

Math 4000/6000 Assignment #5
Due: Thursday, February 10, 2005

Instructions: Complete all Level I problems below. There are 5 Level II problems listed below. Math 4000 students should submit 2 of these; Math 6000 students should submit 3 of these. All students are welcome to submit one additional Level II problem, which will count as a bonus problem.

At the beginning of your assignment, include a sentence like:

"I am submitting the Level II problems 21 and 24 to be graded, and problem 26 to be graded as a bonus problem."

Reading: Sections 2.1-2.4 of Shifrin

Level I Problems – submit all problems

Problem 1. Shifrin, p. 41, Exercise 1.4 # 2

Problem 2. Shifrin, p. 41, Exercise 1.4 # 16(a)

Problem 3. Shifrin, p. 41, Exercise 1.4 # 18
(*hint:* there is a multiplicative identity, but it is not 1)

Problem 4. Let S be a set; its **power set** $\mathcal{P}(S)$ consists of all subsets of S . You should convince yourself that if S has n elements, then $\mathcal{P}(S)$ has 2^n elements.

For this problem, let $S = \{a, b\}$. We will define a ring structure on $\mathcal{P}(S)$. Define the "addition" on $\mathcal{P}(S)$ as the *symmetric difference* between two sets, i.e.,

$$A + B = (A - B) \cup (B - A) \quad .$$

Define "multiplication" on $\mathcal{P}(S)$ as the intersection of the two sets:

$$A \times B = A \cap B \quad .$$

- (a) Find the four elements of $\mathcal{P}(S)$.
- (b) Write out addition and multiplication tables for $\mathcal{P}(S)$.
- (c) Show that $\mathcal{P}(S)$ is a commutative ring with unity. Where else have you seen this ring?
- (d) Show that $\mathcal{P}(S)$ is a Boolean ring.

Problem 5. Shifrin, p. 49, Exercise 2.1 # 3

Problem 6. Shifrin, p. 49, Exercise 2.1 # 5

Level II Problems.

Math 4000 students: submit 2 of these 5 problems. Math 6000 students: submit 3 of these 5 problems.

Problem 21. Show for any finite set S , its power set $\mathcal{P}(S)$ is a Boolean ring. (It is also true for infinite sets S , but you need not prove this.)

Problem 22. Shifrin, p. 41, Exercise 1.4 # 16(b).

Problem 23. Shifrin, p. 41, Exercise 1.4 # 20

Problem 24. Let p be prime. For $q \in \mathbb{Q}$, write $q = \frac{a}{b}p^n$, where p does not divide either a or b . Then the **p-adic absolute value** on the rational numbers \mathbb{Q} is a function defined as follows:

$$\left| \frac{a}{b}p^n \right|_p = p^{-n}$$

Fix a prime p . Let q, x, y be in \mathbb{Q} . Show

(a) There is a unique way to write any nonzero $q = \frac{a}{b}p^n$ such that a and b are relatively prime and p divides neither one.

(b) $|xy|_p = |x|_p |y|_p$

(c) $|x + y|_p \leq \max(|x|_p, |y|_p)$

Problem 25. Fix a prime p . Consider the set \mathcal{P} of rational numbers whose *p-adic absolute value* (defined in problem 24) is less than or equal to one, i.e.,

$$\mathcal{P} = \{q \in \mathbb{Q} : |q|_p \leq 1\} \quad .$$

(a) Show that the integers are a proper subset of \mathcal{P} .

(b) Show that \mathcal{P} forms a commutative ring with unity. (As a subset of the rational numbers, \mathcal{P} automatically inherits the associativity and commutativity of both multiplication and addition.) You need only show that \mathcal{P} is closed under addition and multiplication, contains both the additive and multiplicative identities, and contains the additive inverses.

(c) Is \mathcal{P} a field? Why or why not?

Aside on p-adic numbers: The rational numbers are not complete; if we complete the rationals with respect to the usual Euclidean distance, we obtain a larger set, the real numbers. However, if we complete them with respect to the p-adic absolute value, we obtain another larger set, the **field of p-adic numbers**, \mathbb{Q}_p . (If you don't know what the *completion* of a space means, don't worry; if you like, you can think of it as the way of going from rationals to reals.)

The **ring of p-adic integers** is the set of p-adic numbers whose (p-adic) absolute value is less than one. This ring includes the ring \mathcal{P} discussed in problem 25.