

MATH 3200 PRACTICE PROBLEMS 1

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In all of the following questions, let x, y, z be objects and A, B, C be sets.

1) Let A and B be sets.

a) What is the meaning of $A = B$? Of $A \subset B$? Of $A \subsetneq B$?

Solution: $A = B$ means that the sets A and B have exactly the same elements: any object x is an element of A if and only if it is an element of B . $A \subset B$ means that every element of A is an element of B . $A \subsetneq B$ means that every element of A is an element of B and there exists at least one element of B which is not an element of A .

b) What is the meaning of $A \cup B$? Of $A \cap B$? Of $A \setminus B$?

Solution: $A \cup B$ is the set of all elements x such that $x \in A$ or $x \in B$ (or both). $A \cap B$ is the set of all elements x such that $x \in A$ and $x \in B$. $A \setminus B$ is the set of all elements x such that $x \in A, x \notin B$.

2) Define the **symmetric difference** of A and B as $(A \setminus B) \cup (B \setminus A)$.

In this problem, the symmetric difference will be denoted as $A \Delta B$.

a) Draw a Venn diagram indicating the symmetric difference.

Solution: Two of the four regions should be shaded: the one representing elements of A which are not in B , and the one representing elements of B which are not in A .

b) Let $A = \{2k \mid k \in \mathbb{Z}^+\}$ and let B be the set of prime numbers. What is $A \Delta B$?

Solution: First, recall that by a standard (among mathematicians) convention, 1 is not regarded as a prime. Now, $A \Delta B$ is the set of all positive integers which are either even or prime but not both. Thus, it is the collection of all odd prime numbers together with the collection of all even positive integers greater than 2.

c) Show that $(A \Delta B) \cup (A \cap B) = A \cup B$.

Solution: First we show $(A \Delta B) \cup (A \cap B) \subset A \cup B$. For this, it is most efficient to assume that $x \in (A \Delta B) \cup (A \cap B) \setminus A$ and deduce that $x \in B$. Since $x \notin A$, certainly $x \notin A \cap B$, so $x \in A \Delta B$. Since, again, $x \notin A$, we conclude $x \in B \setminus A$, so $x \in A \cup B$.

Second we show that $A \cup B \subset (A \Delta B) \cup (A \cap B)$. Suppose that $x \in A$. Then, if x is also in B , $x \in A \cap B$ hence is in $(A \Delta B) \cup (A \cap B)$. Otherwise $x \in A \setminus B \subset A \Delta B \subset (A \Delta B) \cup (A \cap B)$. The case of $x \in B$ is the same – indeed,

demonstrably so, because there is a symmetry between A and B in the statement. (So it is not necessary to repeat the argument with $x \in B$, although it would be perfectly permissible to do so.)

3) a) What is meant by a universal set?

Solution: A universal set is a set which contains every other set which is being considered in the context of a certain discussion. For instance, if one is working with sets of real numbers, then \mathbb{R} is a universal set. Note that there is no true universal set – that is, there is no set X such that $x \in X$ for all objects X . b) What is meant by \overline{A} , and when is it defined?

Solution: This is the complement of A , and it is defined exactly when A is a subset of a fixed universal set X , in which case it means $X \setminus A$.

4) a) What is a partition of a set S ?

Solution: A partition of a set S is a family \mathcal{F} of subsets of S satisfying:

- (i) For all $A \in \mathcal{F}$, $A \neq \emptyset$.
- (ii) $\bigcup_{A \in \mathcal{F}} A = S$.
- (iii) If $A, B \in \mathcal{F}$ and $A \neq B$, then $A \cap B = \emptyset$.

b) Are there any sets S which have no partitions? Which sets S have exactly one partition?

Solution: No, every set admits at least one partition. We saw in class that the empty set admits the empty partition $\mathcal{F} = \emptyset$ (and it is easy to see that it admits no other partition). Any nonempty set admits a unique partition with one part, namely $\mathcal{F} = \{S\}$. If S has exactly one element, then this is the only possible partition of S . If S has at least two elements, then the partition $\mathcal{F} = \{\{x\}_{x \in S}\}$ into one-element sets is a different partition. Thus a set S has exactly one partition iff it is empty or has exactly one element.

4) Decide whether each of the following is true or false, and briefly explain.

a) Any partition of a finite set must have a finite number of elements.

Solution: True. The parts of a partition are elements of the power set $\mathcal{P}(S)$ of the set S . If S has n elements, then $\mathcal{P}(S)$ has 2^n elements, so any partition of S has at most 2^n elements, hence finitely many.

b) Any partition of an infinite set must have an infinite number of elements.

Solution: False. As we saw above, any nonempty set admits a partition with exactly one element, the set itself.

5) a) Define: implication, contrapositive, converse, inverse, biconditional.

Solution: implication: $A \implies B$. If A is true, then B is true. More precisely,

this means that A is false or B is true.

Contrapositive: $\sim B \implies \sim A$: If B is false, then A is false. More precisely, this means that A is false or B is true, hence it is equivalent to the implication.

Converse: $B \implies A$: If B is true, then A is true. More precisely, this means that B is false or A is true.

Inverse: $\sim A \implies \sim B$: if A is false, then B is false. More precisely, this means that B is false or A is true, hence it is equivalent to the inverse.

b) What is meant by the “converse fallacy”?

This is a term in logic, philosophy and rhetoric: it means confusion of $A \implies B$ with $B \implies A$. It is perhaps the most common logical mistake made in everyday reasoning.

c) Show that $A \implies B$ is logically equivalent to $\sim B \implies \sim A$ in two different ways: (i) using a truth table, and (ii) using an argument in plain English.

Solution: I did this in class, so I will leave it to you now.

6) True or false: Neither the inverse nor the converse of an implication is logically equivalent to the implication, but the inverse and the converse are logically equivalent to each other.

True. See the solution to 5).

7) Negate the following sentence: “You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time.”

The negation is: “At least one of the following holds:

(i) There exists at least one time when you can’t fool anyone, or (ii) At all times, there exists at least person that you can’t fool, or (iii) At all times, you can fool all of the people.”

8) What does it mean for an implication to hold trivially? To hold vacuously?

An implication of the form $\forall x \in S, P(x) \implies Q(x)$ holds trivially if for all $x \in S$, $Q(x)$ is true. It holds vacuously if for all $x \in S$, $P(x)$ is false.

10) For which of the following famous theorems does the converse also hold?¹

a) (Pythagorean Theorem) Let a , b and c be the side lengths of a triangle T . If T is a right triangle and c is the length of the hypotenuse, then $c^2 = a^2 + b^2$.

b) (Intermediate Value Theorem) Let $f : [a, b] \rightarrow \mathbb{R}$ be a function and let $f(a) < C < f(b)$. If f is continuous, then there exists $a < c < b$ with $f(c) = C$.

c) (Rational Roots Theorem) Let $P(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial with integer coefficients a_0, \dots, a_n , and let $r = \frac{c}{d}$ be a rational number written in

¹Moral: when you see a theorem for the first time, ask yourself whether the converse is true!

lowest terms. If $P(r) = 0$, then $d \mid a_n$ and $c \mid a_0$.

Solution: A better question would have been: first state the converse and then determine whether the converse is also true. I will answer this better question.

a) The converse is: Let a, b, c be the side lengths of a triangle T . If $c^2 = a^2 + b^2$, then T is a right triangle. This is true: for instance it follows from the law of cosines.

b) This one is tricky. As stated, the converse seems to be: let $f : [a, b] \rightarrow \mathbb{R}$ be a function and let $f(a) < C < f(b)$. If there exists c with $a < c < b$ such that $f(c) = C$, then f is continuous. This is clearly false: just because the function satisfies this property for a single value C doesn't make it continuous.

Probably what I actually intended though is the following version: say that a function $f : [a, b] \rightarrow \mathbb{R}$ has the **Intermediate Value Property** if whenever $f(x_1) \neq f(x_2)$, then for any value C in between $f(x_1)$ and $f(x_2)$ there exists at least one c , $x_1 < c < x_2$ with $f(c) = C$. Then the Intermediate Value Theorem can be stated as the fact that every continuous function has the Intermediate Value Property. (The difference is one of more explicit quantification: we require the conclusion of IVT to hold for all possible choices of an intermediate value.) Then it is less clear that there exist functions satisfying IVT but are not continuous on $[a, b]$. But, for example, the function

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

satisfies the Intermediate Value Property despite being discontinuous at $x = 0$. There is also a surprising theorem: if f is any differentiable function, then its derivative f' satisfies the Intermediate Value Property, even though f' need not be continuous.

c) The converse is: if $d \mid a_n$ and $c \mid a_0$ then $P(\frac{c}{d}) = 0$. This is false. For instance, for $P(x) = x^2 - 2$, it says that $P(2) = P(-2) = 0$, which is certainly not the case.

11) True or false: it is not possible to prove a theorem by giving a single example. Discuss.

Solution: False. It is possible, but unusual. Most theorems in mathematics are of the form $\forall x \in S, P(x) \implies Q(x)$, where S is some infinite set. In this case giving a single value of x for which $P(x) \implies Q(x)$ certainly does not prove the theorem. However, occasionally one does encounter theorems of the form $\exists x \in S \mid P(x)$, e.g.:

“In 1903, a mathematician by the name of Frank Nelson Cole (1861 - 1926) gave a ‘lecture’ to the American Mathematical Society entitled ‘On the Factorisation of Large Numbers’. Without saying a word, Cole proceeded to write on a blackboard the calculations for 2 to the power of 67, then carefully subtracted 1. This was M_{67} and it is equal to 147,573,952,589,676,412,927. He then wrote out 193,707,721 x 761,838,257,287 on the other side of the board and proceeded to work out the multiplication by hand. The final product was written at the bottom and was seen to be identical to M_{67} ...He received a standing ovation.”

In other words, Cole proved that $\exists x, y \in \mathbb{Z} \mid (x, y > 1 \wedge xy = 2^{67} - 1)$ by exhibiting $x = 193707721$, $y = 751838257287$.

12) Suppose x is an integer such that $x \equiv 3 \pmod{6}$. Show that $3 \mid x$ and $2 \nmid x$.

Solution: The hypothesis means that $x = 6k + 3$ for some $k \in \mathbb{Z}$. Thus $x = 3(2k + 1)$, so $3 \mid x$, and $x = 2(3k + 1) + 1$, so x is odd, which as we have seen, means that it is not even: $2 \nmid x$.

13) Let $x \in \mathbb{Z}$. Show that if $7 \mid x^2 + 1$, then $13 \mid x^3 + 5x^2 + 17x - 100$.

Solution: I claim that it is *never* the case that for an integer x , $x^2 + 1$ is divisible by 7, so the implication holds vacuously. To see this, it's enough to consider the possible remainders upon division by 7, so there are 7 cases in all.

Case 1: $x = 7k$. Then $x^2 + 1 = 49k^2 + 1 = 7(7k^2) + 1$ is not divisible by 7.

Case 2: $x = 7k + 1$. Then $x^2 + 1 = 49k^2 + 14k + 1 + 1 = 7(7k^2 + 2k) + 2$ is not divisible by 7.

Case 3: $x = 7k + 2$. Then $x^2 + 1 = 49k^2 + 28k + 4 + 1 = 7(7k^2 + 4k) + 5$ is not divisible by 7.

Case 4: $x = 7k + 3$. Then $x^2 + 1 = 49k^2 + 42k + 9 + 1 = 7(7k^2 + 6k + 1) + 2$ is not divisible by 7.

Case 5: $x = 7k + 4$. Then $x^2 + 1 = 49k^2 + 56k + 16 + 1 = 7(7k^2 + 8k + 2) + 3$ is not divisible by 7.

Case 6: $x = 7k + 5$. Then $x^2 + 1 = 49k^2 + 70k + 25 + 1 = 7(7k^2 + 10k + 3) + 5$ is not divisible by 7.

Case 7: $x = 7k + 6$. Then $x^2 + 1 = 49k^2 + 84k + 36 + 1 = 7(7k^2 + 12k + 5) + 2$ is not divisible by 7.

Yes, this problem is a bit out of hand for an exam question. As a number theorist, I happen to know that for any prime number p of the form $4k + 3$, then for all integers x , $x^2 + 1$ is *not* divisible by p . Probably I was trying to sneak in too much number theory here.

14) Suppose that $3x + 2$ is odd. Show that $x^4 + 2x + 2009$ is even.

Solution: If $3x + 2$ is odd, then $3x$ is odd minus 2, or odd minus even, or odd. Moreover, if $3x$ is odd, then x must be odd (since if x were even, so would $3x$ be even). Then $x^4 + 2x + 2009$ is:

$$\begin{aligned} (\text{odd} \cdot \text{odd} \cdot \text{odd} \cdot \text{odd}) + (\text{even} \cdot \text{odd}) + \text{odd} &= \text{odd} + \text{even} + \text{odd} = (\text{odd} + \text{even}) + \text{odd} \\ &= \text{odd} + \text{odd} = \text{even} . \end{aligned}$$