HOMEWORK PROBLEMS FOR MATH 8320: 29-47

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Here are some homework problems for the 8320 course. Some of them are repeated from places in the lecture notes. Some of them are not specifically tied to any one topic. Some of them are easy, and some are quite difficult, etc.[1] There are more problems than any one student will want to work out, but every student should work at least some of them and be prepared to present solutions to some of them.

Unless mention is made to the contrary, k denotes an arbitrary field, but you have my permission to assume that k is perfect if you don't like inseparable field extensions.[2]

If an exercise is labelled with an "L", that means that substantial help on that exercise can be found in Liu's book. (The converse does not necessarily hold...)

Exercise 29: Determine which PIDs have the Hilbert-Jacobson condition. Then (if you know a bit about Dedekind domains), conclude that a Dedekind domain is either a Hilbert-Jacobson ring or a PID (or both).

Exercise 30: For an affine k-algebra A, show that the following are equivalent: (i) A is finitely generated as a k-module. (ii) A has dimension zero. (iii) The set MaxSpec(A) of maximal ideals of A is finite.

Exercise 31: Classify all affine k-algebras A for which every ideal is a radical ideal.

Exercise 32L: Suppose that k is a field which is separably closed (but not necessarily algebraically closed), and let V be a geometrically integral affine k-variety. Show that V necessarily has a k-rational point, i.e., a maximal ideal whose residue field is just k itself. (Hint: use part b) of the Noether normalization theoem.)

Comment: A field k is called pseudoalgebraically closed (PAC) if every geometrically integral k-variety has a k-rational point. Thus this exercise demonstrates that any separably closed field is PAC. There are, however, many other PAC fields, which are important in the branch of arithmetic geometry known as Field Arithmetic.

[1] N.B.: There are fewer "unpack the definitions" type problems here than in the first batch.

[2] Of course, eventually you will see that you need to know about inseparable field extensions to do geometry in characteristic p, and it is not unheard of for questions in characteristic 0 to be reduced to characteristic p, so you will probably need to deal with these issues eventually, but maybe not for a while.

Exercise 33: For any scheme S, construct affine n-space $\mathbb{A}^n_S$ by glueing affine n-spaces over an affine covering of S.

Exercise 34: If $S = \operatorname{Spec} A$ is affine, show that any affine S-scheme is of the form $X = \operatorname{Spec} R$, where R is a quotient of $A[t_1, \ldots, t_n]$. (In particular the two uses of the term "affine" coincide.)

Exercise 35L: By suitably glueing together $n+1$ copies of affine n-space over A, construct $\mathbb{P}^n_{/A}$, projective n-space over the ring A.

Exercise 36: For any field K, show that $\operatorname{Hom}(\operatorname{Spec} K, X) = \operatorname{Hom}(\operatorname{Spec} K, X^{\mathrm{red}})$, where X^{red} is the reduced subscheme of X. Can you give a condition on a scheme Y such that $\operatorname{Hom}(Y, X) = \operatorname{Hom}(Y, X^{\mathrm{red}})$ for all X?

Exercise 37: Let R be a DVR with uniformizing element π. Let $a \in \mathbb{N}$ be a non-negative integer. Show:
a) $\operatorname{Spec} R[x,y]/(xy - \pi^a)$ is smooth over R – that is, both fibers are geometrically regular – iff $a = 0$.
b) Show that $\operatorname{Spec} R[x,y]/(xy - pi^a)$ is regular iff $a \leq 1$.

Remark: This calculation will serve us well throughout the rest of the course.

Exercise 38: Show that $\operatorname{Spec} \mathbb{Z}[x,y]/(xy - m)$ is regular iff m is squarefree. (Suggestion: reduce to Exercise 37.)

Exercise 38.5: Let $f : X \rightarrow Y$ be a morphism (i.e., continuous map) of sober topological spaces, with generic points η_X and η_Y. Show that the following are equivalent:
(i) $f(\eta_X) = \eta_Y$.
(ii) $f(X)$ is dense in Y.
A morphism satisfying these equivalent conditions is called **dominant**.

Exercise 39: Let $f : X \rightarrow Y$ be a morphism of integral schemes, and let η_X, η_Y be the generic points of X and Y respective. Show that TFAE:
(i) f is dominant.
(ii) f induces an extension $K(Y) \hookrightarrow K(X)$.[3]

Exercise 40: Let X and Y be integral k-varieties (or, more generally, finite type schemes over any base S). Show that TFAE:
(i) There exists a birational map (i.e., a birational morphism with domain a dense open subset of X) $f : X \rightarrow Y$.
(ii) There exist nonempty (hence dense!) open subsets U of X and V of Y such that the open subschemes $(U, \mathcal{O}_U)$ and $(V, \mathcal{O}_Y)$ are isomorphic k-schemes.

Comment: This is a doubly corrected version of Exercise 3.2.6 in Liu's book. You might want to look at that exercise and understand why the statement is wrong

[3]I could be more precise, but I think it is instructive to figure out exactly how this works for yourself.

(the error in his statement has nothing to do with the distinction between birational morphism and birational map that came up in class).

Exercise (Liu, 3.2.7) 41: Let k be a field with algebraic closure $\overline{k}$. Let $\overline{X}$, $\overline{Y}$ be varieties over $\overline{k}$, and $\overline{f} : \overline{X} \to \overline{Y}$ a morphism. Show that everything can be defined over a finite subextension l: that is, there exists a finite extension l/k, varieties X, Y defined over l and a morphism $f : X \to Y$ defined over l such that after base change we have isomorphisms $\iota_X : X_{\overline{k}} \cong \overline{X}$, $\iota_Y : Y_{\overline{k}} \cong \overline{Y}$

$$\overline{f} = \iota_Y \circ .f_{\overline{k}} \circ \iota_X^{-1}.$$

Can the finite extension l always be taken to be normal? separable?

Exercise (Liu, 3.2.9) 42: Let X, Y be varieties over a field k, with X geometrically reduced. Suppose $f, g : X \to Y$ are morphisms which induce the same map $X(\overline{k}) \to Y(\overline{k})$ on $\overline{k}$-valued points. Show that $f = g$. (Suggestion: a plan of attack is given in Liu's book.) Does the conclusion hold if X is not assumed to be geometrically reduced?

Exercise (Liu, 3.2.10) 43: Let k be a field and l/k be a finite Galois extension with Galois group G. Let X be an irreducible variety over k.
a) Show that G acts transitively on the irreducible components of X_l. Deduce that $X_{\overline{k}}$ is equidimensional (i.e., all irreducible components have the same dimension.)
b) Suppose X is connected, and show G acts transitively on the connected components of X_l.

Exercise 44: Let X be a variety over a field k.
a) Suppose X is connected and $X(k) \neq \emptyset$. Show that X is geometrically connected.
b) Deduce that the following conditions on a group variety $G_{/k}$ are equivalent:
(i) G is integral.
(ii) G is geometrically integral.
(iii) G is reduced and conected.
c) Does part a) hold if "connected" is replaced by "irreducible"?

Exercise 45: Show that for each of the following properties "P" of algebraic varieties over a(ny) field k, being "geometrically P" implies being "absolutely P": (i) connectedness, (ii) reducedness, (iii) irreducibility. (Suggestion: consult Exercise 3.2.14 of Liu's book.)

Exercise 46: Let R be an integral domain with fraction field k, and let X be an R-scheme.
a) Explain why there is a canonical morphism $X(R) \to X(k)$ from R-valued points to k-valued points.
b) Suppose R is integrally closed (or "normal") and $X \to \operatorname{Spec} R$ is separated. Show that the above morphism is injective. (Hint: first do the case of R a DVR.)
c)* Find an example where the map $X(R) \to X(k)$ is not injective. Can this happen when $X \to \operatorname{Spec} R$ is separated?

Exercise 47: Let k be any field, and consider the morphism of affine varieties $\mathring{A}^1 \to \mathring{A}^1$ which is dual to the k-algebra homomorphism $k[x] \to k[x]$ by $x \mapsto x^2$.

Show that this is a finite morphism and compute the fiber of the morphism over an arbitrary point (including the generic point) of $\mathring{A}^1$. Give particular attention to the fiber over 0 (i.e., over the prime ideal (x) of $k[x]$).