

HOMEWORK PROBLEMS FOR MATH 8320: 1-28

PETE L. CLARK

Here are some homework problems for the 8320 course. Some of them are repeated from places in the lecture notes. Some of them are not specifically tied to any one topic. Some of them are easy, and some are quite difficult, etc. There are more problems than any one student will want to work out, but every student should work at least some of them and be prepared to present solutions to some of them.

Unless mention is made to the contrary, k denotes an arbitrary field.

Exercise 1 (Properties of the category of affine k -algebras):

- Show that any quotient of an affine k -algebra is an affine k -algebra.
- Show that an integral affine k -algebra can be expressed as the quotient of a polynomial ring by a prime ideal.
- Show that the product of two (hence also any finite number) affine k -algebras is an affine k -algebra.
- Show that the tensor product, over k , of two affine k -algebras is an affine k -algebra.¹
- When is an infinite product of affine k -algebras an affine k -algebra?²
- Show by example that a k -subalgebra of an affine k -algebra need not be an affine k -algebra.

Exercise 2:

- Give an example of a integral k -algebra (necessarily infinitely generated!) A for which the Krull dimension $\dim A$ is not equal to the transcendence degree of the fraction field K of A over k .
- Can one at least say either that the Krull dimension is always at most the transcendence degree or that the transcendence degree is always at most the Krull dimension?

Exercise 3: We would like to say that an affine variety is the “union” of its irreducible components, but at the moment we have to phrase everything in terms of the opposite category of k -algebras. What is a true statement about an affine k -algebra and its irreducible components A/\mathfrak{p}_i (where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal prime ideals) that would plausibly be the dual of this geometric statement?

Exercise 4:

- Show that the \mathbb{R} -algebra $A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is the affine \mathbb{R} -algebra of a nice affine curve.
- Can you describe the maximal ideals of A ?

¹Suggestion: do the case of a tensor product of two polynomial rings first, and then reduce to this.

²Answer: almost never, but state and prove a precise result.

Exercise 5: Show that the dimension of an affine k -algebra is an absolute property: that is, for any field extension K/k , we have $\dim(A) = \dim(A \otimes_k K)$.

Exercise 6: Can you think of a property of affine algebras that is geometric but *not* absolute? (Answer: yes; e.g. the cardinality. But can you think of an interesting property?)³

Exercise 7: Let l/k be a finite field extension. Show TFAE:

- (i) l/k is separable.
- (ii) $l \otimes_k l$ is reduced.
- (iii) For any field extension K/k , $l \otimes_k K$ is reduced.

Exercise 8: Let l/k be a finite field extension. Show TFAE:

- (i) l/k is purely inseparable.
- (ii) $l \otimes_k l$ is irreducible.
- (iii) For any field extension K/k , $l \otimes_k K$ is irreducible.

Exercise 9: Deduce from Exercises 7 and 8 that if l/k is a finite field extension, then the affine k -algebra l is geometrically integral iff $l = k$. In particular, if k is not algebraically closed, there exists an integral affine k -algebra which is not geometrically integral.

Exercise 10: Let k be a field, and consider the affine algebra $A = k[x, y]/(x^2 + y^2)$.

- a) Show that A is geometrically connected.
- b) Show that A is reduced iff the characteristic of k is different from 2.
- c) Show that A is integral iff -1 is not a square in k .
- d) Conclude that A is never geometrically integral.

Exercise 11: Let A be an affine k -algebra.

- a) Show that there exist finitely many connected affine k -algebras A_1, \dots, A_r such that

$$A \cong A_1 \times \dots \times A_r.$$

- b) Show that this decomposition is unique in the sense that if also

$$A \cong B_1 \times \dots \times B_s,$$

then $r = s$ and there is a permutation σ of $\{1, \dots, r\}$ such that $B_i \cong A_{\sigma(i)}$ for all i . The A_i 's may be called the **connected components** of A .

Exercise 12: Let k be a field and consider the affine k -algebra $A = k[x, y]/(x^2 + y^2)$.

- a) Show that A is geometrically connected.
- b) Show that A is geometrically reduced iff the characteristic of k is not 2.
- c) Show that A is integral iff -1 is not a square in k .
- d) Conclude that A is never geometrically integral.

Exercise 13: Give an example of a 2-dimensional affine variety which is normal but not regular.

³This is related to the Lefschetz principle in algebraic geometry and model theory.

Exercise 14: Let k be a **imperfect** field, so that k has positive characteristic p and there exists an element $a \in k \setminus k^p$. Let A be the affine k -algebra $k[x, y]/(y^2 - (x^p - a))$. Show that A is normal and regular but not smooth.

Exercise 15: Give a geometric description of the affine \mathbb{C} -algebra $\mathbb{C}[t] \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 16R:⁴ If l/k is a finite extension, there is also a functor from affine varieties over l to affine varieties over k , called **Weil restriction** or **restriction of scalars**. How does this work?

Exercise 17: a) Suppose K, L are linearly disjoint field extensions of k . Show that $K \cap L = k$.

b) Suppose K and L are both finite over k and at least one is Galois. Show that linear disjointness is equivalent to $K \cap L = k$.

c) Suppose K and L are conjugate finite separable⁵ extensions of k – i.e., they have a common Galois closure M and there exists $\sigma \in \text{Aut}(M/k)$ such that $\sigma(K) = L$. Show that K and L are **not** linearly disjoint over k , even though we may well have $K \cap L = k$ (give an example).

Exercise 18: Let K and L be field extensions of k , and suppose that K/k is algebraic. Show that K and L are linearly disjoint over k iff $K \otimes_k L$ is a field. (Hint: first do the case where K/k is finite.)

Exercise 19: Prove Proposition 1 of Handout 3 (the one on formal properties of Galois connections).

Exercise 20: Prove that in all cases asserted in Handout 3, the Zariski topology is indeed a topology. (Suggestion: try to give a unified argument.)

Exercise 21: a) Show that the Zariski topology on $\mathbb{A}_{/k}^n$ coincides with the topology it inherits as a subset of $\mathbb{A}_{/\bar{k}}^n$.

b) Show that the Zariski topology is T_1 : i.e., singleton subsets are closed.

c) Show that when $n = 1$, the Zariski topology is the coarsest T_1 topology on k : namely, the topology in which a proper subset is closed iff it is finite.

d) For any $n \geq 1$, show that the Zariski topology on k^n is discrete iff k is finite.

e) For any infinite field and $m, n \geq 1$, show that the Zariski topology on k^{m+n} is strictly finer than the product of the Zariski topologies on k^m and k^n .

Exercise 22: Let k be a field, $n \in \mathbb{Z}^+$ as above. Explicitly compute the ideal $I(k^n)$, i.e., the set of all polynomials which vanish at *every* point of k^n . Do we necessarily have $I(k^n) = \{0\}$? If not, then what exactly is $I(k^n)$ (i.e., give a set of generators)?

⁴This is a problem that requires some outside reading.

⁵This hypothesis is here to make it easier for you to understand why $K \cap L = k$ need not imply linear disjointness, even in characteristic 0. By all means think about the nonseparable case if you like.

Exercise 23: In Handout 3, deduce Theorem 2 from Theorem 3 and Theorem 4.

Exercise 24: a) Let R be a Hilbert-Jacobson ring (i.e., one in which every prime ideal is the intersection of maximal ideals). Write out the details of the Galois connection between ideals of R and subsets of $\text{MaxSpec}(R)$. Especially, what is the induced bijection?

b) Let R be an arbitrary commutative ring. Write out the details of the Galois connection between ideals of R and subsets of $\text{Spec } R$. What is the induced bijection?

Exercise 25: Let R be a ring and $S \subset \text{Spec } R$. Show that the Zariski closure of S is equal to the set of prime ideals which contain every element of S . (Note that the formalism of Galois connections makes this especially straightforward.) Deduce that a singleton set $\{\mathfrak{p}\}$ is closed iff \mathfrak{p} is a maximal ideal.

Exercise 26: Show that for a commutative ring, the following are equivalent:

- (i) R has Krull dimension zero (i.e., all prime ideals are maximal).
- (ii) $\text{Spec } R$ is T_1 .
- (iii) $\text{Spec } R$ is Hausdorff.

Exercise 27: Let k be an imperfect field with separable algebraic closure k^{sep} and algebraic closure \bar{k} . Show that every element of $G = \text{Aut}(k^{\text{sep}}/k)$ extends to a unique automorphism of \bar{k} , giving a canonical isomorphism from G to $\text{Aut}(\bar{k}/k)$. Nevertheless it is not the case that $(\bar{k}^n)^G = k^n$ in this case: what happens instead?

Exercise 28: Prove Theorem 8 in Handout 3.