Bernstein-Bezier Splines on the Unit Sphere

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Abstract

I will introduce scattered data fitting problems on the sphere and discuss applications of spherical splines. I will define Bernstein-Bezier polynomials and describe methods based on spherical splines.

Error Bounds for Minimal Energy Interpolating Spherical Splines

Problem: Given locations $v_i, = 1, ..., n$ on the unit sphere and function values f_i associated with these locations we would like to find a spline function s in $S_d^r(\Delta)$ such that

- s interpolates f_i's
- s minimizes $\mathcal{E}(f)$

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} \sum_{|\alpha|=2} (D^\alpha f)^2$$

Recall:

$$S_d^r(\Delta) = \{s \in C^r(\mathbb{S}^2), s|_\tau \in \mathcal{H}_d, \tau \in \Delta\}$$

Define

$$U_f := \{s \in S_d^r(\Delta) : s(v) = f(v), v \in \mathcal{V}\}$$

- Is U_f non-empty? In which spaces?

- How well the minimizer approximates f?

Natural Radial Projection

In order to obtain bounds on convergence of the minimal energy splines, we need to constrain spherical triangulations. Let us introduce a concept of a quasi-uniform triangulation on $\mathbb{S}^2$ similar to the planar case.

Define a diameter of a spherical cap C as

$$\sup_{u,v\in C} \arccos(u \cdot v).$$

Given a spherical triangle τ, let $|\tau|$ denote the diameter of the smallest spherical cap containing τ, and let ρ_τ denote the diameter of the largest spherical cap contained in τ. Then

$$|\Delta| = \max\{|\tau|, \quad \tau \in \Delta\} \text{ and } \rho_\Delta = \min\{\rho_\tau, \quad \tau \in \Delta\}$$

are correspondingly the diameter of the largest triangle in Δ and the diameter of the smallest spherical cap inscribed in Δ.

Definition: Let β be a positive real number. A triangulation Δ is said to be β-quasi-uniform provided that

$$\frac{|\Delta|}{\rho_\Delta} \leq \beta.$$

It is well known that in the planar case, the smallest angle of a quasi-uniform triangulation is bounded below by $1/\beta$.

Fix a spherical triangle τ with $|\tau| \leq 1$. Define r_τ to be the center of a spherical cap of smallest possible radius containing τ, and let $\mathbf{T}_\tau$ be the tangent plane touching $\mathbb{S}^2$ at r_τ. We define the radial projection from $\mathbf{T}_\tau$ into $\mathbb{S}^2$ by

$$w := R_\tau \bar{w} := \frac{\bar{w}}{|\bar{w}|} \in \mathbb{S}^2, \qquad \bar{w} \in \mathbf{T}_\tau.$$

Since R_τ is one-to-one, R_τ^{-1} is well-defined. Let $\bar{\tau}$ be the image of τ under R_τ^{-1}.

The sizes of τ and $\bar{\tau}$ are comparable

$$|\tau| \leq |\bar{\tau}| \leq K_1|\tau|, \qquad K_2^{-1}\rho_\tau \leq \rho_{\bar{\tau}} \leq K_2\rho_\tau,$$

for some positive constants K_1 and K_2.

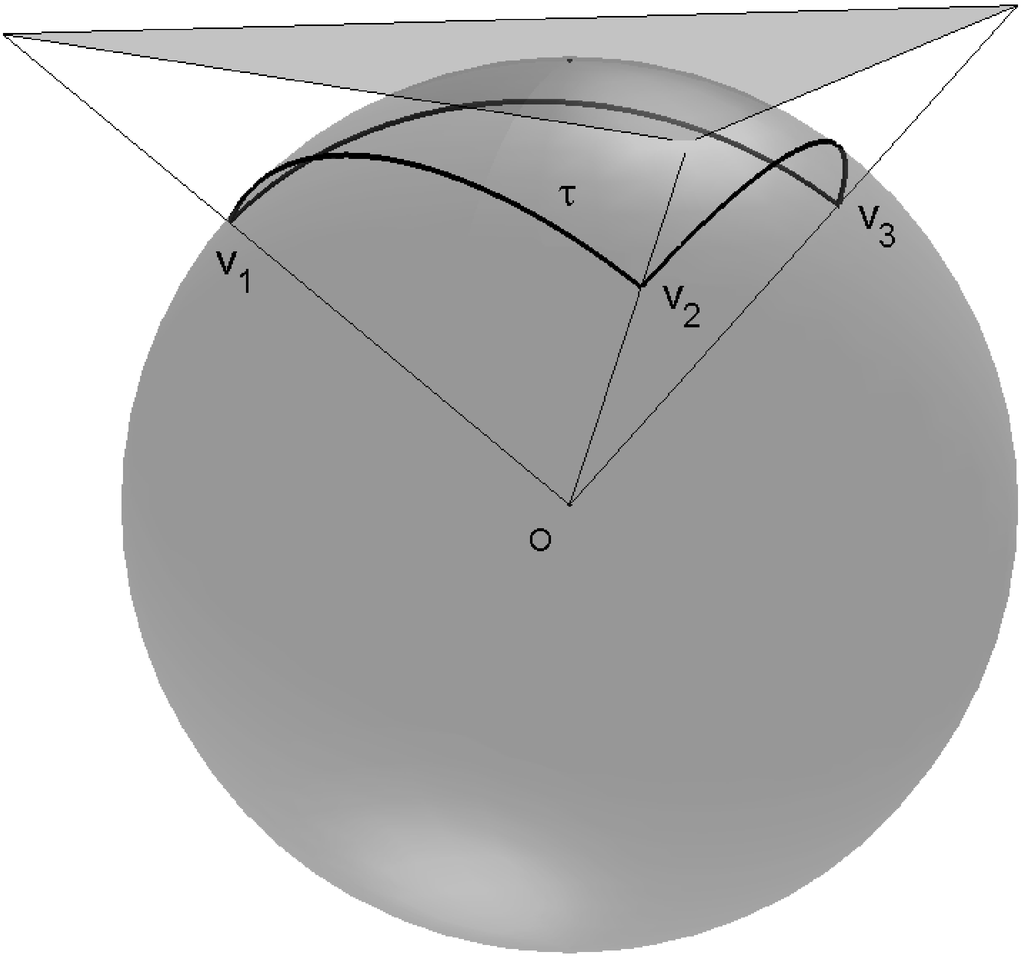


Figure 1: Spherical triangle and its planar projection.

Note that great circles are mapped into straight lines under the inverse of the radial projection R_τ, and any cluster of spherical triangles ω with $|\omega| \leq 1$ is mapped into a planar triangulation $\bar{\omega}$.

Lemma: Let Δ be a β-quasi-uniform triangulation of the unit sphere with $|\Delta| \leq 1$. Let Θ_Δ denote the smallest angle of Δ. There exists a constant A_1 such that

$$\Theta_\Delta \geq \frac{1}{A_1 \beta}.$$

Lemma: For every spherical triangle $\tau \in \Delta$ with $|\Delta| \leq 1$

$$\frac{\pi \rho_\Delta^2}{5} \leq A_\tau \leq \frac{\pi |\Delta|^2}{4}.$$

Another result that we need concerning β-quasi-uniform triangulations is a bound on the number of triangles n_k in the k-th disk around τ. We denote $\mathrm{star}^1(v)$ the union of all triangles in Δ that share the vertex v, $\mathrm{star}^\ell(v) := \cup\{\mathrm{star}^1(w) : w \text{ is a vertex of } \mathrm{star}^{\ell-1}(v)\}$, $\ell > 1$, and $\mathrm{star}^\ell(\tau) := \cup\{\mathrm{star}^\ell(w) : w \text{ is a vertex of } \tau\}$, $\ell > 1$.

Lemma: Suppose Δ is a β-quasi-uniform triangulation such that $|\Delta| \leq 1$. Then for any triangle $\tau \in \Delta$ and any $k \geq 0$, the number n_k of triangles in $\mathrm{star}^k(\tau)$ is

$$n_k \leq \frac{5\beta^2}{4}(2k+1)^2.$$

If, in addition, Δ is regular, then

$$n_k \geq \frac{2}{\pi\beta^2}(2k+1)^2.$$

Spherical Sobolev Space Seminorms

Definition: Given any spherical function f and any integer n, the homogeneous extension of f of degree n to $\mathbb{R}^3\backslash\{0\}$ is a function f_n defined by

$$f_n(v) = |v|^n f(\frac{v}{|v|}).$$

We next recall that a trivariate function $f(v)$ is homogeneous of degree n if

$$f(\alpha v) = \alpha^n f(v), \qquad \forall \alpha \in \mathbb{R}.$$

Fix $0 \leq p \leq \infty$, k nonnegative integer. A Sobolev-type seminorm is defined as

$$|f|_{k,p,\Omega} := \sum_{|\alpha|=k} ||D^{\alpha} f_{k-1}||_{p,\Omega}$$

Here $||D^{\alpha} f_{k-1}||_{p,\Omega}$ is understood as the L_p-norm of the restriction of the trivariate function $D^{\alpha} f_{k-1}$ to Ω. For $k = 0$ the above seminorm reduces to the usual L_p-norm.

A Sobolev norm is defined as

$$||f||_{k,p,\Omega} := \sum_{|\alpha|\leq k} ||D^{\alpha} f_{k-1}||_{p,\Omega}$$

Basic Inequalities

Let $\mathcal{H}_d$ denote the space of trivariate homogeneous polynomials of degree d.

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, \qquad i+j+k=d$$

of Bernstein-Bézier basis polynomials of degree d forms a basis for $\mathcal{H}_d$.

Here $b_1(v),\ b_2(v),\ b_3(v)$ are spherical barycentric coordinates of a point $v \in \mathbb{R}^3$ satisfying and uniquely defined by

$$v = b_1(v)v_1 + b_2(v)v_2 + b_3(v)v_3$$

in terms of a triple of linearly independent unit vectors v_1, v_2 and v_3.

Any homogeneous polynomial P of degree d and its restriction to a spherical triangle τ have a Bernstein-Bézier (BB-) representation with respect to τ

$$P(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v).$$

Given a homogeneous trivariate polynomial P in BB form, let c be a vector of its coefficients. Let $||c||_{\infty,\tau}$ and $||c||_{p,\tau}$ denote its ℓ_∞ and ℓ_p norms on a spherical triangle τ respectively.

Lemma: Any homogeneous polynomial P of degree d in Bernstein-Bézier form with respect to a spherical triangle τ with $|\tau| \leq 1$ satisfies the property

$$A_2 \, ||c||_{\infty,\tau} \leq ||P||_{\infty,\tau} \leq A_3 ||c||_{\infty,\tau}$$

and

$$A_4 A_\tau^{1/p} \, ||c||_{p,\tau} \leq ||P||_{p,\tau} \leq A_5 A_\tau^{1/p} ||c||_{p,\tau}$$

for any $1 \leq p < \infty$. Here A_2, A_3 and A_5 are positive constants independent of τ, P and p. A_4 depends d, p and the smallest angle of τ.

Lemma: Let P be a trivariate homogeneous polynomial of degree d defined on a spherical triangle τ with $|\tau| \leq 1$. There exist constants A_6 depending on d and Θ_τ only, and A_7 depending on d, such that

$$|P|_{k,\infty,\tau} \leq \frac{A_6}{(\tan \frac{\rho_\tau}{2})^k} ||P||_{\infty,\tau},$$

and

$$|P|_{k,p,\tau} \leq \frac{A_7}{(\tan \frac{\rho_\tau}{2})^k} ||P||_{p,\tau}$$

for $1 \leq p < \infty$. Here ρ_τ is a the diameter of the largest spherical cap contained in τ.

Lemma: Let τ be a spherical triangle such that $|\tau| \leq 1$, and suppose $f \in W^{2,p}(\tau)$ vanishes at the vertices of τ, that is $f(v_i) = 0, i = 1, 2, 3$. Then for all $v \in \tau$,

$$|f(v)| \leq A_8 (\tan \frac{|\tau|}{2})^2 |f|_{2,\infty,\tau}$$

for some positive constant A_8 independent of f and τ. Moreover, if f is a homogeneous polynomial of degree d, then

$$|f(v)| \leq A_9 A_\tau^{-1/p} (\tan \frac{|\tau|}{2})^2 |f|_{2,p,\tau}$$

for some positive constant A_9 dependent only on d, p and the smallest angle in τ.

Stable Local Basis and Existence of a Quasi-Interpolant

We now describe the stable local bases. We shall use the spline spaces that have a local basis to solve the interpolation problem on the sphere. Let

$$\mathcal{D} := \cup_{\tau \in \Delta} \{\xi_{ijk}^{\tau}, i + j + k = d\},$$

with

$$\xi_{ijk}^{\tau} := \frac{iu + jv + kw}{\|iu + jv + kw\|}$$

for $\tau =< u, v, w >$ be the set of domain points associated with Δ and d. It is well known that each spline in $S_d^0(\Delta)$ is uniquely determined by associating one Bézier coefficient with each domain point. A subset $\mathcal{M} \subset \mathcal{D}$ is called a minimal determining set for $S_d^r(\Delta)$ if the values of the coefficients of $s \in S_d^r(\Delta)$ associated with domain points in $\mathcal{M}$ uniquely determine all of the coefficients of s.

Definition: A basis $\{B_\xi\}_{\xi\in\mathcal{M}}$ for a space $\mathcal{S}$ of splines on a triangulation Δ is a stable local basis, if there exists an integer ℓ and constants $0 < C_1 < C_2 < \infty$ depending only on d and the smallest angle θ_Δ in the triangulation Δ such that

1) for each $\xi \in \mathcal{M}$, $\mathrm{supp}(B_\xi) \subseteq \mathrm{star}^\ell(v_\xi)$ for some v_ξ of Δ,

2) for all $\{c_\xi\}_{\xi\in\mathcal{M}}$,

$$C_1 \max_{\xi\in\mathcal{M}} |c_\xi| \leq || \sum_{\xi\in\mathcal{M}} c_\xi B_\xi ||_{\infty,\mathbb{S}^2} \leq C_2 \max_{\xi\in\mathcal{M}} |c_\xi|.$$

A construction of a stable local basis using the Bernstein-Bézier representation of splines in $S_d^r(\Delta)$ is possible when $d \geq 3r + 2$. Given a minimal determining set, we can construct a basis $\{B_\xi\}_{\xi \in \mathcal{M}}$ for $S_d^r(\Delta)$ by requiring

$$\mu_\eta B_\xi = \delta_{\xi,\eta}, \quad \eta \in \mathcal{M},$$

where μ_η is the linear functional which picks the coefficient associated with the domain point η. In particular, B_ξ has the property that the coefficient associated with ξ is 1 while the coefficients associated with all other points in $\mathcal{M}$ are zero. The remaining coefficients of B_ξ are computed using smoothness conditions.

For any given spline space $S_d^r(\Delta)$, there are many possible choices for a minimal determining set $\mathcal{M}$.

Lemma: There exist constants $C_3, ..., C_9$ depending only on d, p and the minimal angle in Δ such that for each $\xi \in \mathcal{M}$,

1) there exists a vertex $v_\xi \in \Delta$ such that $\Omega_\xi \subseteq \text{star}^3(v_\xi)$,
2) $||B_\xi||_{\infty,\mathbb{S}^2} \leq C_3$,
3) $|\mu_\xi s| \leq C_4 ||s||_{\infty,\tau_\xi}$, for all $s \in S_d^r(\Delta)$,
4) $|\mu_\xi s| \leq C_5 A_{\tau_\xi}^{-1/p} ||s||_{p,\tau_\xi}$, for all $s \in S_d^r(\Delta)$, and for every $\tau \in \Delta$,
5) $||B_\xi||_{p,\tau} \leq C_6 A_\tau^{1/p}$,
6) $\# I_\tau \leq C_7$, where $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$,
7) $|B_\xi|_{k,\infty,\tau} \leq C_8 \rho_\tau^{-k}$, for all $0 \leq k \leq d$
8) $|B_\xi|_{k,p,\tau} \leq C_9 \rho_\tau^{-k} A_\tau^{1/p}$, for all $0 \leq k \leq d$.

Lemma: For each $f \in L_p(\mathbb{S}^2)$, let

$$Qf := \sum_{\xi \in \mathcal{M}} (\mu_\xi f) B_\xi.$$

Then $Qg = g$ for all $g \in \mathcal{H}_d(\mathbb{S}^2)$. Moreover, there exists a constant C_{10} depending only on d, p and the smallest angle in Δ such that for each triangle $\tau \in \Delta$,

$$|Qf|_{k,p,\tau} \leq C_{10} (\tan \frac{\rho_\tau}{2})^{-k} ||f||_{p,\Omega_\tau},$$

where $\Omega_\tau := \cup_{\xi \in I_\tau} \Omega_\xi$ and $I_\tau := \{\xi : \tau \subset \Omega_\xi\}$.

Theorem: Suppose $\tau \in \Delta$ is a spherical triangle with $|\tau| \leq 1$. Let $f \in W^{m+1,p}(\tau)$ for $0 \leq m \leq d$ such that $(d - m)$mod $2 = 0$. There exists a spherical homogeneous polynomial s of degree d such that for every $0 \leq k \leq m$

$$|f - s|_{k,p,\tau} \leq C_{11}(\tan\frac{|\tau|}{2})^{m+1-k}|f|_{m+1,p,\tau}.$$

Here C_{11} is a constant that depends on p, m and θ_Δ. Moreover

$$|f - s|_{k,p,\Omega_\tau} \leq C_{11}(\tan\frac{|\Delta|}{2})^{m+1-k}|f|_{m+1,p,\Omega_\tau}.$$

Theorem: Let Δ be a β-quasi-uniform spherical triangulation with $|\Delta| \leq 1$. Let $1 \leq p \leq \infty$, $d \geq 3r+2$, and $0 \leq k \leq d$. Then there exists a constant C_{12} depending only on d, p and the smallest angle in Δ, such that

$$|f - Qf|_{k,p,\tau} \leq C_{12}(\tan\frac{|\Delta|}{2})^{m+1-k}|f|_{m+1,p,\Omega_\tau},$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $\tau \in \Delta$. Moreover, there exists a constant C_{13} such that

$$|f - Qf|_{k,p,\mathbb{S}^2} \leq C_{13}(\tan\frac{|\Delta|}{2})^{m+1-k}|f|_{m+1,p,\mathbb{S}^2},$$

for all $f \in W^{m+1,p}(\mathbb{S}^2)$ and all $0 \leq k \leq d$ such that $Qf \in W^{k,p}(\mathbb{S}^2)$. Here m is taken between 0 and d with $(d-m) \bmod 2 = 0$.

Minimal Energy Interpolating Splines

Suppose we are given values $\{f(v), v \in \mathcal{V}\}$ of an unknown function f at a set $\mathcal{V}$ of scattered points on the unit sphere. To approximate f, we choose a linear space $\mathcal{S} \subseteq S_d^r(\Delta)$ of polynomial splines of degree d defined on a triangulation Δ with vertices at the points of $\mathcal{V}$. Recall that

$$U_f := \{s \in \mathcal{S} : s(v) = f(v), v \in \mathcal{V}\}$$

is the set of all splines in $\mathcal{S}$ that interpolate f at the points of $\mathcal{V}$. Recall a commonly used way to create an approximation of f is to choose a spline S_f such that

$$\mathcal{E}(S_f) = \min_{s \in U_f} \mathcal{E}(s),$$

where for a spherical triangle $\tau \in \Delta$

$$\mathcal{E}_\tau(s) := \sum_{|\alpha|=2} ||D^\alpha s||_{2,\tau}^2 \text{ and } \mathcal{E}(s) := \sum_{\tau \in \Delta} \mathcal{E}_\tau(s).$$

Let

$$\mathcal{X} := \{f \in B(\mathbb{S}^2) : f|_\tau \in C^2(\tau), \forall \tau \in \Delta\},$$

where $B(\mathbb{S}^2)$ is the set of all bounded real-valued functions on the sphere. For each triangle $\tau \in \Delta$, let

$$\langle f, g \rangle_\tau := \int_\tau \sum_{|\alpha|=2} D^\alpha f \; D^\alpha g.$$

Then

$$\langle f, g \rangle := \langle f, g \rangle_{\mathbb{S}^2} = \sum_{\tau \in \Delta} \langle f, g \rangle_\tau$$

is a semidefinite inner product on $\mathcal{X}$. Let $||f||_\tau$ and $||f||$ be the associated seminorms. We refer to them as energy or $\mathcal{X}$-norms.

It is easy to see that $\langle\cdot,\cdot\rangle$ is an inner product on the linear space

$$\mathcal{W} := \{s \in \mathcal{S} : s(v) = 0, v \in \mathcal{V}\}.$$

Indeed, if $\langle w, w\rangle = 0$ for some $w \in \mathcal{W}$, then w is a linear homogeneous polynomial on Δ and since w vanishes at all vertices, $w \equiv 0$. Since $\mathcal{W}$ is finite-dimensional, it follows that $\mathcal{W}$ equipped with the inner product $\langle\cdot,\cdot\rangle$ is a Hilbert space.

Given f, suppose s_f is any spline in the set U_f defined above. Then it is easy to see that the solution S_f to the minimal energy problem is equal to $s_f - \mathcal{P}s_f$, where $\mathcal{P}$ is the linear projector $\mathcal{P} : \mathcal{X} \to \mathcal{W}$ defined by

$$\mathcal{E}(f - \mathcal{P}f) = \min_{w\in\mathcal{W}} \mathcal{E}(f - w),$$

for all $f \in \mathcal{X}$.

Since $\mathcal{W}$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle$, $\mathcal{P}f$ is uniquely defined and is characterized by

$$\langle f - \mathcal{P}f, w \rangle = 0, \qquad \forall w \in \mathcal{W}.$$

Moreover

$$||\mathcal{P}f|| \leq ||f||$$

We now establish a lemma showing the equivalence of certain seminorms on the space $\mathcal{X}$ defined above.

Lemma: Let τ be a spherical triangle with $|\tau| \leq 1$ and $f \in \mathcal{X}$. There exists a constant D_1 such that

$$D_1 |f|_{2,2,\tau}^2 \leq \mathcal{E}_\tau(f) \leq |f|_{2,2,\tau}^2.$$

Next we establish the reproductive property of the energy functional $\mathcal{E}_\tau$.

Lemma: Let τ be a spherical triangle with $|\tau| \leq 1$. Suppose $f \in \mathcal{X}$. Then $\mathcal{E}_\tau(f) = 0$ if and only if f is a trivariate homogeneous constant polynomial on τ.

Theorem: Suppose $S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \le 1$, and let $\mathcal{W}$ be the associated Hilbert space . Then there exist constants $0 < D_2 \le D_3 < \infty$ depending only d and β such that

$$D_2 ||f||_{2,\mathbb{S}^2}^2 \le (\tan \frac{|\Delta|}{2})^4 ||f||^2 \le D_3 ||f||_{2,\mathbb{S}^2}^2,$$

for all $f \in \mathcal{W}$.

Next we want to show that under certain conditions on $\mathcal{S}$, the $\mathcal{X}$-norm on the Hilbert space $\mathcal{W}$ is also equivalent to a certain coefficient norm.

Theorem: Suppose $S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ, and that $\{B_\xi\}_{\xi\in\mathcal{M}}$ is a stable local basis. There exist constants D_4, D_5 depending on d, β, largest and smallest $A_\tau, \tau \in \Delta$, such that

$$D_4 \sum_{\xi\in\mathcal{N}} |c_\xi|^2 \le (\tan\frac{|\Delta|}{2})^4 || \sum_{\xi\in\mathcal{N}} c_\xi B_\xi ||^2 \le D_5 \sum_{\xi\in\mathcal{N}} |c_\xi|^2,$$

for all $\{c_\xi\}_{\xi\in\mathcal{N}}$.

Theorem: There exist constants $0 \leq \sigma \leq 1$ and D_6, depending only on ℓ, d, β, such that for any triangle $T \in \Delta$ and any function $f \in \mathcal{X}$ with $\operatorname{supp}(f) \subseteq T$

$$||\mathcal{P}f||_\tau \leq D_6 \sigma^k ||f||,$$

whenever $\tau \in \mathrm{star}^{2(k+2)\ell+1}(T) \backslash \mathrm{star}^{2(k+1)\ell+1}(T)$ with $k \geq 1$.

Theorem: There exists a constant D_7 depending only on d, ℓ and β, such that for every $f \in \mathcal{X}$

$$|\mathcal{P}f|_{2,\infty,\mathbb{S}^2} \leq D_7 |f|_{2,\infty,\mathbb{S}^2}.$$

Theorem: Suppose $\mathcal{S} \subseteq S^r_d(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \leq 1$ and $d \geq 3r+2$. For d odd there exists a constant D_8 depending only on d and β, such that the minimal energy interpolant S_f, satisfies

$$||f - S_f||_{\infty,\mathbb{S}^2} \leq D_8(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbb{S}^2},$$

for all $f \in C^2(\mathbb{S}^2)$. For d even there exist constants D_9 and D_{10} depending only on d and β, such that the minimal energy interpolant S_f satisfies

$$||f - S_f||_{\infty,\mathbb{S}^2} \leq D_9(\tan\frac{|\Delta|}{2})^2|f|_{2,\infty,\mathbb{S}^2} + D_{10}(\tan\frac{|\Delta|}{2})^3|f|_{3,\infty,\mathbb{S}^2},$$

for all $f \in C^3(\mathbb{S}^2)$.

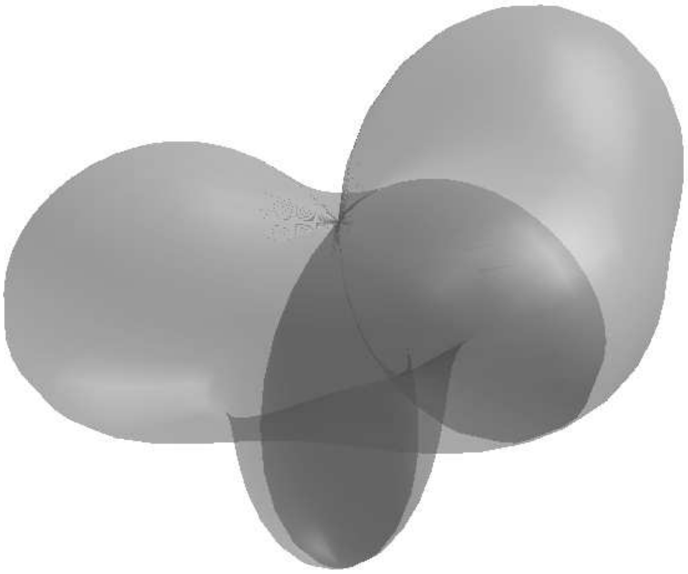
Numerical Experiments

The initial triangulation Δ_1 in the first two examples consists of 8 triangles with vertices being unit coordinate vectors and their antipodes. Its first refinement Δ_2 is obtained by connecting the midpoints of all edges in Δ_1 such that each triangle is split in four subtriangles. Similarly Δ_3 and Δ_4 are obtained from Δ_2 and Δ_3 correspondingly. Note that with such a refinement the size $|\Delta_{i+1}|$ of a refinement is not a half of $|\Delta_i|, i = 1, 2, 3$ as it happens in the planar case. Instead $\tan \frac{|\Delta_i|}{2}, i = 1, 2, 3$ is reduced in half as illustrated in Table below

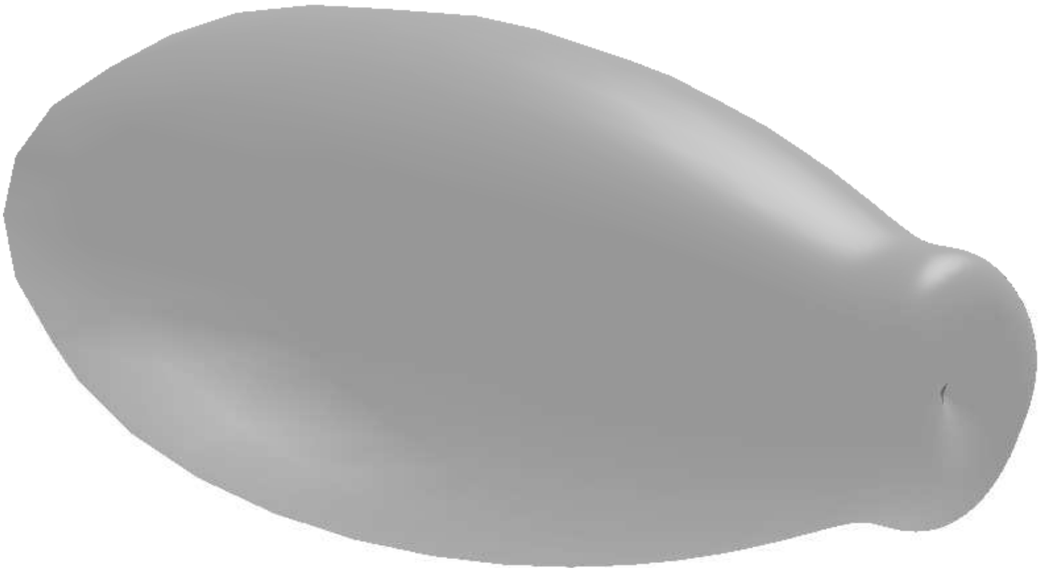
i	1	2	3	4
$\lvert\Delta\rvert$	1.91063	1.23095	0.67967	0.34994
$(\tan \frac{\lvert\Delta\rvert}{2})^2$	2	0.5	0.125	0.03125

Table 1: Triangulation parameters.

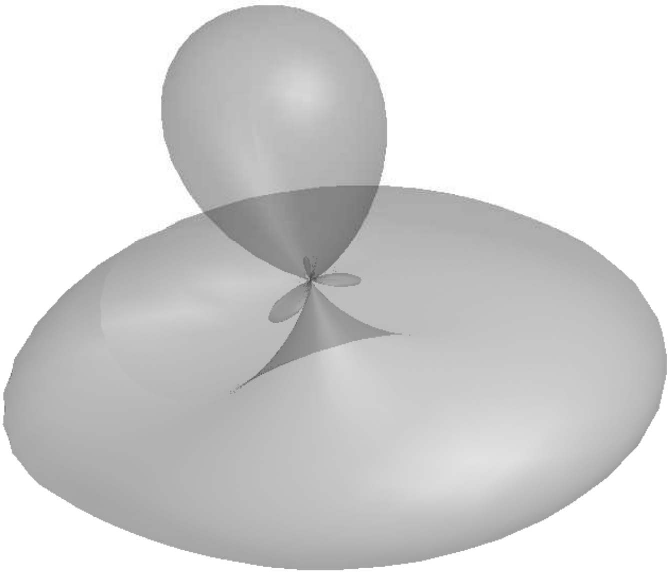
$$f_1(x, y, z) = x^2 - (y^3 + z^7)$$



$$f_2(x, y, z) = 0.1x^8 + e^{2y^3}$$



$$f_3(x, y, z) = \ln(2 + x^2) - \sin(4z - y).$$



Example 1. We use spherical splines of degree 5 and smoothness 1 to find the minimal energy interpolants over the triangulation $\Delta_1, \Delta_2, \Delta_3$ and Δ_4. Then we evaluate the splines at 5120 evenly spaced points w and record the relative errors for these three test functions. The relative error on Δ_i is defined by $e(\Delta_i) := \frac{||s(w)-f(w)||_\infty}{||f(w)||_\infty}$, $s \in S_d^r(\Delta_i)$. The errors are listed in Table 2. In Table 3 we list ratios of the form $\frac{e(\Delta_i)}{e(\Delta_{i+1})}, i = 1, 2, 3$ for all three functions. The numerical convergence rates are close to the convergence rate we derived in the previous section.

$f\backslash i$	1	2	3	4
f_1	$1.0317e-00$	$1.8164e-01$	$2.8386e-02$	$2.8290e-03$
f_2	$0.3834e-00$	$0.6344e-01$	$1.7466e-02$	$2.5500e-03$
f_3	$1.0496e-00$	$4.3803e-01$	$0.5142e-01$	$0.5150e-02$

Table 2: Experimental errors for C^1 quintic splines.

$f\backslash i$	1	2	3
f_1	5.6799	6.3989	10.0339
f_2	6.0435	3.6322	6.8494
f_3	2.3962	8.5187	9.9845

Table 3: Convergence rates of the C^1 quintic splines.

Example 2. In this example we work with one function only and vary the degree d of the spline space. That is, we use $S_d^1(\Delta_i)$, $d = 3, 4, 5, 6, 7$, $i = 1, 2, 3, 4$.

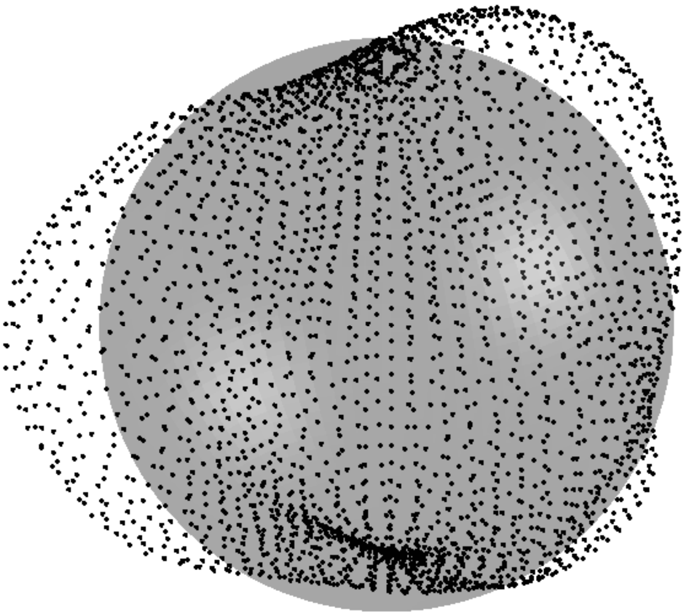
$d\backslash i$	1	2	3	4
5	$3.8334e-01$	$0.6344e-01$	$1.7466e-02$	$2.5500e-03$
6	$3.8057e-01$	$0.6315e-01$	$1.8148e-02$	$2.6871e-03$
7	$3.8286e-01$	$0.6254e-01$	$1.8429e-02$	$2.7489e-03$

Table 4: Splines of various degrees interpolating $f_2(x, y, z) = 0.1x^8 + e^{2y^3}$.

$d\backslash i$	1	2	3
5	6.0435	3.6322	6.8494
6	6.0265	3.4797	6.7538
7	6.1225	3.3934	6.7045

Table 5: Convergence rates for splines of various degrees interpolating f_2.

Example 3. We present an example of scattered data interpolation over the earth. We are given a set of locations with geo-potential values collected by a satellite. The total amount of data values is 5760. We use C^1 cubic spherical splines since the data set is very large.



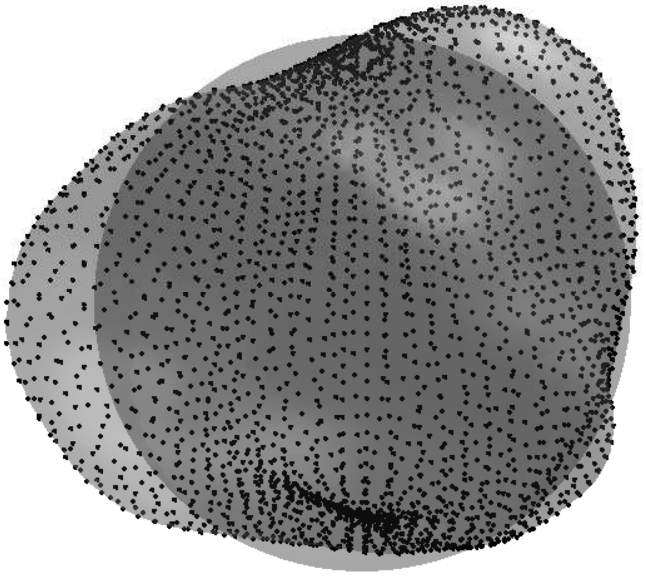


Figure 2: Geopotential data and minimal energy C^1 cubic interpolant.