HOMEWORK PROBLEMS FOR MATH 8320: 48-71

PETE L. CLARK

This is the third set of homework problems for the 8320 course. As usual, unless mention is made to the contrary, k is assumed to be an arbitrary field. However, we are content to develop the theory of hyperelliptic curves only in characteristic different from 2, and there is only one problem where the imperfection of the ground field is addressed in an important way.

Note: Perhaps unfortunately, this problem set contains few problems of medium difficulty. Rather, most of the problems are either easy, hard or (so far as I know), unsolved. In order to make your work more efficient, I have labelled parts of problems with a designation of (E), (M), (H) or (U), accordingly. Do understand that this classification is highly surjective.

Exercise 48(E): Show that the index of a variety is equal to the greatest common divisor of all degrees of closed points P on V. (Recall that our definition of the index was as the cardinality of the cokernel of the degree map from $Z_0(V)$ to $\mathbb{Z}$.)

Exercise 49: (a(E)) If k is not perfect, we can also define the **separable index** i_s to be the gcd of all degrees of closed points with separable residue field extensions. What is an equivalent definition in terms of divisors? (Hint: you want to define a subgroup of $Z_0(X)$ generated by "separable" divisors.)
(b(E)) Show that $i(V) \mid i_s(V)$ for all V.
(c(U)) Find a nice variety V over some (imperfect) field k such that $i_S(V) > i(V)$.

Exercise 50(M): Show that the Weierstrass points on a hyperelliptic curve of genus $g \geq 2$ are precisely the ramification points of the covering $C \to C/\iota = \mathbb{P}^1$. In particular, there are exactly $2g + 2$ Weierstrass points.

Exercise 51(H): In this exercise you will fill in the details of a statement and proof of Hurwitz's *Lückensatz*.

Exercise 52: Let k be an imperfect field and $C_{/k}$ a nice (in particular, geometrically regular) curve.
(a(E)) Must every automorphism of C be defined over a separable extension of k?
(b(H)) What about if we assume that the genus is at least 2?

Exercise 53: Let $C_{/k}$ be a nice algebraic curve, and D a divisor on C whose divisor class is invariant under $\mathrm{Aut}(C)$ (important example: any integer multiple of the canonical class).
(a(E)) Show that there is a representation $\rho : \mathrm{Aut}(C) \to GL(H^0(D) \otimes_k \overline{k})$.

(b(E)) Show that this representation is induced from a k-rational representation

$$\rho_k : \mathrm{Aut}(C)(k) \to GL(H^0(D)).$$

(c(M)) Suppose that C has genus $g \geq 2$ and $D = K$ is a canonical divisor. In this case ρ is called the **canonical representation** of $\mathrm{Aut}(C)$. Show that the representation ρ is faithful (i.e., injective) iff C is canonical; in the case where C is biconic, determine the kernel of ρ.
(c(H)) Find an example of a canonical algebraic curve $C_{/\mathbb{Q}}$ such that the canonical representation cannot be defined over $\mathbb{Q}$. Deduce that $\mathrm{Aut}(C)(\mathbb{Q}) \subsetneq \mathrm{Aut}(C)$, i.e., that the Galois action on automorphisms is necessarily nontrivial.

Exercise 54(M): Let $E = (C, O)$ be an elliptic curve over a perfect field k. Describe explicitly the Galois action on $\mathrm{Aut}(E)$.

Exercise 55(U)[1]: Let C be a curve (of genus $g \geq 2$, as ever) and φ an automorphism of C which fixes exactly $2g+2$ geometric points. Prove/disprove: C is hyperelliptic and $\varphi = \iota$ is the hyperelliptic involution.

Exercise 56: (a(E)) Show that the action of $GL_2(k)$ on $\mathbb{P}^1$ by linear fractional transformations factors through an effective action of PGL_2 on $\mathbb{P}^1$. (In other words, the kernel of the action is precisely the subgroup $\mathbb{G}_m$ of diagonal matrices.)
(b(E)) Show that this action is triply transitive: given any two triples (P_1, P_2, P_3), (Q_1, Q_2, Q_3) of distinct geometric points of $\mathbb{P}^1$, there exists a linear fractional transformation τ such that $\tau(P_i) = Q_i$ for $1 \leq i \leq 3$.
(c(E)) Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. Show that there exists $\tau \in PGL_2$ such that $\varphi \circ \tau^{-1}$ fixes 0, 1, and ∞ pointwise. Apply Proposition **??** to deduce that $\varphi = \tau$.

Exercise 57(H): What is the maximum cardinality of the automorphism group of a genus 2 curve C defined over the complex numbers? (Recall that we exhibited a curve with equation $y^2 = x^6 - x$, which had automorphism group of cardinality 24. Is this the largest one?)

Exercise 58: Consider the Klein quartic curve

$$C_{/Q} : X^3Y + Y^3Z + Z^3X = 0.$$

(a(H)) Show that $\mathrm{Aut}(C) \cong PSL_2(\mathbb{F}_7)$. In particular, its cardinality is 168, so C attains the Hurwitz bound.
(b(H)) What is the subgroup $\mathrm{Aut}(C)(\mathbb{Q})$? Can you explicitly compute the canonical representation ρ of C?

Exercise 59(U): Suppose that instead of a number field, K is a global function field, i.e., the function field of a nice curve over a finite field. Can one show that there exist genus one curves C of every index over K?[2]

Exercise 60: Let p be a prime number. Consider the plane curve $C_p/\mathbb{Q}$ given

[1] I don't know the answer, but probably someone does.

[2] The analogous result in the number field case was proved in `http://math.uga.edu/ pete/crelle.pdf` .

by the equation

$$C_p : X^3 + pY^3 + p^2Z^3 = 0.$$

(a(E)) Show that C_p is a smooth curve of genus one.
(b(M)) Show that the index of C_p is 3. In particular, C_p is not hyperelliptic.
(c(M))* Suppose that $p \equiv 1 \pmod 3$. Show that C_p has no rational points over any abelian number field. [3]

Exercise 61(E): Show that any nice curve of genus 0 is hyperelliptic.

Exercise 62(M): Let k be an arbitrary field (possibly of characteristic 2), and let $K/k(t)$ be a quadratic extension such that k is algebraically closed in K. Then K is the function field of a regular curve $C_{/k}$. Suppose that the genus of C is positive. Show that the extension $K/k(t)$ is separable.

Exercise 63(E) Show that every hyperelliptic curve over an algebraically closed field k has a defining equation of the form $y^2 = P_{2g+1}$.[4]

Exercise 64(M): Let $y^2 = ax^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0$ be a defining equation for a hyperelliptic curve $C_{/k}$ (we assume that the characteristic of k is not 2). Because the point at ∞ on $\mathbb{P}^1$ is not a branch point, the preimage of ∞ on C consists of two geometric points, say ∞_1 and ∞_2. Show that TFAE:
(i) The residue field of ∞_1 is k.
(ii) The points ∞_1 and ∞_2 are distinct closed points on C.
(iii) The leading coefficient a is a square in k.

Exercise 65: (a(E)) Let $\varphi : X \to Y$ be a finite morphism of nice curves over k, of degree d. Letting, as before, $i(C)$ denote the index of a curve, show

$$i(Y) \leq i(X) \leq di(Y).$$

(b(E)) Deduce that the index of a hyperelliptic curve is 1 or 2. (In fact, show that the least degree of a rational point on a hyperelliptic curve is at most 2. Why is this a stronger statement?)

Exercise 66: (a(E)) Show that any elliptic curve is hyperelliptic.
(b(E)) Exhibit a genus one curve without rational points which is hyperelliptic.
(c(E)) Show that there is a genus one curve (over some field) which is not hyperelliptic.[5]

Exercise 67(U): Prove or disprove: let k be a field for which there exists a conic curve $V_{/k}$ without a rational point. (N.B.: equivalently, $\mathrm{Br}(k)[2] \neq 0$. This holds e.g. for all locally compact fields except $\mathbb{C}$, and also for all global fields and all infinite, finitely generated fields.) Show that for any odd g there exists a genus g

---

[3]Hint: apply the Kronecker-Weber theorem, which asserts that every finite abelian extension of $\mathbb{Q}$ is contained in some cyclotomic field $\mathbb{Q}(\zeta_N)$. Or see `http://math.uga.edu/ pete/plclarkarxiv8v2.pdf`.

[4]Hint: use an automorphism of ¶[1] to map one of the branch points to infinity.

[5]Hint: this might be difficult, but you have already seen one...

curve $C_{/k}$ with a $2:1$ map $C \to V$ such that C is **not** hyperelliptic.

Exercise 68: (a(E)) Find the smallest integer g such that no smooth curve of genus g can be a complete intersection in projective space.
(b(U)) Prove or disprove: let $S \subset \mathbb{Z}^+$ be the set of positive integers g such that there exists some nice curve $C_{/k}$ (take $k = \mathbb{C}$ if you like; the question does not depend on the choice of k) of genus g whic is a complete intersection in projective space. Then

$$\delta(S) = \lim_{N \to \infty} \frac{\#\{S \cap [1, N]\}}{N} = 0.$$

In other words, the claim is that for a "randomly chosen" positive integer g, no curve of genus g is a complete intersection in projective space.

Exercise 69: Suppose $C_{/k}$ is a nice curve.
(a(E)) Show that the index of C divides the gonality of C.
(b(M)) Show that the gonality of C divides $2g(C) - 2$.

Exercise 70 (M): Prove Theorem of Accola and Namba, according to the following outline:
(a) Note that it suffices to assume $k = \overline{k}$.
(b) Adjust g by a linear automorphism so that the branch loci of x and y are distinct. Consider the map $\phi : C \to \mathbb{P}^2$ by $p \mapsto [x(p) : y(p) : 1]$. Show that the independence of x and y implies that the map $\phi : C \to \phi(C)$ is birational. Equivalently, the image $C' = \phi(C)$ is a plane curve and $\phi : C \to C'$ is its normalization.
(c) Show that C has a singularity at $[1 : 0 : 0]$ (resp. $[0 : 1 : 0]$) of multiplicity d_1 (resp. d_2). Using the formula for the genus of a singular plane curve, show that – with $d = d_1 + d_2$ –

$$g(C) \leq \frac{(d-1)(d-2)}{2} - \frac{(d_1-1)(d_1-2)}{2} - \frac{(d_2-1)(d_2-2)}{2}.$$

Exercise 71 (M): For which genera can you construct a superelliptic, nonhyperelliptic curve of genus g?