PROF. CLARK'S MATH 2200 THIRD MIDTERM, FALL 2009

You will have 75 minutes to complete the exam. It is a closed book exam – no outside study materials of any kind are permitted. Calculators are not permitted. In order to be sure to receive full credit, please justify your final answer by a reasonable amount of work.

1) In medieval Europe, a horse-drawn caravan containing a human noblewoman is crossing the forest in the middle of the night seeking to reach a vampire castle. Unfortunately, the forest is replete with werewolves who want to attack the caravan. The vampires know this all too well, so they arrange for a retinue of horse-mounted soldiers to be sent along to protect the caravan. But what size should the retinue be? Last time, they sent out a retinue of 50 soldiers, 40% of whom were killed by the werewolves. Further, the vampire elders' long experience suggests that for each additional soldier they send out, 2% more of the soldiers will be killed by werewolves (and for each fewer soldier, 2% fewer of the soliders will be killed by werewolves). What size retinue should the vampires request so as to maximize the number of soldiers who survive to reach the castle?

(Hint: Let n be the number of soldiers. Then the number of soldiers who survive is obtained by multiplying n by the percentage chance that each soldier survives, which also depends upon n in a way that you will have to figure out.)

Solution: Following the hint, let n be the number of soliders sent out. Then the percentage chance that a soldier survives is equal to $(60 - 2(n - 50))\%$, or $\frac{160-2n}{100}$. Therefore the number of surviving soldiers is

$$S(n) = n \cdot \frac{160 - 2n}{100} = \frac{1}{100}(160n - 2n^2).$$

Thus

$$S'(n) = \frac{1}{100}(160 - 4n),$$

so $S'(n) = 0$ if and only if $160 - 4n = 0$, or $n = 40$. Since the function $S(n)$ to be maximized is just a downward opening parabola, its maximum value occurs at its unique critical point, namely at $n = 40$. Thus the vampires should request a retinue of 40 soldiers.

For one point of extra credit: To what film does this question allude?

Answer: Underworld: Rise of the Lycans.

2) Bender B. Rodriguez is given a girder which is 10 meters in length. His task is to bend the girder into an L shape so as to minimize the distance between the two tips of the girder. At what point x, with $0 \leq x \leq 10$, should he bend the girder, and what is this minimum distance?

Solution: If we bend the girder at a point x, $0 \leq x \leq 10$, the two pieces of the girder form the legs of a right triangle, with lengths x and $10 - x$. The distance between the endpoints is the length of the hypotenuse of this right triangle, so is $D(x) = \sqrt{x^2 + (10 - x)^2}$. Thus we seek to minimize $D(x)$ on the interval $[0, 10]$. Note that $x = 0$ and $x = 10$ corresponding to not bending the girder at all, so (as you can check!) $D(0) = D(10) = 10$. To find critical points, we take $D'(x)$ and set it equal to 0:

$$D'(x) = \frac{2x - 2(10 - x)}{2\sqrt{x^2 + (10 - x)^2}} = 0.$$

As usual, a fraction equals zero exactly when its numerator equals 0, so we set

$$2x - 2(10 - x) = 2x + 2x - 20 = 0$$

and get $x = 5$. Since $D(5) = 5\sqrt{2} < 5\sqrt{4} = 10$, this must be the absolute minimum value. In other words, the answer is to bend the girder exactly in the middle (one might have expected this), and the minimum distance is $5\sqrt{2}$.

For one point of extra credit: To what television show does this question allude?

Answer: Futurama.

Remark: I will give everyone two extra points: it didn't seem reasonable to reward those who have recently watched mediocre fantasy/horror movies and/or fantastic sci-fi cartoons over those who haven't.

In each of the remaining problems, sketch the graph of the function given, including all critical points, inflection points and asymptotes.

3) $f(x) = x^3 - 12x$.

Solution: Step 1: The domain is $(-\infty, \infty)$.

Step 2: Since f is defined and continuous on $(-\infty, \infty)$, there are no vertical asymptotes. Since $\lim_{x \to \infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$, there are no horizontal asymptotes.

Step 3: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$.
This is equal to 0 at $x = -2$ and at $x = 2$. So f' is positive on $(-\infty, -2)$, negative on $(-2, 2)$ and positive again on $(2, \infty)$: therefore f is increasing on $(-\infty, -2)$ and on $(2, \infty)$ and decreasing on $(-2, 2)$, and there is a local maximum at $x = -2$ and a local minimum at $x = 2$.

Step 4: $f''(x) = 6x$, so it is positive, negative or 0 exactly when x is. Thus f is concave down on $(-\infty, 0)$, concave up on $(0, \infty)$ and has an inflection point at 0.

Step 5: I can't bring myself to scan in drawn pictures, so I can't show you the graph, but is a standard "two-humped cubic curve".

4) $f(x) = \frac{x-1}{x+2}$.

Solution: Step 1: The domain is all real numbers except $x = -2$.

Step 2: We have a vertical asymptote at $x = -2$. Moreover,

$$lim_{x \to -2^-} \frac{x-1}{x+2} = \frac{-3}{0^-} = +\infty,$$

$$\lim_{x \to 2^+} \frac{x-1}{x+2} = \frac{-3}{0^+} = -\infty.$$

Also, since $\lim_{x \to \pm\infty} \frac{x-1}{x+2} = 1$, we have a "two-sided" vertical asymptote at $y = 1$: the function approaches the line $y = 1$ as x gets either very large or very small. Note however that the equation $f(x) = 1$ entails $x - 1 = x + 2$, which has no solution, so the graph of f never *crosses* the asymptote $y = 1$. (This is useful when drawing the graph. It is also possible for the function to cross its horizontal asymptote one or more times: wait for it!)

Step 3: $f'(x) = \frac{x+2-(x-1)}{(x+2)^2} = \frac{1}{(x+2)^2}$.
Thus $f'(x)$ is never equal to zero, so there are no horizontal tangent lines. Moreover, since $(x+2)^2$ is always positive (except for $x = -2$, where the function is not defined), we have $f'(x)$ for all x in the domain: thus f is increasing on $(-\infty, -2)$ and again on $(-2, \infty)$.

Step 4: We compute $f''(x)$. For this, perhaps it's easiest to rewrite $f'(x) = (x+2)^{-2}$ and apply the power rule: $f''(x) = \frac{-2}{(x+2)^3}$. Again this is never zero, so there are no inflection points, but since $(x + 2)$ is now cubed, it is negative when $x < -2$ and positive when $x > -2$. Keeping in mind the minus sign in the numerator, this means that $f''(x)$ is positive on $(-\infty, -2)$ – so f is concave up there – and is negative on $(-2, \infty)$ – so f is concave down there.

Step 5: The graph is a *rectangular hyperbola*, i.e., just a translated copy of the graph of $y = \frac{1}{x}$.

5) $f(x) = \frac{x}{x^2+1}$. You may take as given that

$$f'(x) = \frac{-(x-1)(x+1)}{(x^2+1)^2}, \quad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Solution: Step 1: The domain is $(-\infty, \infty)$, since the denominator is never 0.

Step 2: Since f is defined and continuous on $(-\infty, \infty)$ there cannot be any vertical asymptotes. However, since f is a rational function whose denominator has larger degree (2) than the degree of the numerator (1), we have

$$\lim_{x \to \pm\infty} f(x) = 0,$$

so $y = 0$ is a "two-sided horizontal asymptote". Note that this time the graph does cross the asymptote $y = 0$, at $x = 0$.

Step 3: The hard work has been done for us, so just by looking carefully at the expression for f' we can see that we have critical points at $x = 1$, $x = -1$, and that f' is negative on $(-\infty, -1)$, positive on $(-1, 1)$ and negative on $(1, \infty)$, so f is decreasing on $(-\infty, -1)$, increasing on $(-1, 1)$ and decreasing again on $(1, \infty)$. So we have a local min at $x = -1$ and a local min at $x = 1$.

Step 4: We have inflection points at $x = 0$, $x = -\sqrt{3}$ and $x = \sqrt{3}$, so f'' changes sign across each of these points. When $x < -\sqrt{3}$, $f''(x)$ is negative, so we get: concave down on $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, concave up on $(\sqrt{3}, \infty)$.

Step 5: Again, I must leave the sketch to you. But two comments: first, f is an odd function, so its graph should have symmetry about the origin. Second, consideration of the graph shows that our local min at $x = -1$ is in fact an *absolute minimum*, and similarly we have an *absolute maximum* at $x = 1$.