

THE CATEGORY OF AFFINE k -VARIETIES

PETE L. CLARK

1. THE CATEGORY OF AFFINE VARIETIES OVER A FIELD k

Let k be an arbitrary field. We wish to define a **category** of affine varieties over k .

Recall the most classical definition: if $k = \bar{k}$, then an affine subvariety of k^n is given by the zero locus of a set (which, by the Hilbert basis theorem, can be taken to be finite) of polynomials $P_1(\mathbf{x}), \dots, P_r(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_n)$.

Even in the classical case, this is a somewhat creaky definition since it is so extrinsic: it is like defining a complex manifold as a certain type of subset of \mathbb{C}^n . But in the case of a general field, it is manifestly inadequate: we would like to regard the equation $x^2 + y^2 = -1$ as defining an affine curve in \mathbb{R}^2 , but this curve has no (real) points.

There are several rather elaborate remedies for this: we could switch to the theory of schemes, or we could introduce the theory of Galois descent. But let us try something more modest: we switch our arrows around. Namely, let's try to define a category Aff_k^{op} , the opposite of the category of affine varieties.

In this category, the objects will be **affine k -algebras**, i.e., finitely generated k -algebras A . In other words, A is a ring, equipped with a morphism of rings $k \rightarrow A$, which is isomorphic, as a k -algebra to $k[x_1, \dots, x_n]/I$ for some $n \in \mathbb{Z}^+$. Of course there will be many such isomorphisms, and we do not privilege any one of them as part of the definition. A morphism of affine algebras is just what it sounds like: a morphism of rings which respects the k -module structure.

Then an **affine variety** V_k will be identified with a corresponding affine algebra A_V . However, a morphism $V \rightarrow W$ of affine varieties is precisely a morphism $W_k \rightarrow V_k$, i.e., a morphism of the corresponding affine algebras, but in the other direction.

Exercise: Show that the category of affine k -algebras is closed under quotients, finite Cartesian products, and finite tensor products (over k), but is not closed under passage to arbitrary subalgebras.

Before we go any farther, let us introduce the following construction: let $k \hookrightarrow k'$ be any field extension. Then we have a functor from the category of affine k -algebras to the category of affine k' -algebras: we map A to $A_{k'} := A \otimes_k k'$. The only thing we need to check is that if A is finitely generated over k , then $A_{k'}$ is finitely generated over k' , but this is immediate: we can write $A = k[\mathbf{x}]/I$ and then $A_{k'} \cong k'[\mathbf{x}]/I$.

We refer to this process as **base extension**.

Now we can use commutative algebra to define “geometric properties” of affine varieties. For instance, recall that a commutative ring A is called **reduced** if it has no nonzero nilpotent elements: equivalently, its **nilradical** $\text{rad}(A)$ – the intersection of all prime ideals – is equal to 0. If we have an affine algebra which is not reduced but we want it to be, we simply mod out by the radical: note, by the way, that any k -algebra quotient of an affine algebra remains an affine algebra.

Another basic fact from commutative algebra is that any Noetherian ring has only finitely many **minimal** prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. An affine algebra is **irreducible** if it has a unique minimal prime \mathfrak{p} . If so, then evidently $\mathfrak{p} = \text{rad}(A)$. Note that a ring is an integral domain iff 0 is the unique minimal prime. Thus:

Proposition 1. *For an affine algebra A (and in fact for any Noetherian ring), TFAE:*

- (i) A is an integral domain.
- (ii) A is reduced and irreducible.

In geometric language we simply speak of **integral** affine varieties.

Exercise: Show that affine k -algebra which is an integral domain can be expressed as the quotient of a polynomial ring by a prime ideal.

Let A be an affine algebra and \mathfrak{p} be a minimal prime. We define the **irreducible component** of A at \mathfrak{p} to be the quotient algebra A/\mathfrak{p} . Notice that this is a domain: in particular it is reduced. Dually then, to each of the finitely many minimal primes, we associate a “subvariety” of the corresponding affine variety $\text{Irr}(V, \mathfrak{p})$. Exercise: We would like to say, at least unofficially, that an affine variety V is the “union” of its finitely many irreducible components. State and prove a corresponding “dual” property relating the set of minimal primes to an affine algebra.

The **dimension** of an irreducible affine variety is equal to the Krull dimension of $A/\text{rad}(A)$, i.e., the supremum over all lengths of chains of prime ideals (this supremum is indeed finite). The dimension is also equal to the transcendence degree of the fraction field of $A/\text{rad}(A)$ over k , a fact which requires proof.

Exercise: Give an example of a field k and an integral k -algebra A for which the Krull dimension of A is not equal to the transcendence degree of $K(A)/k$. Is there an inequality in either direction?

The dimension of an arbitrary affine variety is the supremum of the dimensions of its irreducible components. A variety is said to be **equidimensional** if all of its irreducible components have the same dimension. An **affine curve** is an affine variety which is equidimensional of dimension 1.

Exercise: Show that $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is an integral affine \mathbb{R} -curve.

Let P be a property of affine k -varieties. We say that an affine variety A is **geometrically P** if the base extension from A to the algebraic closure \bar{k} has property

P . A **geometric property** is one for which P and “geometrically P ” coincide. A property is said to be **absolute** if A has property P iff for any algebraically closed field K containing k , A_K has property P .

Remark: The formalization of this terminology is nonstandard, but it is indeed consistent with many uses of “geometrically P ” in arithmetic geometry. The notable exception here is the “geometric genus”, which can be changed by inseparable algebraic extensions. More on this later.

Proposition 2. *Let k be a field, K/k a field extension and A an affine variety. Then $\dim(A) = \dim(A_K)$. In particular dimension is an absolute property.*

Exercise: Convince yourself that this is true. (Probably easiest is the characterization in terms of transcendence degrees.)

Example: Let l/k be a finite-dimensional field extension of degree greater than 1. Then l is an affine k -algebra, which is integral and 0-dimensional. Consider the base extension to l : $l_l = l \otimes_l l$. Then l_l is not a field: indeed, the map $(a \otimes b) \mapsto ab$ is a unital non-injective k -algebra homomorphism to l , which is not possible for a field. Therefore it is not an integral domain either (easy algebra exercise), so that l_l is not an integral variety. Exactly why this is the case depends upon l/k :

Exercise: Let l/k be a finite field extension. Show that TFAE:

- (i) l/k is separable.
- (ii) $l_l = l \otimes_k l$ is reduced.
- (iii) $l_{\bar{k}} = l \otimes_k \bar{k}$ is reduced.
- (iv) For any field extension K/k , l_K is reduced.

Exercise: Let l/k be a finite field extension. Show that TFAE:

- (i) l/k is purely inseparable.
- (ii) $l_l = l \otimes_k l$ is irreducible.
- (iii) ...
- (iv) ...

A commutative ring A is **connected** if its only idempotent elements are 0 and 1. Equivalently, A is disconnected if it can be expressed as a nontrivial product $A_1 \times A_2$.

Exercise: Let A_1 and A_2 be affine k -algebras. Find the prime ideals of $A_1 \times A_2$ in terms of the primes of A_1 and A_2 . Deduce that irreducible implies connected.

Exercise: Let l/k be a finite field extension. Notice that l/k is connected. When is it geometrically connected?

Exercise: Let k be a field, and consider the affine algebra $A = k[x, y]/(x^2 + y^2)$.

- a) Show that A is geometrically connected.
- b) Show that A is reduced iff the characteristic of k is different from 2.
- c) Show that A is integral iff -1 is not a square in k .
- d) Conclude that A is never geometrically integral.

A commutative ring A is **normal** if it is integral and integrally closed in its fraction field. Recall:

Theorem 3. *For an integral domain A , TFAE:*

- (i) A is integrally closed.
- (ii) For every prime ideal \mathfrak{p} of A , the localization $A_{\mathfrak{p}}$ is integrally closed.
- (iii) For every maximal ideal \mathfrak{m} of A , the localization $A_{\mathfrak{m}}$ is integrally closed.

Theorem 4. *(Finiteness of normalization) Let A be an integral domain with fraction field $K(A)$.*

- a) *The set A^n of all elements of $K(A)$ which are integral over A is a subring of $K(A)$ which contains A and is integrally closed.*
- b) *If A is an affine k -algebra, then A^n is finitely generated as an A -module. In particular A^n is again an affine k -algebra. In the dual category of affine varieties we have $V^n \rightarrow V$, and V^n is called the **normalization** of the integral variety A .*

Warning: In general the integral closure of a domain A need not be finitely generated as an A -module. This occurs already for one-dimensional Noetherian domains. (At least in dimension one the integral closure of a Noetherian domain remains Noetherian; even this can fail in higher dimensions!) This is one of the most basic pathologies in commutative algebra and leads to the theory of **excellent rings**, which we may (or may not) say a few words about later on.

Theorem 5. *Let (A, \mathfrak{m}) be a local Noetherian domain of dimension one. TFAE:*

- (i) A is integrally closed.
- (ii) The dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over A/\mathfrak{m} is 1.
- (iii) A is a DVR.

An affine k -algebra A is **regular** if all of its local rings are regular local rings. Recall that a Noetherian local ring with maximal ideal \mathfrak{m} is regular if $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ is equal to the Krull dimension of A .

From these two theorems we immediately deduce: an integral curve is normal iff it is regular.

Exercise: Give an example of a two-dimensional integral affine variety which is normal but not regular.

Theorem 6. *Let A/k be a regular affine k -algebra.*

- a) *There exists a finite decomposition $A = A_1 \times \dots \times A_n$, with each A_i an integral domain.*
- b) *Therefore a connected, regular affine k -variety is integral, and thus a geometrically connected, geometrically regular affine k -variety is geometrically integral.*

Proof: See e.g. [QPoints, Prop. 2.6.26].

Unfortunately, if the ground field k is not perfect, normality and regularity are not geometric properties. Let p be the characteristic of k , let $a \in k \setminus k^p$, and let $A = k[x, y]/(y^2 - (x^p - a))$.

Exercise: Show that A is normal (equivalently, regular) but A is not geometrically normal (nor geometrically regular).

We say an affine variety is **smooth** if it is geometrically regular.

Remark: Smoothness is one of the most important notions in algebraic and arithmetic geometry. Quite generally, a morphism $X \rightarrow Y$ of schemes is smooth if it is flat and the geometric fibers are all smooth. On the other hand there are many other equivalent conditions for smoothness which are important in their own right: e.g. an analogue of the Jacobian criterion as in the inverse function theorem in differential topology. Stay tuned...

Exercise: It is very often the case that for a property P , the property of being “geometrically P ” is absolute: in other words, if a property holds for an affine k -algebra A iff it holds for $A_{\bar{k}}$, then (often) it holds also for A_K for any field extension K/k . (This is one of the many incarnations of the “Lefschetz principle” in algebraic geometry.) Can you show that this is the case for all the properties we have defined so far?

Definition: An affine k -algebra is **nice** if it is smooth and geometrically integral.

Remark: Later, our definition of a “nice k -variety” will be smooth, geometrically integral and **projective**.

REFERENCES

[QPoints] B. Poonen, *Rational points on algebraic varieties*, online lecture notes.