PRACTICE PROBLEMS FOR 2200 MIDTERM EXAM 2

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1) Give the definition of the derivative of a function f at the point x.

Solution: It is (still)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2) State the Extreme Value Theorem.[1]

Solution: Let f be a continuous function defined on the closed interval $[a, b]$. Then f has a maximum value: that is, there exists a real number M such that: (i) there is at least one x in $[a, b]$ such that $f(x) = M$, and (ii) for all x in $[a, b]$, $f(x) \leq M$. Similarly, f has a minimum value: there exists a real number m such that: there is at least one x in $[a, b]$ such that $f(x) = m$, and (ii) for all x in $[a, b]$, $f(x) \geq m$.

3) In each of the following problems, find $\frac{dy}{dx}$.

a) $y = (x^2 + 4x)^{5/2}$.

Solution: $\frac{dy}{dx} = \frac{5}{2}(x^2 + 4x)^{3/2} \cdot (2x + 4)$.

b) $y = \frac{2+3x}{e^{4x}}$.

Solution: $\frac{dy}{dx} = \frac{e^{4x}(3) - (2+3x)(4e^{4x})}{(e^{4x})^2} = \left(e^{4x}\right) \frac{3-8-12x}{e^{8x}} = \frac{-5-12x}{e^{4x}}$.

Alternate Solution: rewrite y as $y = (2 + 3x)e^{-4x}$ and apply the product rule:

$$\frac{dy}{dx} = 3e^{-4x} + (2 + 3x)(e^{-4x})' = 3e^{-4x} - 4(2 + 3x)e^{-4x} = (-5 - 12x)e^{-4x}.$$

c) $y = \sin(2 \cos 3x)$.

Solution:

$$\frac{dy}{dx} = \cos(2 \cos 3x) \cdot (2 \cos 3x)' = \cos(2 \cos 3x)\,(2 \cdot (-\sin(3x))(3x)')$$

$$= -6 \sin(3x) \cos(2 \cos 3x).$$

d) $y = \ln(xe^{x^2})$.

[1] Your textbook calls it the "Maximum and Minimum Value Property".

Solution: First simplify the function:

$$y = \ln(xe^{x^2}) = \ln x + \ln(e^{x^2}) = \ln x + x^2,$$

so

$$\frac{dy}{dx} = \frac{1}{x} + 2x.$$

Alternate Solution: By brute force,

$$\frac{dy}{dx} = \frac{(xe^{x^2})'}{xe^{x^2}} = \frac{1 \cdot e^{x^2} + xe^{x^2}(x^2)'}{xe^{x^2}} =$$

$$\frac{e^{x^2}(1+2x^2)}{xe^{x^2}} = \frac{1+2x^2}{x} = \frac{1}{x} + 2x.$$

e) $y = \cot x$.

Solution: Recall $\cot x = \frac{\cos x}{\sin x}$, so

$$\frac{dy}{dx} = \frac{(\sin x)(\cos x)' - (cosx)(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

f) $y = (\cos x)(\ln x)e^x$.

Solution: We use the product rule with three factors, which has the general form:

$$(fgh)' = f'gh + fg'h + fgh'.$$

so

$$\frac{dy}{dx} = (-\sin x)(\ln x)(e^x) + (\cos x)(\frac{1}{x})(e^x) + (\cos x)(\ln x)(e^x).$$

4) In each of the following problems, find $\frac{dy}{dx}$ by logarithmic differentiation.

General strategy of logarithmic differentiation: First we take the logarithm and then simplify, then we differentiate both sides, and finally we multiply back by y to obtain a formula for $\frac{dy}{dx}$.

a) $y = \left(\frac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right)^{\frac{1}{3}}$.

$$\ln y = \frac{1}{3}\left(\ln(x+1) + \ln(x+2) - \ln(x^2+1) - \ln(x^2+2)\right).$$

$$\frac{\frac{dy}{dx}}{y} = \frac{d}{dx}\ln y = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2}\right),$$

so

$$\frac{dy}{dx} = y \cdot \left(\frac{d}{dx}\ln y\right) = \frac{1}{3}\left(\frac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right)^{\frac{1}{3}}\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2}\right).$$

b) $y = x^{(e^x)}$.

Solution:

$$\ln y = \ln x^{(e^x)} = e^x \ln x,$$

so

$$\frac{d}{dx} \ln y = e^x \ln x + e^x \frac{1}{x}$$

and

$$\frac{dy}{dx} = y \cdot \left(\frac{d}{dx} \ln y \right) = x^{(e^x)} \left(e^x \ln x + e^x \frac{1}{x} \right).$$

5) A rocket is launched vertically upward from a point 3 miles west of an observer on the ground. What is the speed of the rocket when the angle of elevation (from the horizontal) of the observer's line of sight to the rocket is 47 degrees and is increasing at 3.5 degrees per second? (Hint: remember to convert from degrees to radians: $180^\circ = \pi$ radians.)

Solution: Let $y(t)$ be the rocket's distance from the ground at time t. The quantity that we are trying to find is $\frac{dy}{dt}$. There is a right triangle, whose vertical leg is the distance y from the rocket to the ground, whose horizontal leg is 3 miles, and whose base acute angle is θ. Thus $\tan\theta = \frac{y}{3}$. Differentiating both sides with respect to t gives

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dy}{dt},$$

so

$$\frac{dy}{Dt} = 3\sec^2\theta \frac{d\theta}{dt}.$$

We are ready to plug everything in, but we must do this carefully, i.e., with the correct units. Especially, our formulas for the derivatives of the trigonometric functions presuppose radian measure, so we convert

$$47^\circ = 47^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{47\pi}{180}\text{radians}$$

and

$$\frac{d\theta}{dt} = \frac{3.5^\circ}{\text{sec}} = \frac{3.5^\circ}{\text{sec}} \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{3.5\pi}{180}\frac{\text{radians}}{\text{sec}}.$$

Thus

$$\frac{dy}{dt} = (3\text{miles}) \cdot \left(\frac{3.5\pi}{180}\frac{\text{radians}}{\text{sec}} \right) \cdot \left(\frac{1}{\cos^2(\frac{47\pi}{180})\text{radians}} \right).$$

Without a calculator, we should just leave the answer like this. If we had a calculator, we could compute that it is approximately 0.394 miles per second, or $0.394 \cdot 3600 = 1418.4$ miles per hour.

6) Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute. It forms a pile in the shape of a right circular cone whose base diameter and height are always the same. How fast is the height of the pile increasing when the pile is 18 feet high? Recall that the volume of a right circular cone with height h and base radius r is $\frac{1}{3}\pi r^2 h$.

Solution: We have $V = \frac{1}{3}\pi r^2 h$. Before differentiating with respect to t, we should eliminate one of the variables. Since the base diameter is twice the base radius, we have $h = 2r$, or $r = \frac{h}{2}$. Thus

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}\frac{h^2}{4}h = \frac{\pi}{12}h^3.$$

So

$$\frac{dV}{dt} = \frac{3\pi}{12}h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

and

$$\frac{dh}{dt} = \frac{\frac{dV}{dT}}{\frac{\pi}{4}h^2} = \frac{4\ \frac{dV}{dt}}{\pi h^2}.$$

Plugging in $\frac{dV}{dt} = 30$ and $h = 18$, we get

$$\frac{dh}{dt} = \frac{4 \cdot 30}{\pi \cdot 18^2}\frac{\text{feet}}{\text{minute}}.$$

Again, without a calculator we should leave it like this. The numerical answer happens to be approximately 0.118 feet per minute.

7) Let $f(x) = \frac{4x}{x^2+1}$ be defined on the interval $[-4, 0]$. Find the absolute maximum and absolute minimum values of f.

Solution: We first plug in the endpoints: $f(-4) = \frac{4(-4)}{(-4)^2+1} = \frac{-16}{17}, f(0) = 0$. Notice that for any $x < 0$, the numerator, $4x$, is negative, whereas for any real x the denominator, $x^2 + 1$, is positive. Therefore for all $x < 0$, $f(x)$ is negative, so the maximum value *must* be $y = 0$, occurring only at the right endpoint $x = 0$. It remains to find the minimum value. For this, notice that f is differentiable on its entire domain (and indeed for all real numbers), so that there are no "Type III" points. It suffices then to find the stationary points: i.e., interior points x for which $f'(x) = 0$. We compute:

$$f'(x) = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4 - 4x^2}{(x^2+1)^2} = 0.$$

As always, a fraction $\frac{A}{B}$ is equal to 0 iff its numerator, A equals 0. So we get

$$4 - 4x^2 = 0,$$

which has solutions $x = \pm 1$. Only $x = -1$ lies in the interval $[-4, 0]$, so we throw out $x = +1$ and plug in $x = -1$:

$$f(-1) = \frac{4(-1)}{(-1)^2 + 1} = -2.$$

Certainly $-2 < \frac{-16}{17}$, so -2 is the minimum value, occurring at $x = -1$.

8) Let $g(x) = |x^2 - 5|$ be defined on $[-8, 9]$. Find the absolute maximum and minimum values of g.

Solution: First we test the endpoints:

$$g(-8) = |(-8)^2 - 5)| = 59, \;\; g(9) = |9^2 - 5| = 76.$$

Next, recall from our in class discussion that in the case of $g(x) = |f(x)|$ where $f(x)$ is differentiable, the critical points are where $f'(x) = 0$ – since this also implies

$g'(x) = 0$ – and where $f(x) = 0$ – since, if $f'(x) \neq 0$, the derivative of $|f(x)|$ will not exist. Here $f(x) = x^2 - 5$, so $f'(x) = 2x$. Thus $f'(x) = 0$ when $x = 0$, and $f(x) = 0$ when $x = \pm\sqrt{5}$. Plugging in all these points, we get $f(0) = 5$, $f(\pm\sqrt{5}) = 0$. Thus the minimum y-value is $y = 0$, occurring at $x = \pm\sqrt{5}$, and the maximum y-value is 76, occurring at $x = 9$.

9) Find the tangent line to $3x^2 + e^x \sin y = 2x + y + 8$ at the point $(x, y) = (2, 0)$.

Solution: Take the derivative of both sides with respect to x:

$$6x + e^x \sin y + e^x \cos y \frac{dy}{dx} = 2 + \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$6x + e^x \sin y - 2 = (1 - e^x \cos y)\frac{dy}{dx},$$

$$\frac{dy}{dx} = \frac{6x + e^x \sin y - 2}{1 - e^x \cos y}.$$

Plugging in $(x, y) = (2, 0)$, we get

$$\frac{dy}{dx} = \frac{6(2) + e^2 \sin 0 - 2}{1 - e^2 \cdot \cos 0} = \frac{10}{1 - e^2}.$$

This is the slope of the tangent line, so its equation is

$$y - 0 = \frac{10}{1 - e^2}(x - 2).$$