MATH 3200 SECOND MIDTERM EXAM

Directions: Do all five problems. Always justify your reasoning completely. No calculators are permitted (nor would they be helpful in any way that I can see).

1) a) State the principle of mathematical induction.

Solution: Let $P(n)$ be a statement with domain $\mathbb{Z}^+$. Suppose that:
(i) $P(1)$ holds, and
(ii) for all $n \in \mathbb{Z}^+$, $P(n) \implies P(n+1)$.
Then, for all $n \in \mathbb{Z}^+$, $P(n)$ holds.

b) State the principle of strong/complete induction.

Solution: Let $P(n)$ be a statement with domain $\mathbb{Z}^+$. Suppose that:
(i) $P(1)$ holds, and
(ii) for all $n \in \mathbb{Z}^+$, $P(1) \wedge \ldots \wedge P(n) \implies P(n+1)$.
Then, for all $n \in \mathbb{Z}^+$, $P(n)$ holds.

2) Prove or disprove:
a) There exist nonzero rational numbers a and b such that a^b is irrational.

Solution: This is true: take $a = 2$, $b = \frac{1}{2}$, so $a^b = 2^{\frac{1}{2}} = \sqrt{2}$.

b) For all nonzero rational numbers a and b, a^b is irrational.

Solution: This is false: take $a = b = 1$, so $a^b = 1^1 = 1$.

3) a) Let a be a real number. Prove that if a^2 is irrational, then a is irrational.

Solution: We prove the contrapositive: if a is rational, then a^2 is rational. Indeed, if $a = \frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $y \neq 0$, then $a^2 = \frac{x^2}{y^2}$ with $x^2, y^2 \in \mathbb{Z}$ and $y^2 \neq 0$.

b) Prove that $\sqrt{77}$ is irrational.
(Hint: you may use Euclid's Lemma: if a prime p divides ab, then $p \mid a$ or $p \mid b$.)

Solution: Seeking a contradiction, we suppose that $\sqrt{77}$ is rational. Since $\sqrt{77} > 0$, this means there exist positive integers a and b, with no common factor greater than 1, such that $\sqrt{77} = \frac{a}{b}$. Squaring both sides gives $77 = \frac{a^2}{b^2}$ and then $77b^2 = a^2$. Thus 7 divides a^2. Since 7 is a prime, by Euclid's Lemma, $7 \mid a$. Put $a = 7A$, so

$$77b^2 = a^2 = (7A)^2 = 49A^2,$$

which implies

$$11b^2 = 7A^2.$$

Thus $7 \mid 11b^2$. By Euclid's Lemma $7 \mid 11$, $7 \mid b$ or $7 \mid b$. The first alternative is manifestly false, so we must have $7 \mid b$. Thus a and b are both divisible by 7, contradicting the assumption that they have no common factor greater than one.

c) Prove that $\alpha = \sqrt{7} + \sqrt{11}$ is irrational. (Hint: use parts a) and b).)

Solution: By part a), it suffices to prove that α^2 is irrational. Suppose for a contradiction that $\alpha^2 = \frac{a}{b}$, for positive integers a and b. Then

$$\frac{a}{b} = \alpha^2 = (\sqrt{7} + \sqrt{11})^2 = 7 + 2\sqrt{77} + 11 = 18 + 2\sqrt{77}.$$

Thus

$$\sqrt{77} = \frac{\frac{a}{b} - 18}{2} = \frac{a - 18b}{2b},$$

so that $\sqrt{77}$ is rational. This contradicts part b).

4) Prove that for all $n \in \mathbb{N}$ (i.e., for all non-negative integers!), 5 divides $3^{2n} - (-1)^n$.

Solution: We go by induction on n.
Base case: $n = 0$: $3^{2\cdot 0} - (-1)^0 = 1 - 1 = 0$, which is divisible by 5.

Inductive Step: Assume that for $n \in \mathbb{N}$, 5 divides $3^{2n} - (-1)^n$. Then

$$3^{2n+2} - (-1)^{n+1} = 3^{2n}3^2 + (-1)^n = 9 \cdot 3^{2n} + (-1)^n$$
$$= (10 - 1) \cdot 3^{2n} + (-1)^n = 10 \cdot 3^{2n} - \left(3^{2n} - (-1)^n\right).$$

The first term is certainly divisible by 5, and by induction, the second term is also divisible by 5, so the entire expression is divisible by 5.

5) Define a sequence of natural numbers by:

$$x_0 = 2,\ x_1 = 5, \forall n \geq 1,\ x_{n+1} = 5x_n - 6x_{n-1}.$$

Prove that for all $n \in \mathbb{N}$, $x_n = 2^n + 3^n$.

Proof: We go by strong/complete induction on n.

Base cases: $n = 0$: $2^0 + 3^0 = 1 + 1 = 2 = x_0$.
$n = 1$: $2^1 + 3^1 = 5 = x_1$.

Inductive step: let $n \in \mathbb{Z}^+$ and assume that for all $0 \leq k \leq n$ we have $x_k = 2^k + 3^k$. Especially, we assume that $x_{n-1} = 2^{n-1} + 3^{n-1}$ and that $x_n = 2^n + 3^n$. Then

$$x_{n+1} = 5x_n - 6x_{n-1} = 5(2^n + 3^n) - 6(2^{n-1} + 3^{n-1})$$
$$= 5 \cdot 2^n - 3 \cdot 2^n + 5 \cdot 3^n - 2 \cdot 3^n = 2 \cdot 2^n + 3 \cdot 3^n = 2^{n+1} + 3^{n+1}.$$