REVIEW FOR 3200 MIDTERM II. . . WITH SOLUTIONS

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1) Give definitions of $a \mid b$ and of $a \equiv b \pmod{c}$.

Solution: For $a, b \in \mathbb{Z}$, $a \mid b$ means there exists an integer c such that $ac = b$. $a \equiv b \pmod{c}$ means that $c \mid (a - b)$.

2) Suppose that $m \mid n$ and $ma \equiv mb \pmod{n}$. Show that $a \equiv b \pmod{\frac{n}{m}}$.

Solution: If $ma \equiv mb \pmod{n}$, then $n \mid (ma - mb) = m(a - b)$, so there exists $c \in \mathbb{Z}$ such that $nc = m(a - b)$. Equivalently, $c(\frac{n}{m}) = a - b$, so the integer $\frac{n}{m}$ divides $a - b$ and $a \equiv b \pmod{\frac{n}{m}}$.

3) Let $x, y \in \mathbb{Z}$, and suppose that $x \equiv 3 \pmod{9}$.
a) Show that $2x^2 + 54y$ is divisible by 9.

Solution: By hypothesis we may write $x = 9k + 3$, so

$$2x^2 + 54y = 2(81k^2 + 54y + 9) + 54y = 9(18k^2 + 12k + 6y + 2).$$

Alternate solution: Going mod 9, we get

$$2x^2 + 54y \equiv 2 \cdot (3)^2 + 0 \equiv 9 \equiv 0 \pmod{9}.$$

b) Show that $2x^2 + 54y$ is not divisible by 27.

Solution: It follows from part a) that $2x^2 + 54y$ is of the form $9(18k^2 + 12k + 6y + 2)$ for some $k \in \mathbb{Z}$. Therefore it is divisible by 27 iff $N = 18k^2 + 12k + 6y + 2$ is divisible by 3, but since $N = 3(6k^2 + 4k + 2y) + 2$, this is not the case.

Alternate solution: Applying part a) and problem 2), it suffices to compute $\frac{2 \cdot 3^2 + 54y}{9}$ modulo 3: we get $2 + 54y \equiv 2 \pmod{3}$.

4) Let $x \in \mathbb{Z}$. Prove or disprove each of the following statements:
a) If $4 \mid x^2$, then $4 \mid x$.

Solution: It is false: $4 \mid 2^2$ but $4 \nmid 2$.

b) If $5 \mid x^2$, then $5 \mid x$.

Solution: By contraposition, suppose that $5 \nmid x$. Then $x \equiv 1, 2, 3, 4 \pmod{5}$, and we check:

$$1^2 = 1 \equiv 1 \not\equiv 0 \pmod{5}.$$

$$2^2 = 4 \equiv 4 \not\equiv 0 \pmod{5}.$$

$$3^2 = 9 \equiv 4 \not\equiv 0 \pmod 5.$$
$$4^2 = 16 \equiv 1 \not\equiv 0 \pmod 5.$$

Thus in all cases $5 \nmid x^2$.

5) Show that there do not exist positive integers x, y such that $x^2 - y^2 = 1$.

Solution: We have

$$1 = (x+y)(x-y).$$

Since $x+y, x-y \in \mathbb{Z}$, there are only two ways this can happen.
Case 1: $x+y = x-y = 1$. Then $2 = (x+y)+(x-y) = 2x$, so $x = 1$ and $y = 0$, which is not a positive integer.
Case 2: $x+y = x-y = -1$. Then $-2 = (x+y)+(x-y) = 2x$, so $x = -1$ and $y = 0$, which is (still!) not a positive integer.

Comment: As was pointed out in the problem session, since x, y are both positive, so is $x+y$ and therefore the case $x+y = -1$ need not be considered.

6) Let $r \neq 1$ be a real number. Show that for all $n \in \mathbb{Z}^+$, $1 + r + \ldots + r^n = \frac{r^{n+1}-1}{r-1}$.

Proof: By induction on n.

Base Case ($n = 1$): The left hand side is $1 + r$. The right hand side is $\frac{r^2-1}{r-1} = \frac{(r+1)(r-1)}{r-1} = r + 1$: OK.

Induction step: Suppose that for some positive integer n we have

$$1 + r + \ldots + r^n = \frac{r^{n+1}-1}{r-1}.$$

Adding r^{n+1} to both sides, we get

$$1 + r + \ldots + r^n + r^{n+1} = \frac{r^{n+1}-1}{r-1} + r^{n+1}$$
$$= \frac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r-1} = \frac{r^{n+2}-1}{r-1} = \frac{r^{(n+1)+1}-1}{r-1}.$$

Comment: A "better" proof is the usual telescoping sum trick.

7) Show that for all $n \in \mathbb{N}$, $3 \mid n^3 + 2n$: a) using congruences; b) by induction.

Solution: a) Put $N = n^3 + 2n = n(n^2+2)$. If $n = 3k$, $N = 3k(9k^2+2)$ is divisible by 3. If $n = 3k+1$, $N = (3k+1)(9k^2+6k+1+2) = 3(3k+1)(3k^2+2k+1)$ is divisible by 3. If $n = 3k+2$, $N = (3k+1)(9k^2+12k+4+2) = 3(3k+1)(3k^2+4k+2)$ is divisible by 3.

Alternate solution: Modulo 3 there are only three choices for n: $0, 1, 2$.

$$0(0^2+2) = 0 \pmod 3.$$
$$1(1^2+2) = 3 \equiv 0 \pmod 3.$$
$$2(2^2+2) = 12 \equiv 0 \pmod 3,$$

so in all cases $3 \mid n^3 + 2n$.

b) By induction on n.
Base case: $n = 1$: indeed $3 \mid 1^3 + 2 \cdot 1 = 3$.
Induction step: assume that $3 \mid n^3 + 2n$. Then
$N = (n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + (3n^2 + 3n + 3)$.
By the induction hypothesis, $3 \mid n^3 + 2n$, whereas visibly $3 \mid (3n^2 + 3n + 3)$, and so N – being the sum of two integers each divisible by 3 – is itself divisible by 3.

Comment: Probably the noninductive proof is preferable here.

8) A student has been asked to prove: $\forall x \in \mathbb{Z}, P(x) \implies Q(x)$.[1] For each of the following openers, give comments: is a valid proof technique? Which one?

Example: "Let $x \in \mathbb{Z}$, and suppose $P(x)$ is true."
Comment: This is the beginning of a direct proof.

a) "Let $x \in \mathbb{Z}$, and suppose $P(x)$ is false."

Comment: Assuming the negation of what is given is an unhelpful way to begin a proof. The strategy can succeed only if the implication is trivially true: i.e., if $Q(x)$ holds for all $x \in \mathbb{Z}$.

b) "Let $x \in \mathbb{Z}$, and suppose that $Q(x)$ is true."

Comment: This is totally invalid! Philosophers call it **begging the question**.[2]

c) "Let $x \in \mathbb{Z}$, and suppose $Q(x)$ is false."

Comment: This is the beginning of a proof by contraposition.

d) "Let $x = 1$. Then" [the student shows that $P(1)$ is true and $Q(1)$ is true].

Comment: The student has only shown the implication for a single element of the domain. Stopping here would certainly not give a valid proof. However, we should read on: it is often the case that certain exceptional cases need to be dealt with explicitly (we have ourselves seen examples of this, as when proving $2^n > n$ for all $n \in \mathbb{N}$). If the next line reads something like, "Now we assume that $x \neq 1$", then everything is fine.

e) "Let $x = 2$. Then" [the student shows that $P(2)$ is false and $Q(2)$ is false].

Comment: Same as above. In itself this is comparatively little progress, but it could be part of a valid argument.

[1] Here $P(x)$ and $Q(x)$ are sentences involving an arbitrary integer x.

[2] Unfortunately in modern times "begs the question" is more commonly used as a synonym for "raises the question" or "leads us to ask the question". Careful writers look upon this usage with extreme disdain.

f) "Let $x = 3$. Then" [the student shows that $P(3)$ is true and $Q(3)$ is false].

Comment: If the argument is correct, the student has **disproved** the statement. (Sometimes what you are asked to prove is not true.)

g) Let $x \in \mathbb{Z}$, and suppose that $P(x)$ is true and $Q(x)$ is false.

Comment: This is the beginning of a proof by contradiction.

9) Give a complete statement of the principle of mathematical induction.

Solution: Let $P(n)$ be a sentence with domain $\mathbb{Z}^+$. Suppose:
(i) $P(1)$ is true, and
(ii) for all $n \in \mathbb{Z}^+$, $P(n) \implies P(n+1)$.
Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Comment: I also stated a version of the principle of mathematical induction involving a subset of the positive integers. The two versions are seen to be logically equivalent (i.e., each can be deduced from the other), so either one would be fine. The one I stated above is the form that is more directly useful in doing induction proofs, whereas the other one is, perhaps, more useful in understanding mathematical induction itself.

10) Let $x, y, z \in \mathbb{Z}$ be such that $x^2 + y^2 = z^2$. Show that $3 \mid xy$.

Solution: Suppose that $3 \nmid x$. We will show that $3 \mid y$, which implies that $3 \mid xy$. Indeed, if $3 \nmid x$, then (e.g. by considering $(3k+1)^2 = 9k^2 + 6k + 1$ and $(3k+2)^2 = 9k^2 + 12k + 3 + 1$) $x^2 \equiv 1 \pmod 3$. Assuming for a contradiction that $3 \nmid y$, we have $y^2 \equiv 1 \mod 3$, but then $z^2 = x^2 + y^2 \equiv 1 + 1 = 2 \pmod 3$, a contradiction.

11) Let $x \geq 1$ be a real number. Show that for all $n \in \mathbb{Z}^+$, $(1+x)^n \geq 1 + nx$. When does equality hold?

Solution: We go by induction on n.
Base case ($n = 1$): $(1+x)^1 = 1 + x = 1 + (1)x$.
Induction step: assume that for some $n \in \mathbb{Z}^+$, $(1+x)^n \geq 1 + nx$. Then

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + nx + x + x^2 > 1 + (n+1)x.$$

As for when we can have equality: certain we do for all $x \geq 1$ when $n = 1$. Otherwise, the strict inequality in the above calculation – true since $x \geq 1 \implies x^2 > 0$ – shows that for all $x \geq 1$ and $n \geq 2$, we have $(1+x)^n \geq 1 + nx$.

Comment: There was a bit of a typo here: instead of $x \geq 1$ in the statement, I had intended $x \geq -1$. For simplicity I solved it as written. Note though that the hypothesis $x \geq -1$ also makes sense, because this is necessary and sufficient for $1 + x \geq 0$, which we used when we multiplied an inequality by $1 + x$.