PRACTICE PROBLEMS FOR 2200 MIDTERM EXAM 3

PETE L. CLARK

1) You have 12 meters of wire and a pair of wire cutters. You will make at most one cut in the wire, bend one piece into a square, and bend the remaining wire into a circle. You wish to do this so as to maximize the sum of the areas of the square and the circle. (To be clear, you are allowed not to cut the wire at all and bend it entirely into a square or entirely into a circle.) What are the lengths of the pieces of wire that maximize the area?

Solution: Suppose we cut the wire x meters from the left, for some $0 \leq x \leq 12$. Then we have x meters of wire to bend into a square, and $12 - x$ meters of wire to bend into a circle. Thus x is the perimeter of the square, so its side length is $\frac{x}{4}$ and its area is $\frac{x^2}{16}$. Similary, $12 - x$ is the circumference of the circle, so its radius is $\frac{12-x}{2\pi}$ and its area is $\pi(\frac{12-x}{2\pi})^2 = \frac{(x-12)^2}{4\pi}$. Thus the function we wish to maximize is

$$A(x) = \frac{x^2}{16} + \frac{(x-12)^2}{4\pi}$$

on the interval $[0, 12]$. First we test the endpoints:

$$A(0) = \frac{36}{\pi},$$

$$A(12) = 9 = 36/4.$$

Note that since $\pi < 4$, $A(0) > A(12)$. Now we look for interior critical points:

$$A'(x) = 8x + \frac{2(x-12)}{4\pi} = 8x + \frac{x-12}{2\pi} = (8 + \frac{1}{2\pi})x - \frac{6}{\pi}.$$

Despite the fact that this is a linear equation, it is a little messy to solve exactly for x and plug back into the function. (If you had a calculator it wouldn't be a problem, but you won't on the exam, so let's discuss alternate approaches.) In fact the function $A(x)$ that we're trying to maximize is visibly an upward opening parabola, so it's one critical point has to be a global minimum. A fancier way of rederiving the same thing is to use the second derivative test: $A''(x) = (+\frac{1}{2\pi}) > 0$ for all x, so the critical point is at least a local min. Therefore the maximum value occurs at the larger of the two endpoint values, which we saw above is $A(0) = \frac{36}{\pi}$.

Interpretation: the best thing to do is not cut the wire at all but bend it entirely into a circle!

2) You need to build a tin can (including the top and bottom) with a volume of 500 cubic centimeters. Suppose tin costs 1 cent per square centimeter. What are the dimensions of the can that minimize the cost?

Solution: The tin can will be a right circular cylinder, say with base radius r

and height h. Then its total surface area is equal to the sum of the areas of its top and bottom disks – which each contribute πr^2 – plus the lateral surface area, which is constructed from taking a rectangle with width $2\pi r$ and height h and soldering the two ends together. Thus the total surface area is

$$A = 2\pi r^2 + 2\pi r h.$$

This is not yet a function of one variable, so we need to do something else before taking the derivative. As usual, the "something else" is writing down an auxiliary *constraint equation* and using that equation to eliminate one of the two variables. Here we know that the volume of the cylinder is $\pi r^2 h = 500$, so $h = \frac{500}{\pi r^2}$. Substituting this into the area function, we get

$$A(r) = 2\pi r^2 + 2\pi r \left(\frac{500}{\pi r^2}\right) = 2\pi r^2 + \frac{1000}{r}.$$

Here we may have $0 < r < \infty$: nothing stops us from building a tall, skinny can or a very short, stubby can except that the surface area will be very large. In the language of calculus, we have

$$\lim_{r \to 0^+} A(r) = \lim_{r \to \infty} A(r) = \infty.$$

Let's compute the derivative and set it equal to zero:

$$A'(r) = 4\pi r - \frac{1000}{r^2} = 0.$$

Multiplying through by r^2 gives

$$4\pi r^3 - 1000 = 0, \quad \text{so} \quad r = (\frac{250}{\pi})^{1/3}.$$

Looking at A', we see that it is negative to the left of this one critical point and positive to the right of it. Therefore the function has a global minimum at $r = (\frac{250}{\pi})^{1/3}$. If this is the value of r, then $h = \frac{500}{\pi r^2} = \frac{500}{\pi \cdot (\frac{250}{\pi})^{2/3}}$.

3) You own a parking lot in Five Points and sell the spots for every home game. Last weekend, you charged \$ 15 per space and sold out all 30 spaces, so this week you have decided to increase the price. Informal research suggests that for every dollar you increase the price per space, you will leave one more space vacant. What price should you charge so as to maximize your profit?

Solution: Let n be the total number of dollars that we charge for a parking space, so $n \geq 15$. Then the number of spaces that we can sell at n dollars apiece is $30 - (n - 15) = 45 - n$, so our profit (which is equal to our revenue here: there's no cost to us!) is

$$P(n) = n \cdot (45 - n) = -n^2 + 45n.$$

Thus

$$P'(n) = -2n + 45 = 0, \text{ or } n = 22.5.$$

It follows that we should charge \$ 22.5 dollars per space.

Alternate solution: To show that it doesn't matter exactly what we do, let's solve it again letting a be the number of additional dollars that we charge for a space.

Then the number of spaces we can sell is $30 - a$ and the price per space is $15 + a$, so the total profit is

$$P(a) = (30 - a)(15 + a) = -a^2 + 15a + 450.$$

Thus

$$P'(a) = -2a + 15 = 0, \text{ or } n = 7.5.$$

Thus we should charge 7.5 *more* dollars than last time, for a total of \$ 22.5 per space.

In each of the following problems, graph the function. Your answer should take into account the following information:

(i) The domain of the function. (If not specified, assume the domain is the largest possible on which the function makes sense.)
(ii) Horizontal and vertical asymptotes, if any.
(iii) Regions on which the function is increasing and on which the function is decreasing.
(iv) Local and global maxima and minima, if any.
(v) concavity and inflection points.

4) $f(x) = \frac{4x}{x^2+1}$.

5) $f(x) = x^4 - 4x^3$.

6) $f(x) = (2 - x)e^x$.

7) $f(x) = \frac{-3}{x^2-4}$.