such for any discrete function  $v^h$  defined on  $\Omega^h$*

$$\left| L_h v^h \right|_{\text{BV}} \leq J_c(Lv^h) + Ch \left( \|\Delta_{xx} v^h\|_{L^1(\Omega^h)} + \|\Delta_{yy} v^h\|_{L^1(\Omega^h)} \right) \quad (30)$$

and for any  $v$  in  $L^1(\Omega)$ ,

$$J_c(P_h v) \leq |v|_{\text{BV}} + Ch(\|D_{xx} v\|_{L^1(\Omega)} + \|D_{yy} v\|_{L^1(\Omega)}). \quad (31)$$

We remark that for  $v \notin W^{2,1}(\Omega)$ , the right hand side of (31) is infinite, thus the inequality still holds.

We also give properties of the minimizers of the continuous ROF model (3) and the discrete ROF model (8). The proofs for these lemmas can be easily adapted from the results in [19].

In the following lemma, we need to consider discrete variational functionals for discrete functions defined on  $2\Omega^h$ . For these purposes we define

$$J_c^{2\Omega^h} = \sum_{(i,j) \in 2\Omega^h} \left( \left| \frac{u_{i+1,j}^h - u_{i-1,j}^h}{2h} \right|^2 + \left| \frac{u_{i,j+1}^h - u_{i,j-1}^h}{2h} \right|^2 \right)^{1/2} h^2.$$

We also define the  $L^2$  norm on  $L^2(2\Omega^h)$  by

$$\|v^h\|_{L^2(2\Omega^h)} = \left( \sum_{(i,j) \in 2\Omega^h} |v_{i,j}^h|^2 h^2 \right)^{1/2}.$$

We note that for discrete variations and  $L^2$  norms defined on  $2\Omega^h$ , the weights at each point  $(i, j)$  is always  $h^2$  which is slightly different from  $J_c$  defined in (9).

**Lemma 3.4** (Extension of Minimizers) *If  $u^h$  is the minimizer of the functional*

$$E_h(v^h) = \frac{1}{2\lambda} \|v^h - g^h\|_{L^2(\Omega^h)}^2 + J_c(v^h), \quad (32)$$

*then  $\text{Ext}_h u^h$  is the minimizer over all discrete functions  $v^h$  defined on  $2\Omega^h$  of the functional*

$$E_h^{2\Omega^h}(v^h) = \frac{1}{2\lambda} \|v^h - \text{Ext}_h g^h\|_{L^2(2\Omega^h)}^2 + J_c^{2\Omega^h}(v^h) \quad (33)$$

*with periodic boundary conditions.*

*Similarly, if  $u$  is the minimizer of*

$$E(v) = \frac{1}{2\lambda} \|v - g\|_{L^2(\Omega)}^2 + |v|_{\text{BV}(\Omega)} \quad (34)$$

*then  $\text{Ext } u$  is the minimizer of*

$$E^{2\Omega}(v) = \frac{1}{2\lambda} \|v - \text{Ext } g\|_{L^2(2\Omega)}^2 + |v|_{\text{BV}(2\Omega)}, \quad (35)$$

*again with periodic boundary conditions.*

Recall that if  $u$  and  $w$  are minimizers of (34) with data  $g$  and  $h$ , respectively, then

$$\|u - w\|_{L^2(\Omega)} \leq \|g - h\|_{L^2(\Omega)}.$$

(See [18] for a proof.) Similarly, for the two periodic problems (33) and (35) we have

$$\|\text{Ext}_h u^h - T_\ell \text{Ext}_h u^h\|_{L^2(2\Omega^h)} \leq \|\text{Ext}_h g^h - T_\ell \text{Ext}_h g^h\|_{L^2(2\Omega^h)} \quad (36)$$

and

$$\|\text{Ext } u - \mathcal{T}_\tau \text{Ext } u\|_{L^2(2\Omega)} \leq \|\text{Ext } g - \mathcal{T}_\tau \text{Ext } g\|_{L^2(2\Omega)}. \quad (37)$$

where  $T$  and  $\mathcal{T}$  are translation operators

$$(T_\ell(u^h))_{i,j} := u^h_{(i,j)+\ell} \quad \text{for any } \ell = (\ell_1, \ell_2) \in \mathbb{Z}^2 \quad (38)$$

and

$$(\mathcal{T}_\tau(u)) := u((x, y) + \tau) \quad \text{for any } \tau = (\tau_1, \tau_2) \in \mathbb{R}^2. \quad (39)$$

With (36) and (37), we can immediately obtain the following property of the smoothness of the minimizers.

**Lemma 3.5** (Smoothness bounds) *Suppose that  $u$  is the minimizer of  $E$  in problem (3) for input data  $g$ . Let  $\text{Ext } u$  be the extended  $u$  over  $\mathbb{R}^2$  as defined in Section 2.2. Then, for  $\epsilon > 0$*

$$\|\text{Ext } u(x + h) - u(x)\|_{L^2(\Omega)} \leq C\omega(g, \epsilon)_{L^2(\Omega)}, \quad |h| \leq \epsilon \quad (40)$$

and

$$\omega(u, \epsilon)_{L^2(\Omega)} \leq C\omega(g, \epsilon)_{L^2(\Omega)}. \quad (41)$$

Moreover, suppose that  $u^h$  is the minimizer of  $E_h$  in (8). Let  $\text{Ext}_h u^h$  be the extended  $u^h$  over  $\mathbb{Z}^2$ . Then, for any integer  $L > 0$ ,

$$\|T_\ell(u^h) - u^h\|_{L^2(\Omega^h)} \leq C\omega(g^h, L)_{L^2(\Omega^h)}, \quad |\ell| \leq L \quad (42)$$

and

$$\omega(u^h, L)_{L^2(\Omega^h)} \leq C\omega(g^h, L)_{L^2(\Omega^h)} \leq C\omega(g, Lh)_{L^2(\Omega)}. \quad (43)$$

In addition, the following properties about the minimizers  $u$  and  $u^h$  can be proved easily.

$$|u|_{\text{BV}} \leq \frac{1}{2\lambda} \|g\|^2, \quad (44)$$

$$\|u - g\| \leq \|g\|, \quad (45)$$

$$J_c(u^h) \leq \frac{1}{2\lambda} \|g^h\|_{L^2(\Omega^h)}^2 \leq \frac{1}{2\lambda} \|g\|^2, \quad (46)$$

and

$$\|u^h - g^h\|_{L^2(\Omega^h)} \leq \|g^h\|_{L^2(\Omega^h)} \leq \|g\|. \quad (47)$$

Finally we need the following

**Lemma 3.6** *If  $u$  is the minimizer of  $E$  in (49), then for any  $v \in \text{BV}$ ,*

$$\|v - u\|^2 \leq 2\lambda(E(v) - E(u)). \quad (48)$$

**Proof.** The result is classical. We give a proof for completeness. By the definition of  $E$ ,

$$E(v) - E(u) = |v|_{\text{BV}} - |u|_{\text{BV}} + \frac{1}{2\lambda} (\|v - g\|^2 - \|u - g\|^2).$$

Since  $u$  is the minimizer,  $(g - u)/\lambda$  is in the sub-differential  $\partial|u|_{\text{BV}}$ , i.e., for any  $v$ ,

$$\left\langle \frac{g - u}{\lambda}, v - u \right\rangle \leq |v|_{\text{BV}} - |u|_{\text{BV}}.$$

Then

$$\begin{aligned} E(v) - E(u) &\geq \left\langle \frac{g - u}{\lambda}, v - u \right\rangle + \frac{1}{2\lambda} (\|v - g\|^2 - \|u - g\|^2) \\ &= \left\langle \frac{g - u}{\lambda}, v - u \right\rangle + \frac{1}{2\lambda} (\|v - u\|^2 + 2\langle v - u, u - g \rangle) \\ &= \frac{1}{2\lambda} \|v - u\|^2. \end{aligned}$$

■

## 4 Proof of the Main Results

Recall the ROF continuous and discrete energy functionals are defined by

$$E(v) = |v|_{\text{BV}} + \frac{1}{2\lambda} \|v - g\|^2; \quad (49)$$

$$E_h(v^h) = J_c(v^h) + \frac{1}{2\lambda} \|v^h - g^h\|_{L^2(\Omega^h)}^2 \quad (50)$$

with input image  $g^h = P_h g$ . We first give the bound on the difference of the continuous energy and the discrete energy.

### 4.1 Bound On the Difference of the Energy Functionals

We begin with the following property of modulus of smoothness. The proof is trivial and hence is omitted here.

**Lemma 4.1** *If  $v$  is in  $L^2(\Omega)$ , for  $h > 0$ ,*

$$\omega(v, h)_{L^1(\Omega)} \leq C\omega(v, h)_{L^2(\Omega)}. \quad (51)$$

*If  $v^h$  is a discrete function on  $\Omega^h$ , for interger  $m > 0$ ,*

$$\omega(v^h, m)_{L^1(\Omega^h)} \leq C\omega(v^h, m)_{L^2(\Omega^h)}, \quad (52)$$

*where in both inequalities,  $C$  depends on  $\Omega$ .*

**Lemma 4.2** *Suppose that  $g \in \text{Lip}(\alpha, L^2(\Omega))$ . If  $u^h, u$  are the minimizers of  $E_h$  and  $E$  in (50) and(49), respectively. If  $L = \lfloor h^{-1/(\alpha+1)} \rfloor$ , then*

$$\begin{aligned} E(u) &\leq E(L_h S_L u^h) \\ &\leq E_h(u^h) + C(1 + \frac{1}{\lambda})(\|g\|_{\text{Lip}(\alpha, L^2(\Omega))} + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2)h^{\alpha^2/(\alpha+1)}. \end{aligned}$$

**Proof.** We first have  $E(u) \leq E(L_h S_L u^h)$  by the definition of  $u$ . Now we shall bound the energy  $E(L_h S_L u^h)$ . By (30) and the properties of discrete smoothing operator (24) and (25) in Lemma 3.2,

$$\begin{aligned} \left| L_h S_L u^h \right|_{\text{BV}} &\leq J_c(S_L u^h) + Ch \left( \|\Delta_{xx} S_L u^h\|_{L^1(\Omega^h)} + \|\Delta_{yy} S_L u^h\|_{L^1(\Omega^h)} \right) \\ &\leq J_c(u^h) + \frac{C}{L} \|\nabla u^h\|_{L^1(\Omega^h)}. \end{aligned}$$

Note

$$\begin{aligned}
\|\nabla u^h\|_{L^1(\Omega^h)} &\leq \frac{C}{h}\omega(u^h, 1)_{L^1(\Omega^h)} \leq \frac{C}{h}\omega(u^h, 1)_{L^2(\Omega^h)} && \text{by (52)} \\
&\leq \frac{C}{h}\omega(g^h, 1)_{L^2(\Omega^h)} && \text{by (43)} \\
&\leq C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}h^{\alpha-1}.
\end{aligned}$$

Hence, we have

$$\left|L_h S_L u^h\right|_{\text{BV}} \leq J_c(u^h) + \frac{C}{L}\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}h^{\alpha-1}.$$

On the other hand, the  $L^2$  term of  $E(L_h S_L u^h)$  can be written as

$$\begin{aligned}
\|L_h S_L u^h - g\| &= \|(L_h S_L u^h - I_h S_L u^h) + (I_h S_L u^h - I_h u^h) \\
&\quad + (I_h u^h - I_h g^h) + (I_h g^h - g)\|.
\end{aligned}$$

We apply (21), (18) and (20) to obtain

$$\begin{aligned}
\|L_h S_L u^h - g\| &\leq C\omega(S_L u^h, 1)_{L^2(\Omega^h)} + \|S_L u^h - u^h\|_{L^2(\Omega^h)} + \\
&\quad \|u^h - g^h\|_{L^2(\Omega^h)} + C\omega(g, h).
\end{aligned}$$

Then by (22), (23) and (43), we have

$$\|L_h S_L u^h - g\| \leq \|u^h - g^h\|_{L^2(\Omega^h)} + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}(Lh)^\alpha + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}h^\alpha.$$

Note by (47),

$$\|u^h - g^h\|_{L^2(\Omega^h)} \leq \|g\|.$$

We have

$$\begin{aligned}
\|L_h S_L u^h - g\|^2 &\leq \|u^h - g^h\|_{L^2(\Omega^h)}^2 + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2((Lh)^{2\alpha} + h^{2\alpha}) \\
&\quad + (Lh)^\alpha h^\alpha + (Lh)^\alpha + h^\alpha,
\end{aligned}$$

where  $C$  is a positive constant. It follows that

$$\begin{aligned}
E(L_h S_L u^h) &= \left|L_h S_L u^h\right|_{\text{BV}} + \frac{1}{2\lambda}\|L_h S_L u^h - g\|^2 \\
&\leq J_c(u^h) + \frac{C}{L}\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}h^{\alpha-1} + \frac{1}{2\lambda}\|u^h - g^h\|_{L^2(\Omega^h)}^2 + \\
&\quad \frac{C}{\lambda}\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2((Lh)^{2\alpha} + h^{2\alpha} + (Lh)^\alpha h^\alpha + (Lh)^\alpha + h^\alpha)
\end{aligned}$$

Setting the largest error terms  $(Lh)^\alpha$  and  $h^{\alpha-1}/L$  equal, i.e., setting

$$L = \lfloor h^{-1/(\alpha+1)} \rfloor,$$

we obtain the result. ■

Using similar method we can prove the following

**Lemma 4.3** *Suppose that  $g \in W^{1,2}$ . Let  $u, u^h$  be the minimizers of  $E, E_h$  in (49) and (50), respectively. If  $\epsilon = h^{1/(\alpha+1)}$ , then*

$$E_h(u^h) \leq E_h(P_h(\mathcal{S}_\epsilon u)) \leq E(u) + \frac{C}{\lambda} \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 h^{\alpha/(\alpha+1)}.$$

**Proof.** First of all, we have  $E_h(u^h) \leq E_h(P_h \mathcal{S}_\epsilon u)$  by the definition of  $u^h$ . By (31)

$$J_c(P_h \mathcal{S}_\epsilon u) \leq |\mathcal{S}_\epsilon u|_{\text{BV}} + Ch(\|D_{xx} \mathcal{S}_\epsilon u\|_{L^1} + \|D_{yy} \mathcal{S}_\epsilon u\|_{L^1}).$$

By (31), (28), (29) and (44)

$$J_c(P_h \mathcal{S}_\epsilon u) \leq |u|_{\text{BV}} + \frac{Ch}{\epsilon} |u|_{\text{BV}} \leq |u|_{\text{BV}} + \frac{Ch}{\lambda \epsilon} \|g\|^2.$$

The  $L^2(\Omega^h)$  term of  $E_h(P_h \mathcal{S}_\epsilon u)$  can be estimated as follows,

$$\begin{aligned} \|P_h \mathcal{S}_\epsilon u - g^h\|_{L^2(\Omega^h)} &= \|I_h P_h(\mathcal{S}_\epsilon u) - I_h g^h\| \\ &\leq \|I_h P_h(\mathcal{S}_\epsilon u) - \mathcal{S}_\epsilon u\| + \|\mathcal{S}_\epsilon u - u\| \\ &\quad + \|u - g\| + \|g - I_h g^h\|. \end{aligned}$$

We apply (20), (27) and (43)

$$\begin{aligned} \|P_h \mathcal{S}_\epsilon u - g^h\|_{L^2(\Omega^h)} &\leq C\omega(\mathcal{S}_\epsilon u, h) + \omega(u, \epsilon) + \|u - g\|_{L^2} + C\omega(g, h) \\ &\leq \|u - g\|_{L^2} + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}(\epsilon^\alpha + h^\alpha). \end{aligned}$$

Note that  $\|u - g\|_{L^2} \leq \|g\|_{L^2}$  by (45), we have

$$\|P_h \mathcal{S}_\epsilon u - g^h\|_{L^2(\Omega^h)}^2 \leq \|u - g\|_{L^2}^2 + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2(\epsilon^{2\alpha} + h^{2\alpha} + \epsilon^\alpha h^\alpha + \epsilon^\alpha + h^\alpha).$$

We now summarize above to get

$$\begin{aligned} E_h(u^h) &\leq E_h(P_h \mathcal{S}_\epsilon u) \\ &\leq E(u) + \frac{Ch}{\lambda \epsilon} \|g\|_{L^2}^2 \\ &\quad + \frac{C}{\lambda} \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2(\epsilon^{2\alpha} + h^{2\alpha} + \epsilon^\alpha h^\alpha + \epsilon^\alpha + h^\alpha). \end{aligned}$$

Again, setting the largest error term  $\epsilon^\alpha$  equal  $h/\epsilon$ , we have

$$\epsilon = h^{1/(\alpha+1)}.$$

This completes the proof. ■

## 4.2 Proofs of Main Theorems

We are now ready to prove our main results in this paper.

**The Proof of Theorem 1.** Combining Lemma 4.2 and Lemma 4.3 immediately yields the following

$$|E(u) - E_k(u^h)| \leq C\left(1 + \frac{1}{\lambda}\right) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)}$$

which is the result of Theorem 1.

**The Proof of Theorem 2.** We shall prove the following

$$\|I_h u^h - u\|^2 \leq C(1 + \lambda) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)}.$$

To do so, we first use Lemma 3.6 to get

$$\|L_h S_L u^h - u\|^2 \leq 2\lambda(E(L_h S_L u^h) - E(u))$$

where  $L = \lfloor h^{-1/(\alpha+1)} \rfloor$ . By Lemma 4.2 and Lemma 4.3,

$$\begin{aligned} E(L_h S_L u^h) &\leq E_h(u^h) + C\left(1 + \frac{1}{\lambda}\right) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)} \\ E(u) &\geq E_h(u^h) - C\left(1 + \frac{1}{\lambda}\right) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)}. \end{aligned}$$

Thus,

$$\|L_h S_L u^h - u\|^2 \leq C(1 + \lambda) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)}. \quad (53)$$

Next we use inequality (53), Lemma 3.5, Lemma 3.2 to obtain

$$\begin{aligned} \|I_h u^h - u\|^2 &= \|I_h u^h - I_h S_L u^h + I_h S_L u^h - L_h S_L u^h + L_h S_L u^h - u\|^2 \\ &\leq 3(\|I_h u^h - I_h S_L u^h\|^2 + \|I_h S_L u^h - L_h S_L u^h\|^2 + \|L_h S_L u^h - u\|^2) \\ &\leq 3(\|u^h - S_L u^h\|_{L^2(\Omega^h)}^2 + (C\omega(g^h, 1)_{L^2(\Omega^h)})^2 + \|L_h S_L u^h - u\|^2) \\ &\leq C(\omega(u^h, L)_{L^2(\Omega^h)})^2 + C(\omega(g^h, 1)_{L^2(\Omega^h)})^2 + \|L_h S_L u^h - u\|^2 \\ &\leq C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 (Lh)^{2\alpha} + C\|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 h^{2\alpha} \\ &\quad + C(1 + \lambda) \left( \|g\|_{\text{Lip}(\alpha, L^2(\Omega))}^2 + \|g\|_{\text{Lip}(\alpha, L^2(\Omega))} \right) h^{\alpha^2/(\alpha+1)}. \end{aligned}$$

which can be rewritten in the form of the statement of Theorem 2. Here we have used  $C$  for a generic constant which may be different in the different lines above.

## 5 Conclusion and Remarks

In this paper, we proved the error bound for the discrete ROF model equipped with a central-difference TV term using the same ideas in [19]. This model is simpler in form than the models studied in [19] and [18]. This model is also slightly easier to be computed by Chambolle's method (cf. [3]).

On the other hand, the results in this paper show that the ideas in [19] are powerful which can be extended to deal with other symmetric discrete approximation of TV term. It is also interesting to study further whether the convergent result can be established for other discrete variations, for example

$$\left( \max\left\{ \frac{(u_{i+1,j} - u_{i,j})(u_{i,j} - u_{i-1,j})}{2h}, 0 \right\} + \max\left\{ \frac{(u_{i,j+1} - u_{i,j})(u_{i,j} - u_{i,j-1})}{2h}, 0 \right\} \right)^{1/2}$$

to replace the first order difference

$$\left( \frac{(u_{i+1,j} - u_{i,j})^2}{h^2} + \frac{(u_{i,j+1} - u_{i,j})^2}{h^2} \right)^{1/2}$$

the discrete TV term. We leave the study to the interested reader.

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