PRACTICE PROBLEMS FOR 2200 MIDTERM EXAM 1

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1) Give the definition of the derivative of a function f at the point x.

Solution:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2) Give an equation for the tangent line to $y = f(x)$ at the point $x = a$.

Solution:
$$y = f'(a)(x - a) + f(a).$$

3) Give a definition (in terms of limits) of continuity of f at at the point x.

Solution: Let us call the point $x = c$. Then, f is continuous at x if $\lim_{x \to c} f(x) = f(c)$. In particular, f must be defined at c and the limit must exist.

4) Give a definition of "removable discontinuity".

Solution: A function has a removable discontinuity at the point $x = c$ if f is not continuous at c but $\lim_{x \to c} f(x)$ exists. This means that either f is not defined at c or $\lim_{x \to c} f(x) \neq f(c)$.

5) State the Intermediate Value Theorem.

Solution: Let f be a continuous function defined on the closed interval $[a, b]$. Suppose that L is a number which lies in between $f(a)$ and $f(b)$. Then there exists a number c, with $a < c < b$, such that $f(c) = L$.

6) For each of the following, mark as true or false, and give brief explanations.
a) A polynomial function $P(x)$ is continuous at every point at which it is defined.

Solution: True. (Any polynomial is built up out of continuous functions $f(x) = x$ and $f(x) = C$ (constant) by addition and multiplication.)

b) A rational function $\frac{P(x)}{Q(x)}$ is continuous at every point at which it is defined.

Solution: Also true. A quotient of two continuous functions is continuous except where the denominator is zero, in which case the function is not defined.

c) For every rational function $\frac{P(x)}{Q(x)}$, $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ exists.

Solution: False. The limit may be infinite.

d) For every rational function $\frac{P(x)}{Q(x)}$ for which the numerator and the denominator are polynomials of the same degree, $\lim_{x\to\infty}\frac{P(x)}{Q(x)}$ exists and is nonzero.

Solution: True. This follows from the method of factoring out the largest power of x from both the numerator and the denominator.

e) If rational function $\frac{P(x)}{Q(x)}$ has a vertical asymptote at $x = c$, then $Q(c) = 0$.

Solution: True. If $Q(c)$ were not equal to 0, then the function would be continuous at that point and take a finite vlaue there.

f) If $Q(c) = 0$, then the rational function $\frac{P(x)}{Q(x)}$ has a vertical asymptote at $x = c$.

Solution: False (i.e., not necessarily true). For example $f(x) = \frac{x^2+x}{3x^2-4x}$. Then $Q(0) = 0$, but also $P(0) = 0$, and for $x \neq 0$, $f(x) = \frac{x+2}{3x-4}$, which is continuous at 0. Thus the original function has a removable discontinuity at $x = 0$, not a vertical asymptote there.

g) It is possible for a function to be differentiable at x but not be continuous at x.

Solution: False. We proved this in class.

h) It is possible for a function to be continuous at x but not differentiable at x.

Solution: True, e.g. $f(x) = |x|$ is continuous everywhere but not differentiable at $x = 0$. (It is even possible for a function to be defined and continuous on all real numbers but not to be differentiable at *any* real number. Such functions were discovered in the late 19th century.)

7) Let $y = f(x) = 7x^2 + 5x - 11$.
a) Compute the derivative y' and the second derivative y''.

Solution: We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{7(x+h)^2+5(x+h)-11-(7x^2+5x-11)}{h}$$

$$= \lim_{h\to 0}\frac{7x^2+14xh+7h^2+5x+5h-11-7x^2-5x+11}{h} = \lim_{h\to 0}\frac{14xh+7h^2+5h}{h}$$

$$= \lim_{h\to 0} 14x+7h+5 = 14x+5.$$

To find the second derivative, we set $g(x) = f'(x) = 14x+5$. Then $f''(x) = g'(x) =$

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{14(x+h)+5-(14x+5)}{h} = \lim_{h\to 0}\frac{14h}{h} = \lim_{h\to 0} 14 = 14.$$

b) Compute the equation of the tangent line to $y = f(x)$ at a general point $x = a$.
Solution: It is $y = f'(a)(x-a) + f(a) = (14a+5)(x-a) + (7a^2+5a-11)$.

c) Find all points a such that the tangent line to $y = f(x)$ is horizontal.

Solution: The tangent line is horizontal at a if and only if $f'(a) = 0$. Thus:

$$f'(a) = 14a + 5 = 0,$$

getting the solution $a = \frac{-5}{14}$.

8) Evaluate the following limits (or show that they do not exist):
a) $\lim_{x\to 2} \frac{x^2+4x-12}{x-2}$.

Solution:

$$\lim_{x\to 2} \frac{x^2+4x-12}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+6)}{x-2}$$
$$= \lim_{x\to 2} x+6 = 2+6 = 8.$$

b) $\lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$.

Solution:

$$\lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = 2 \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)} \cdot \frac{1}{h}$$
$$= 2 \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-2}{x^2}.$$

Note that evaluating this limit shows that the derivative of $f(x) = \frac{2}{x}$ is $f'(x) = \frac{-2}{x^2}$.

c) $\lim_{x\to 0} \frac{\sin 3x(2-2\cos x)}{\sin 11x}$.

Solution: For any $a \neq 0$, we have $\lim_{x\to 0} \frac{\sin(ax)}{ax} = 1$. From this it follows that whenever we are evaluating a limit as $x \to 0$, any expression $\sin ax$ can be replaced by ax. Using this trick we get

$$\lim_{x\to 0} \frac{\sin 3x(2-2\cos x)}{\sin 11x} = \lim_{x\to 0} \frac{(3x)(2-2\cos x)}{11x}$$
$$= \frac{3}{11} \lim_{x\to 0} 2 - 2\cos x = \frac{3}{11}(2-2\cos 0) = \frac{3}{11}(0) = 0.$$

9) For which values of x are the following functions continuous?

a) $f(x) = \frac{4x}{\sqrt{49-x^2}}$.

Solution: In order for $f(x)$ to be defined, we need $49 - x^2 \geq 0$, so $x^2 \leq 49$, or $-7 \leq x \leq 7$. Moreover we need $\sqrt{49-x^2} \neq 0$ (otherwise we are dividing by 0), so the domain of f is in fact the open interval $(-7, 7)$. On this domain f is continuous.

b) $f(x) = (x-8)^{\frac{1}{3}}$.

Solution: $f(x)$ is defined and continuous for all real numbers. (Note that we can take the cube root, or any odd-numbered root, of any real number, whereas we can only take the square root, or any even-numbered root, of a non-negative number.)

c) $f(x) = |\sin x|$.

Solution: f is defined and continuous for all real numbers.

10) Find an integer n such that the polynomial $f(x) = x^4 - 4x^3 + 2x^2 + 2$ has at least one root c with $n \leq c \leq n + 1$.

Solution: We have $f(1) = 1$, $f(2) = -6$. Since f is continuous, $f(1)$ is positive and $f(2)$ is negative, by the Intermediate Value Theorem f must have a root between 1 and 2.

11) Show that there exists a real number x such that $2^x = \cos x$.

Solution: In fact we can take $x = 0$: $2^0 = \cos 0 = 1$.

A better problem would have been: show that there exists a real number x such that $2^x = \sin x$. For this, we define $f(x) = 2^x - \sin x$. Since $f(x)$ is continuous, by the Intermediate Value Theorem it suffices to find numbers a and b such that $f(a) > 0$ and $f(b) < 0$. We have $f(0) = 2^0 - \sin 0 = 1 - 0 = 1 > 0$ and $f(-3\pi/2) = 2^{-3\pi/2} - \sin(-3\pi/2) = 2^{-3\pi/2} - 1$. Since $2^x < 1$ for all $x < 0$, the expression $2^{-3\pi/2} - 1$ is negative. Thus, by IVT there exists a solution which is in between $-3\pi/2$ and 0.

12) Sketch a graph of each of the following functions.
a) $f(x) = \sin x$.
b) $f(x) = |\sin x|$.
c) $f(x) = x \sin x$.
d) $f(x) = |x \sin x|$.

Solution: Try this for yourself!