REVIEW PROBLEMS FOR THIRD 3200 MIDTERM

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1) Show that for all integers $n \geq 2$ we have

$$1^3 + \ldots + (n-1)^3 < \frac{1}{4}n^4 < 1^3 + \ldots + n^3.$$

Solution: Scroll to the bottom of
`http://en.wikibooks.org/wiki/Algebra/Proofs/Exercises#ix.29`.

2) The distributive law for the real numbers states that for any real numbers a, b, c, $a \cdot (b + c) = a \cdot b + a \cdot c$. Assuming this, show by induction that for all $n \in \mathbb{Z}^+$ and real numbers $a, b_1, \ldots, b_n$, $a \cdot (b_1 + \ldots + b_n) = a \cdot b_1 + \ldots + a \cdot b_n$.

Solution: There is nothing to show if $n = 1$, and the base case $n = 2$ is our given assumption. So assume that for a given $n \geq 2$ and all real numbers $a, b_1, \ldots, b_n$ we have

$$a \cdot (b_1 + \ldots + b_n) = a \cdot b_1 + \ldots + a \cdot b_n.$$

Now let $a, b_1, \ldots, b_{n+1}$ be any real numbers. Then

$$a \cdot (b_1 + \ldots + b_n + b_{n+1}) = a \cdot ((b_1 + \ldots + b_n) + b_{n+1})$$

$$= a \cdot (b_1 + \ldots + b_n) + a \cdot b_{n+1} = a \cdot b_1 + \ldots + a \cdot b_n + b_{n+1}.$$

In the penultimate inequality we used the distributive law for $n = 2$, and in the last equality we used the induction hypothesis.

3) State the principle of mathematical induction, the principle of strong/complete induction and the well-ordering principle.

Solution: See `http://www.math.uga.edu/ pete/3200induction.pdf`.

4) Define all of the following terms: a **relation** R between sets X and Y, the **domain** of a relation; the **inverse relation** R^{-1}; a **partial ordering**; an **equivalence relation**; a **function**; the **domain of a function**; the **codomain of a function**; the **range of a function**; an **injective function**; a **surjective function**; a **bijective function**.

Solution: A relation R on a set is a subset of the Cartesian product $X \times Y$. The domain of a relation R is the set of all $x \in X$ such that there exists at least one $y \in Y$ with $(x, y) \in R$. The inverse relation is the subset $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$. A function is a relation in which for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in R$. The range of a function is the set of all $y \in Y$ such that $y = f(x)$ for at least one $x \in X$. A function $f : X \to Y$ is injective (or one-to-one) if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2) \implies x_1 = x_2$. A function $f : X \to Y$ is surjective (or onto) if its range is all of Y. A function is bijective if it is both injective and surjective.

5) $f : X \to Y$ and $g : Y \to X$ be functions.
a) Say what it means for f and g to be inverse functions.
b) Suppose $\forall x \in X$, $g(f(x)) = x$. Prove/disprove: f and g are inverse functions.
c) Same question as part b), but now assume that f is surjective.
d) Same question as part b), but now assume that g is injective.

Solution:
a) This means that $g \circ f = 1_X$, $f \circ g = 1_Y$.
b) We have seen in class several times that this is not true. For instance, take $X = \{a\}$ and $Y = \{1, 2\}$, $f : a \mapsto 1$, $g : 1 \mapsto a,\ 2 \mapsto a$. Then $g \circ f = 1_X$ but $f \circ g : 1 \mapsto 1,\ 2 \mapsto 1$.
c) It now follows that f and g are inverse functions, as discussed in class. E.g., by the green and brown fact, f is injective, whereas by our assumption f is surjective, so f is bijective and therefore f^{-1} exists. Applying f^{-1} to the equation $g \circ f = 1_X$, we get $g = f^{-1}$, so f and g are inverse functions.
d) Again, as discussed in class it does follow that f and g are inverse functions. E.g., by the green and brown fact, g is surjective, whereas by our assumption g is injective, so g is bijective and therefore g^{-1} exists. Applying g^{-1} to the equation $g \circ f = 1_X$, we get $f = g^{-1}$, so f and g are inverse functions.

6) Let $f : X \to Y$ and $g : Y \to Z$ be functions.
a) Define $g \circ f$.
b) Suppose that f and g are injective. Show that $g \circ f$ is injective.
c) Suppose that f and g are surjective. Show that $g \circ f$ is surjective.
d) Suppose that f and g are bijective. Show that $g \circ f$ is bijective.

Solution:
a) $g \circ f$ is the function from X to Z defined by $x \mapsto g(f(x))$.
b) Suppose that $x_1, x_2 \in X$ are such that $g(f(x_1)) = g(f(x_2))$. We wish to show that $x_1 = x_2$. But since g is injective, we have $f(x_1) = f(x_2)$, and since f is injective, we have $x_1 = x_2$.
c) Suppose that $z \in Z$. We want to show that there exists $x \in X$ such that $g(f(x)) = z$. But since g is surjective, there exists $y \in Y$ such that $g(y) = z$, and since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then $(g \circ f)(x) = g(f(x)) = g(y) = z$.
d) If f and g are bijective, then f and g are both injective, so by part b) $g \circ f$ is injective. Similarly, f and g are both surjective, so by part c) $g \circ f$ is surjective. Therefore $g \circ f$ is bijective.

7) Let X be a set. Prove or disprove: there does not exist any function $f : X \to \varnothing$.

Solution: This is true if and only if X is *not* the empty set. Indeed, if there exists $x \in X$, by definition of a function we need there to be exactly one $y \in Y$ such that $f(x) = y$, but Y is empty so there is no such Y! However, if $X = \varnothing$ then the empty relation on $\varnothing \times \varnothing$ does (rather vacuously) satisfy the properties of a function.

8) Let $n \in \mathbb{Z}^+$ and $b \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto x^n + b$. Determine the range of f. Is f injective? Surjective? (Your answer may depend on n and/or b.)

Solution: We claim that f is injective if and only if f is surjective if and only if n is odd. To see this, note that f is differentiable and $f'(x) = nx^{n-1}$. If n is odd, then $n-1$ is even, so $f'(x) \geq 0$ for all $x \in \mathbb{R}$ and is 0 only at the single point 0. From calculus, it follows that f is strictly increasing: $x_1 < x_2 \implies f(x_1) < f(x_2)$. In particular f is injective. Moreover, we have $\lim_{x\to\infty} x^n = \infty$, $\lim_{x\to-\infty} x^n = -\infty$; certainly f is continuous, so it follows from the intermediate value theorem that f assumes all real values.

Inversely, suppose n is even. Then $f(x) \geq 0$ for all $x \in \mathbb{R}$, so that f is not surjective. Moreover, $f(-1) = f(1)$, so f is not injective.

Remark: Note that the "$+b$" was irrelevant in the proof. This is always true in the following sense: for any $b \in \mathbb{R}$ and any function $f : \mathbb{R} \to \mathbb{R}$, the function $g(x) = f(x) + b$ is injective iff $f(x)$ is injective, and $g(x)$ is surjective iff $f(x)$ is surjective.

9)a) Prove/disprove: if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and such that $f'(x) \geq 0$ for all x, then f is injective.

Solution: This is false. A counterexample is $f(x) = C$, any constant function.

b) Same as part a), except with the assumption that $f'(x) > 0$ for all x.

Solution: Now it follows from calculus that f is strictly increasing hence injective. The details are as follows: suppose for a contradiction that there exist $x_1 < x_2$ such that $f(x_1) \geq f(x_2)$. Applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, there exists a c, $x_1 < c < x_2$ with $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f'(c) > 0$ is positive and $\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq 0$, a contradiction.

Remarks: Parts a) and b) use calculus in a nontrivial way so are probably not suitable test questions. Nevertheless these techniques of showing that functions are injective and surjective are quite useful, and you should feel free to use them on the exams without the need for proof.

c) Which of the following functions are injective (on their usual domains): $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$?

Solution: $\sin x$, $\cos x$ and $\tan x$ are **periodic**: there exists a positive real number C (here, $C = \pi$) such that $f(x + C) = f(x)$ for all $x \in \mathbb{R}$. Any periodic function is far from injective: $f(0) = f(C) = f(2C) = \ldots$. On the other hand e^x and $\ln x$ are both injective: again, their derivatives – e^x and $\frac{1}{x}$ respectively – are positive for all x in the domain.

d) Which of the functions of part c) are surjective onto $\mathbb{R}$?

Solution: $\sin x$ and $\cos x$ are *bounded*, hence certainly not surjective. The function $\tan x$ is surjective, as a study of its vertical asymptotes makes clear. The fact that $e^x > 0$ for all real x means it is not surjective. On the other hand, since $\lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x\to\infty} \ln x = \infty$, $\ln x$ is surjective.

10) Let $f : X \to Y$ be a function.
a) Define a relation on X by $x \sim x'$ if $f(x) = f(x')$. Show that $\sim$ is an equivalence relation.

Solution: Reflexivity: Indeed $f(x) = f(x)$ for all $x \in X$.
Symmetry: If $f(x) = f(y)$, then $f(y) = f(x)$.
Transitivity: If $f(x) = f(y)$ and $f(y) = f(z)$, then $f(x) = f(z)$.
In other words, the properties follow easily from the corresponding properties for equality.

b) For any $y \in Y$, the **fiber over y in X** is the set

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Show that for $y \neq y'$ the fibers over y and y' are disjoint subsets of X.

Solution: If x lies in the fiber over y and x' lies in the fiber over y', then $f(x) = y \neq y' = f(x')$. So certainly $x \neq x'$ since otherwise $f(x) = f(x')$.

c) Show that the set of fibers $\{f^{-1}(y) \mid y \in Y\}$ gives a partition of x if and only if f is surjective. Show that in this case the corresponding equivalence relation on X is the same as the $\sim$ relation of part a).

Solution: If f is not surjective, then at least one of the fibers is the empty set, which is not allowed as an element of a partition. Conversely, suppose f is surjective. Then each fiber is nonempty. We saw above that distinct fibers are disjoint sets, so it remains to be seen that the union of all fibers is X itself. But this is clear: for $x \in X$, x lies in the fiber over $f(x)$. The corresponding equivalence relation is $x \sim x'$ iff x and x' lie in the same fiber, but this happens iff $f(x) = f(x')$, which is the definition of $\sim$ in part a).

11) Show that every equivalence relation on a set X arises as the equivalence relation associated to a surjective function as in 10) above. (Hint: let Y be the set of all equivalence classes, and define an appropriate function $f : X \to Y$.)

Solution: As suggested, we take Y to be the set of equivalence classes $C_x = \{y \in X \mid y \sim x\}$ for some equivalence relation $\sim$ on X. The desired function is then just $f : x \mapsto C_x$. In this case the fiber over an element C_x is precisely the equivalence class $C_x \subset X$, so we recover the equivalence relation this way.

12) Let X be the set of all functions $f : \mathbb{R} \to (0, \infty)$. Define a relation $\sim$ on X by $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Show that $\sim$ is an equivalence relation on X. (It

is called **asymptotic equality**).

Solution: Reflexivity: $\lim_{x\to\infty} \frac{f(x)}{f(x)} = \lim_{x\to\infty} 1 = 1$.
Symmetry: Recall that if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$, then $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{L}$. Taking $L = 1$ gives the desired conclusion.
Transitivity: If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to infty} \frac{g(x)}{h(x)} = 1$, then

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \lim_{x\to\infty} \frac{g(x)}{h(x)} = 1 \cdot 1 = 1.$$

13) For each of the following, either give an example, or prove that no such example exists.
a) A relation on a set which is symmetric and transitive but not reflexive.

Example: The relation $\{(0,0)\}$ on the set of real numbers.

b) A partial ordering on a set which is not a total-ordering.

Example: The relation $\subset$ on $\mathcal{P}(X)$, where $X = \{0,1\}$.

c) A total ordering on a set which is not a well-ordering.

Example: The usual $\leq$ on the real numbers.

d) A total ordering on a finite set which is not a well-ordering.

Any total ordering on a finite set is a well-ordering. To see this, let $(X, \leq)$ be a total ordering on a finite set X, and let Y be a nonempty subset of X. So choose $y_1 \in Y$. If y_1 is not the least element there exists $y_2 \in Y$ with $y_2 < y_1$. If y_2 is not the least element, there exists $y_3 \in Y$ with $y_3 < y_2$. Continuing in this manner, if there is no least element we construct an infinite sequence of distinct elements of X. But X was assumed to be finite.

e) A relation $R \subset \mathbb{R} \times \mathbb{R}$ such that each vertical line $x = c$ intersects R in exactly one point, but R is not a function.

This is impossible: that each x-value get assigned to a unique y-value is the definition of a function.

f) A function $R \subset \mathbb{R} \times \mathbb{R}$ such that each horizontal line $y = d$ intersects R in at most one point, but which is not invertible.

Example: $f(x) = e^x$.

14) Suppose $f : X \to Y$ is a surjective function. Show that there exists a function $g : Y \to X$ such that for all $y \in Y$, $f(g(y)) = y$.

Solution: By definition of surjectivity, for each $y \in Y$, there exists at least one element $x \in X$ with $f(x) = y$. So choose one such x, say x_y. Then defining $g : Y \to X$ by $y \mapsto x_y$ does the trick, because $f(g(y)) = f(x_y) = y$.