RESEARCH STATEMENT

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1. Introduction

Lie theory has been a central topic of study for over a century because of its connections to various parts of mathematics and physics. In my research I am interested in problems bridging Lie theory and geometry. One of the starting points of my investigations involves the computation of cohomology for the Frobenius kernels of simple algebraic groups. In studying cohomology, one needs to employ algebraic tools from representation theory and computational algebra, and geometric methods from the theory of nilpotent orbits, cohomology of sheaves, and commutative algebra and algebraic geometry. My research entails a mix of ideas from the aforementioned areas. I am also interested in the passage from using classical methods to study algebraic groups to using non-commutative methods to study quantum groups.

The representation and cohomology theory for the r-th Frobenius kernels of an algebraic group is an area of Lie theory that has received considerable attention. For $r = 1$, representations of the first Frobenius kernel correspond to restricted representations of the corresponding Lie algebra. The cohomology of the first Frobenius kernel of reductive groups was investigated, for example, in the works of Andersen-Jantzen [AJ], Friedlander-Parshall [FP1, FP2], and Kumar-Lauritzen-Thomsen [KLT]. In joint work with Drupieski and Nakano [DNN], we completely described the cohomology ring structure for the first Frobenius kernel of maximal unipotent subgroups of simple algebraic groups. This computation answers a 25-year-old problem concerning the ring structure of $\mathrm{H}^\bullet(U_1, k)$. We also obtained new results on the cohomology modules for the first Frobenius kernels of parabolic subgroups. The state of affairs for higher Frobenius kernels still remains a big mystery. Exploiting the spectral sequence of Andersen-Jantzen, I have obtained some results for certain low degree cohomology modules. This generalizes the work of Bendel-Nakano-Pillen [BNP1, BNP2]. I also computed cohomology modules for the r-th Frobenius kernels of the Borel subgroup and G in the case when $G = SL_2$. Moreover, I wrote a program in MAGMA to compute the character multiplicities for the B_r-cohomology modules.

The geometric connection of cohomology for the Frobenius kernels was first noticed by a result of Friedlander and Parshall in which there is an isomorphism between G_1-cohomology and the coordinate ring of the corresponding nilpotent cone $\mathcal{N}$ [FP2]. A generalization was obtained for the higher Frobenius kernels by Bendel-Friedlander-Suslin in their two papers [BFS1, BFS2]. They demonstrated that studying nilpotent commuting varieties of r-tuples in the corresponding Lie algebra determines the spectrum of the cohomology ring for the r-th Frobenius kernel of an algebraic group. In my thesis, I constructed an explicit homeomorphism from the spectrum of the cohomology ring to the spectrum of $k[G \times^B \mathfrak{u}^r]$ in the case $G = SL_2$.

In general, commuting varieties, especially with $r = 2$, arise as interesting varieties in algebraic geometry. Many mathematicians have worked on the irreducibility of this variety [G, GS, KN]. Showing that the (nilpotent) commuting variety is normal or Cohen-Macaulay is still an open problem [Pr]. In my dissertation, I used determinantal varieties to prove that the variety of commuting r-tuples of 2×2-matrices is irreducible, normal and Cohen-Macaulay. Then combining strategies in geometry and commutative algebra, I proved the analogous result for the commuting variety of r-tuples of nilpotent elements in $\mathfrak{sl}_2$. This not only provides the homeomorphism between the spectrum of G_r-cohomology ring and this variety but also supports the conjecture about Cohen-Macaulayness of nilpotent commuting varieties. I was also able to show that the singularities of the nilpotent commuting variety for $\mathfrak{sl}_3$ are of codimension 2 which is the necessary condition for being normal. I have been considering the generalization to the case of $\mathfrak{sl}_n$.

2. Notation and Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$. Let G be a simple, simply-connected algebraic group over k, defined and split over the prime field $\mathbb{F}_p$. Let h be the Coxeter number of G. Fix a maximal torus T of G. Then associating to this maximal torus, we denote by Φ, Π and X respectively the root system, set of simple roots and weight lattice of G. Let $B \subset G$ be the Borel subgroup of G containing T and corresponding to Φ^+, the set of positive roots, and let $U \subset B$ be the unipotent radical of B. Given $J \subseteq \Pi$, let P_J be the standard parabolic subgroup of G containing B and corresponding to J, let U_J be the unipotent radical of P_J, and let L_J be the Levi factor of P_J. For each r, the r-th Frobenius kernel of certain closed subgroup H of G is defined as the intersection H with kernel of the morphism $F_r : G \to G$ induced from the ring endomorphism $f \mapsto f^{p^r}$ on $k[G]$. Given a rational H-module M, write $M^{(r)}$ for the module obtained by twisting the structure map for M by F_r.

Let X^+ be the subset of X consisting of all dominant weights. Simple G-modules are indexed by $\lambda \in X^+$, and denoted $L(\lambda)$. The simple module $L(\lambda)$ can be identified with the socle of the induced module $H^0(\lambda) = \operatorname{ind}_{B^-}^{G} \lambda$. Set $\mathfrak{g} = \operatorname{Lie}(G)$, the Lie algebra of G, $\mathfrak{b} = \operatorname{Lie}(B)$, $\mathfrak{u} = \operatorname{Lie}(U)$, $\mathfrak{p}_J = \operatorname{Lie}(P_J)$, $\mathfrak{u}_J = \operatorname{Lie}(U_J)$, and $\mathfrak{l}_J = \operatorname{Lie}(L_J)$. Denote by $S^\bullet(\mathfrak{u}_J^*)$ the symmetric algebra over $\mathfrak{u}_J^*$. The bottom p-alcove and its closure are defined, respectively, by

$$C_{\mathbb{Z}} := \left\{ \lambda \in X : 0 < (\lambda + \rho, \beta^\vee) < p \text{ for all } \beta \in \Phi^+ \right\},$$
$$\overline{C}_{\mathbb{Z}} := \left\{ \lambda \in X : 0 \le (\lambda + \rho, \beta^\vee) \le p \text{ for all } \beta \in \Phi^+ \right\}.$$

We use R_{red} to denote the reduced ring $R/\sqrt{(0)}$ where $\sqrt{(0)}$ is the radical ideal of 0 in R which consists of all nilpotent elements of R. Let Spec R be the spectrum of all prime ideals of R. It is a topological space under the Zariski topology. Let X be a variety. We denote by $k[X]$ the algebra of regular functions defined on X. Note that when X is an affine variety, $k[X]$ coincides with the coordinate algebra of X.

Suppose $V_1, ..., V_r$ are subvarieties of $\mathfrak{g}$. Then we define the commuting variety of r-tuples (which is a subvariety of $V_1 \times ... \times V_r$) as follows:

$$C(V_1, ..., V_r) = \{(x_1, ..., x_r) \in V_1 \times ... \times V_r \mid [x_i, x_j] = 0,\ 1 \le i \le j \le r\}.$$

In case $V_1 = ... = V_r = V$, we simply write $C_r(V)$ for this algebraic variety.

3. U_1-cohomology

A major motivation for studying cohomology of the first Frobenius kernels is the equivalence between the category of modules over these objects and the category of (restricted) modules over the corresponding restricted Lie algebra. In joint work with Drupieski and Nakano, we first computed the cohomology of $(U_J)_1$ with coefficients in certain simple G-module. This result strengthens an observation previously made by Friedlander and Parshall [FP1, Remark 2.7(b)].

Theorem 3.1. [DNN, Theorem 2.2.1] *Suppose $\lambda \in C_{\mathbb{Z}}$ and that $p > h$. Then there exists an isomorphism of graded L_J-modules*

$$\mathrm{H}^\bullet((U_J)_1, L(\lambda)) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, L(\lambda)).$$

In 1983, Crane claimed in her Ph.D thesis [Cra] that if G is of type A, then $\mathrm{H}^\bullet(U_1, k)$ can be identified as the tensor product of a polynomial ring and a finite dimensional algebra. By using the theory of support varieties one can conclude that this is false when $p < h$. We were able to verify Crane's claim for type A and for more general reductive groups for large primes p (i.e., $p > 2(h-1)$).

Theorem 3.2. [DNN, Theorem 3.1.1] *Let $p > 2(h-1)$. Then there exists a graded algebra isomorphism*

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k).$$

Moreover, if $p > 3(h-1)$ and $J \neq \emptyset$, then there is an L_J-algebra isomorphism

$$\mathrm{H}^\bullet((U_J)_1, k) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, k).$$

We also applied our calculations to obtain new results on the cohomology of $(P_J)_1$ with coefficients in a simple G-module having highest weight in the closure of bottom p-alcove.

Theorem 3.3. [DNN, Theorem 4.2.1] *If $p > h$ and $\lambda = w \cdot 0 + p\sigma$, i.e. λ is weakly p-linked to 0, then there exists a P_J-module isomorphism*

$$\mathrm{H}^j((P_J)_1, L(\lambda))^{(-1)} \cong \begin{cases} \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] & \text{if } j \equiv l(w) \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}$$

One of the primary ingredients is the calculation for $p > h$ by Kumar, Lauritzen and Thomsen [KLT] of the cohomology of G_1 with coefficients in an induced module, which employs the existence of Frobenius splittings on the cotangent bundle of the flag variety. Our calculations for $(P_J)_1$ significantly extend the earlier calculations of Friedlander and Parshall [FP1, Corollary 2.6 and Remark 2.7(b)].

Finally, we showed that these above results can be adapted to prove results about the cohomology of quantum groups. Different techniques are required here due to the lack of quantum analogs for various tools available in the classical setting.

4. Computations for low degrees and SL_2

4.1. **Low degree cohomology.** While describing the ring structure of $\mathrm{H}^\bullet(G_r, k)$ appears to be unreachable in the near future, it is reasonable to focus our attention on low degree cohomology. In 1986, Friedlander and Parshall considered this problem for groups with underlying root systems of type A.

Theorem 4.1. [FP1, Theorem 1.8] *Let G be a simple, simply connected algebraic group of type A with $p > h$. Then for $r > 1$, the inflation map $\mathrm{H}^i(G_r/G_{r-1}, k) \to \mathrm{H}^i(G_r, k)$ is an isomorphism for all $i < 2p-1$.*

Recently, Bendel-Nakano-Pillen computed the second cohomology of Frobenius kernels for arbitrary type of G and prime p with coefficients in an induced module.

Theorem 4.2. [BNP2, Theorem 6.1] *Let $\lambda \in X^+$ and p be an arbitrary prime. Then for all r, we have*

$$\mathrm{H}^2(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_r, \lambda)^{(-r)}).$$

The main tool in the proofs of both theorems is the use of the Lyndon-Hochschild-Serre spectral sequence. By employing some calculations with the spectral sequence of Andersen-Jantzen for computing B_r-cohomology and applying the induction functor, I have proved the following theorem that generalizes both of the aforementioned results.

Theorem 4.3. [N] *Let p be a very good prime (ref.* [BNPP, 3.1]*). Let c be the largest coefficient in the expression of all positive roots in terms of simple roots. Then for each $n \leq \frac{p}{c}$, there are B-module isomorphisms*

$$\mathrm{H}^n(B_r, k)^{(-r)} \cong \mathrm{H}^n(B_1, k)^{(r-1)} \cong \begin{cases} S^{\frac{n}{2}}(\mathfrak{u}^*) & \text{if } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

and G-module isomorphisms

$$\mathrm{H}^n(G_r,k)^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^n(B_r,k)^{(-r)}) \cong \begin{cases} \mathrm{ind}_B^G(S^{\frac{n}{2}}(\mathfrak{u}^*)) & \text{if } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\mathrm{H}^n(G_r,k) \cong \mathrm{H}^n(G_1,k)^{(r-1)}$.

4.2. SL_2**-computations.** Let $G = SL_2$ and let α be the positive root in the underlying root system A_1 of G. Then the cohomology for G_r is computable. Applying the same strategy as of Bendel-Nakano-Pillen [BNP2], I provided the following description of the G_r-cohomology.

Theorem 4.4. [N] *Let* $G = SL_2$. *Suppose* λ *is a dominant weight. Then we have*

$$\begin{aligned} \mathrm{H}^n(G_r,\mathrm{H}^0(\lambda))^{(-r)} &\cong \mathrm{ind}_B^G(\mathrm{H}^n(B_r,\lambda)^{(-r)}) \\ &\cong \bigoplus \mathrm{ind}_B^G\left[a_r\alpha + \frac{\frac{\frac{\frac{\lambda+b_1\alpha}{p}+(a_1+b_2)\alpha}{p}+(a_2+b_3)\alpha}{p}+..+(a_{r-1}+b_r)\alpha}{p}\right] \end{aligned}$$

where the direct sum is taken over all the tuples $(a_1,...,a_r,b_1,...,b_r) \in \mathbb{N}^r \times \{0,1\}^r$ *satisfying*

$$n = 2(a_1+...+a_r)+b_1+...+b_r.$$

The first isomorphism, in the context of arbitrary reductive groups, is a long-standing conjecture. I plan to work on proving this result at least for type A_n in the near future. On the other hand, it is extremely difficult to describe $\mathrm{H}^\bullet(G_r,k)$ as a ring for arbitrary r. As a geometric object, I discovered the following description for the cohomology ring of G_r.

Theorem 4.5. [N] *Suppose* $G = SL_2$. *Then for each* $r \geq 1$, *we have* $\mathrm{Spec}\,\mathrm{H}^\bullet(G_r,k)_{red}$ *is homeomorphic to* $\mathrm{Spec}\, k[G\times^B \mathfrak{u}^r]$.

5. Commuting varieties

In my thesis, I was motivated by the cohomological work to investigate the structure of varieties of r-tuples of commuting nilpotent elements in a simple Lie algebra. Without the nilpotency condition, the commuting variety $C_r(\mathfrak{gl}_n)$ is a classical object studied by Gerstenhaber, Guralnick-Sethuraman, and Kirillov-Neretin [G, GS, KN]. All of these works focused on irreducibility. The conjecture that the variety of commuting r-tuples of $n\times n$-matrices is Cohen-Macaulay and normal is still a difficult conjecture to verify. Computer verification only works for low rank cases up to $r=2, n=3$ [Hr]. In my dissertation, I verified this conjecture for several cases. The main results are the following.

Theorem 5.1. [N] *For each* $r \geq 1$ *and* $n \geq 2$, *there is an isomorphism between varieties* $C_r(\mathfrak{gl}_n)$ *and* $C_r(\mathfrak{sl}_n)\times\mathbb{A}^r$. *In particular, if* $n=2$, *both* $C_r(\mathfrak{gl}_2)$ *and* $C_r(\mathfrak{sl}_2)$ *are irreducible, normal and Cohen-Macaulay.*

With the nilpotency condition, the problem about commuting varieties turns out to be more difficult. The irreducibility of nilpotent commuting varieties was studied by Baranovsky, Premet, Basili and Iarrobino [Ba, Pr, B, BI]. However, there has not been any successful work on normality and Cohen-Macaulayness of these varieties even in the simple cases. In my thesis, I completely solved this problem for the variety of r-tuples of commuting nilpotent elements in $\mathfrak{sl}_2$.

Theorem 5.2. [N] *Suppose* k *is an algebraically closed field of characteristic* $p \neq 2$. *Let* $\mathcal{N}(\mathfrak{sl}_2)$ *be the nilpotent cone of the Lie algebra* $\mathfrak{sl}_2$. *Then for each* $r \geq 1$, *the variety* $C_r(\mathcal{N}(\mathfrak{sl}_2))$ *is irreducible, normal and Cohen-Macaulay. Moreover, the resolution* $m: G\times^B\mathfrak{u}^r \to C_r(\mathcal{N}(\mathfrak{sl}_2))$ *admits rational singularities.*

In the proof, I studied the structure of the variety $C_r(\mathcal{N}(\mathfrak{sl}_2))$ by intersecting it with a hypersurface. This approach allows us to predict a regular sequence for this variety. In order to obtain the Cohen-Macaulayness of $C_r(\mathcal{N}(\mathfrak{sl}_2))$, I used a concept in commutative algebra, namely *Principal Radical System* [BV][1], to show the geometric structure agrees with ring structure in this case. Next, exploiting Čech cohomology computations, I proved the rest of the theorem. As a consequence of this result, the characters of the coordinate algebra for this variety as a G-module can be determined via the isomorphisms in Theorem 4.4. Combining this with Theorem 4.5, we obtain a homeomorphism between the spectrum of the G_r-cohomology ring and the variety $C_r(\mathcal{N}(\mathfrak{sl}_2))$. The two observations above are summarized in the following corollary.

Corollary 5.3. [N] *Let $G = SL_2$. For each $r \geq 1$, there is an G-algebra isomorphism between $k[G \times^B \mathfrak{u}^r]$ and $k[C_r(\mathcal{N}(\mathfrak{sl}_2))]$. Moreover, Spec $\mathrm{H}^\bullet(G_r, k)_{red}$ is homeomorphic to $C_r(\mathcal{N}(\mathfrak{sl}_2))$ as a topological space.*

This gives an alternative proof for the main result in [BFS1] for the special case $G = SL_2$. It is hard to generalize the proof to arbitrary simple Lie algebras because of the complexity of the defining ideal for $C_r(\mathcal{N}(\mathfrak{g}))$. However, in the case $\mathfrak{g} = \mathfrak{sl}_3$, I proved that the codimension of singularities for $C_r(\mathcal{N}(\mathfrak{sl}_3))$ is greater than or equal to 2. This is the necessary condition for a variety to be normal.

Theorem 5.4. [N] *For each $r \geq 1$, the variety $C_r(\mathcal{N}(\mathfrak{sl}_3))$ is irreducible of dimension $2r + 4$. Moreover, the set of singularities is of codimension greater than or equal to 2, and is contained in $C_r(\overline{\mathcal{O}_{min}})$, where $\mathcal{O}_{min}$ is the minimal nilpotent orbit of $\mathfrak{sl}_3$.*

This theorem shows that $C_r(\mathcal{N}(\mathfrak{sl}_3))$ satisfies Serre's condition (R1) (ref. [SH]). I conjecture that this variety holds not only Serre's condition (S2), which implies the normality, but also the condition for being Cohen-Macaulay.

Conjecture 5.5. *The variety $C_r(\mathcal{N}(\mathfrak{sl}_3))$ is Cohen-Macaulay and normal for each $r \geq 1$.*

6. Ongoing Projects

Along with completing the Conjecture 5.5, I am also interested in working on two other problems at present. The first one is a special case of Theorem 3.2 which remains open.

Conjecture 6.1. *If G is of type A and $p > h$, then there exists a T-algebra isomorphism*

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k).$$

One way to approach this problem is by considering the following system of equations

$$\begin{cases} w_1 \cdot 0 + w_2 \cdot 0 & = w_3 \cdot 0 + p\sigma \\ \ell(w_1) + \ell(w_2) & = \ell(w_3) + 2\deg(\sigma) \end{cases}$$

where $\ell(w)$ is the length of the element w in the Weyl group W and $\deg(\sigma)$ is the degree of σ as a polynomial in $S^\bullet(\mathfrak{u}^*)$. If this system has no solutions except $\sigma = 0$, then a similar argument as in proof of [DNN, Theorem 3.1.1] can be applied to obtain the isomorphism. By listing all of the possibilities, we already verified this for types A_2 and A_3. Computations for higher cases are more complicated. We hope to have some new idea to generalize our works in the near future.

I am also studying the commuting variety for the nilpotent Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$

$$C_2(\mathfrak{u}) = \{(X, Y) \in \mathfrak{u}^2 \mid [X, Y] = 0\}.$$

Not much is known about this variety, even its dimension. Gross stated in his paper [Gr] that $\dim C_2(\mathfrak{u}) = \dim B$, but his proof implicitly assumed that $\mathfrak{u}$ is the union of finitely many B-orbits

[1]This notion was introduced by Hochster to prove certain classes of ideals are radical. It also plays an important role in the proof of Hochster and Eagon that shows determinantal rings are Cohen-Macaulay (ref. [HE]).

which is certainly false. We propose another approach to this problem by introducing a new kind of commuting variety

$$C(\mathcal{V},\mathfrak{u}) = \{(X,Y) \in \mathcal{V} \times \mathfrak{u} \mid [X,Y] = 0\}$$

where $\mathcal{V}$ is an orbital variety associated to some orbit. We also conjecture that such a variety forms an irreducible component of $C_2(\mathfrak{u})$.

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