

INTRODUCTION TO THE GEOMETRY OF SCHEMES

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1. BRIEF REMINDERS ON SHEAVES

It is a shame that one does not meet sheaves earlier in one's study of geometry: they are useful in differential and complex geometry as well, and their introduction in a basic course in one of these areas would cut down on the amount of vocabulary one has to acquire in an algebraic geometry course.

I am not going to give much of an account of sheaf theory either in my lectures or in my lecture notes; recall that on the first day I asked everyone if they knew what a sheaf was, and you all said yes! Liu's book contains all the background on sheaves we will need.

Moreover, the point of a sheaf is that all of its data is encapsulated in the stalks at every point. To be more precise, if \mathcal{F} is any sheaf on X and U is an open subset of X , then $\mathcal{F}(U)$ can be recovered as the set of elements (s_x) of the product group $\prod_{x \in U} \mathcal{F}_x$ such that: for all $x_0 \in U$, there exists an open neighborhood $x_0 \in V \subset U$ and an element $\tilde{s} \in \mathcal{F}(V)$ such that for all $x \in V$, the stalk of \tilde{s} at x is equal to s_x . In particular, if one starts with merely a **presheaf** F on X then the associated sheaf has the same stalks and is constructed from F via precisely the above recipe.

As a good rule of thumb, if ever you find yourself beginning to be snowed under by the formalism of sheaves, just stop, take a breath, and ask yourself "What is happening on the stalks?" This is usually sufficient to clear up all confusion.

Push forward and pullbacks: Let $f : X \rightarrow Y$ be a continuous function, and \mathcal{F} a sheaf on X . Then we can define a sheaf $f_*(\mathcal{F})$ on Y , the **pushforward** of \mathcal{F} in a very simple way: for V open in Y , put $f_*(\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$. Dually, if \mathcal{G} is a sheaf on Y , then we define $f^*(\mathcal{G})$ to be the sheaf associated to the presheaf $U \mapsto \lim_{\rightarrow V \supset f(U)} \mathcal{G}(V)$.¹

Example (open restriction): let \mathcal{F} be a sheaf on a space X , and let $U \subset X$ be an open subset. Writing $\iota : U \hookrightarrow X$ for the natural inclusion map, consider the sheaf $\iota^*(\mathcal{F})$ (actually no sheafification is required here). It is more transparently described as the restriction of \mathcal{F} to U and denoted $\mathcal{F}|_U$. As we don't need the formalism of pullbacks to define $\mathcal{F}|_U$, this is sort of a trivial example.

Example (arbitrary restriction): Now let \mathcal{F} be a sheaf on X and $Y \subset X$ be any subset. Writing $\iota : Y \rightarrow X$ for the natural inclusion map, consider the sheaf $\iota^*(\mathcal{F})$,

¹Note that Hartshorne uses f^{-1} for the pullback and reserves f^* for pullback on \mathcal{O}_X -modules. In my opinion, the context will make clear when this other meaning is intended.

called the restriction of \mathcal{F} to Y . This is more interesting: e.g. in case $Y = \{x\}$ is a point, we get a sheaf on a one-point space – i.e., just an abelian group – which is nothing else than the stalk of \mathcal{F} at x .

Remark 1: Suppose one has a topological space X and a base $\{U_i\}$ for the open sets, i.e., every open set is a union (possibly infinite) of the U_i 's. Then any sheaf \mathcal{F} on X is determined by its values on the basis (Liu, Remark 2.2.6, p. 34). If furthermore the base is closed under finite intersections, then given any assignment F of an abelian group to each element U_i of the base which satisfies the two presheaf axioms with respect to the base, then F extends to a unique sheaf on all of X .²

2. RINGED SPACES AND SCHEMES

A **ringed space** is a pair (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf of (as ever commutative, unital) rings on X . The sheaf \mathcal{O}_X is called the **structure sheaf**, and the sections $\mathcal{F}(U)$ are to be thought of as a ring of functions on the open subset U .

If (X, \mathcal{O}_X) is any ringed space and U is an open subset of X , then we may put $\mathcal{O}_U := \mathcal{O}_X|_U$, and (U, \mathcal{O}_U) is a locally ringed space in its own right.

A ringed space is said to be **locally ringed** if for each $x \in X$, the stalk of the sheaf at x , denoted $\mathcal{O}_{X,x}$, is a local ring, i.e., a ring with a unique maximal ideal. In this case, we write \mathfrak{m}_x for the maximal ideal in $\mathcal{O}_{X,x}$, and $k(x)$ for $\mathcal{O}_{X,x}/\mathfrak{m}_x$, the **residue field at x** . The restriction of a locally ringed space to an open subset is again a locally ringed space.

Example: Let X be a compact Hausdorff space, \mathbf{E} be either \mathbb{R} or \mathbb{C} (i.e., it doesn't matter which, but choose one and stick with it). Then the assignment of $U \mapsto C(U, \mathbf{E})$ of an open subset to the ring of all \mathbf{E} -valued continuous functions on U gives a sheaf \mathcal{O}_X . The maximal ideals in the ring $\mathcal{O}_X(X)$ of global continuous functions correspond to the points of X , and the local ring at $x \in X$ is again the ring of all germs of functions vanishing at x . Thus the stalk of \mathcal{O}_X at x is precisely the localization of the global ring at the maximal ideal \mathfrak{m} , and the residue fields are all canonically isomorphic to \mathbf{E} .

Moreover, let $R = \mathcal{O}_X(X) = C(X, \mathbf{E})$, the ring of global functions. Then each $f \in R$ really does give an \mathbf{E} -valued function on $\text{MaxSpec}(R)$. If we endow $\text{MaxSpec}(R)$ with the **weakest topology** that makes all these functions continuous, then we recover precisely the original topology on X . To do this, observe first that it is immediate that under the identification of $\text{MaxSpec}(R)$ with X , the “weak topology” is indeed weaker than the originally given topology, because by definition elements of R are continuous functions. Using the fact that any two distinct compact Hausdorff topologies on the same set are incomparable, it is therefore enough to show that the “weak topology” is nevertheless a compact Hausdorff topology. This is called **Gelfand duality** and is the main historical antecedent to the theory of schemes and ringed spaces.

²Yes, I am being a bit vague about this. I don't want to get bogged down with the details.

We also need the notion of a **morphism** of ringed spaces (resp. locally ringed spaces). If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are locally ringed spaces, a morphism between them is a pair $(f, f^\#)$, where $f : X \rightarrow Y$ is a continuous map and $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is a morphism of sheaves of rings on Y . When X and \mathcal{O}_X are both locally ringed spaces, let $x \in X$ and $y = f(x) \in Y$. For every open neighborhood V of Y , $f^{-1}(V)$ is an open neighborhood of x , and we have a morphism $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$. Composing with the map to the stalk $\mathcal{O}_{X,x}$ of X , we have a compatible family of morphisms $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_{X,x}$, as V ranges over a base of open neighborhoods for Y . This is easily seen to induce a unique homomorphism from the direct limit $\varphi_x : \mathcal{O}_{Y,y}$ to $\mathcal{O}_{X,x}$ – indeed, this is precisely the universal property of the direct limit. By a **morphism of locally ringed spaces**, we require that for all $x \in X$, each of these maps be a homomorphism of local rings: that is, $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$.

Thus we have a category of locally ringed spaces and morphisms between them. In particular we have the notion of an isomorphism of locally ringed spaces, i.e., a pair $(f, f^\#)$ such that $f : X \rightarrow Y$ is a homeomorphism and $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is an isomorphism of sheaves on Y .

A **open immersion** of lrs is a morphism $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ such that $f : X \rightarrow Y$ is an embedding onto an open subset of Y and $f^\#$ is an isomorphism on all stalks. A **closed immersion** of lrs is as above except that $f : X \rightarrow Y$ is an embedding onto a closed subset of Y and $f^\#$ is a surjection on stalks.

Warning: An open immersion just identifies (X, \mathcal{O}_X) with (U, \mathcal{O}_U) for $U = f(X)$ the corresponding open subset of Y . However, closed immersions are *not* uniquely determined by their image, so are a much richer concept. They can be understood via the concept of an **ideal sheaf**: see Liu, pp. 38-39.

Glueing construction: Suppose (X_1, \mathcal{O}_{X_1}) and (X_2, \mathcal{O}_{X_2}) are locally ringed spaces, $U_i \subset X_i$ are open subsets, and we have an isomorphism of locally ringed spaces $\phi : (U_1, \mathcal{O}_{U_1}) \rightarrow (U_2, \mathcal{O}_{U_2})$. We can define a locally ringed space (X, \mathcal{O}_X) from glueing X_1 and X_2 along $U_1 = U_2$. (Details left to you.) More generally, given an entire indexed family $(X_i, \mathcal{O}_{X_i})_{i \in I}$, and all $(i, j) \in I \times I$ an open subset U_{ij} , and isomorphisms of locally ringed spaces $\varphi_{ij} : U_{ij} \rightarrow U_{ji}$ satisfying:

- (GC1) $\varphi_{ii} = Id$,
- (GC2) $\varphi_{ji} = \varphi_{ij}^{-1}$,
- (GC3) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{jk}$,

we can glue the spaces to get a space X .³

This gives a very powerful and flexible way to generate entire geometric theories. For example, a topological manifold is a locally ringed space which is locally isomorphic to the locally ringed space \mathbb{R}^n , with structure sheaf the ring of all continuous functions. A smooth manifold is the same thing, only this time we take the

³This is Exercise 2.8 of Liu's book.

structure sheaf of \mathbb{R}^n to be the set of smooth (say, C^∞) functions. (We can generate infinitely many other flavors of differentiable structures by imposing smoothness conditions of different strengths.) A \mathbb{C} -manifold is a locally ringed space which is locally isomorphic to a ringed space of the form (U, \mathcal{O}_U) , where $U \subset \mathbb{C}^n$ is an open subset and \mathcal{O}_U is the sheaf of holomorphic functions on U .

Remark: There is an important difference between the smooth and analytic cases. In the former case there are “sufficiently many” global functions, whereas in the latter case there are not. Suppose for instance that X is a connected, compact \mathbb{C} -manifold. Then by the maximal modulus principle, the only global holomorphic functions on X are the constant functions: $\mathcal{O}_X(X) = \mathbb{C}$. On the other hand, the existence of continuous (resp. smooth) partitions of unity shows that the structure sheaf of a topological (resp. smooth) manifold is soft.

If you compare this to the standard definition of, say, a smooth manifold, one’s first reaction is that it can’t be right because it is too easy! We don’t need the notion of a differentiable atlas (which, notice, never seems to get used in any nontrivial way in the theory), we don’t need a separate definition of differentiable functions nor a separate definition for a morphism of differentiable manifolds: all this is taken care of “automatically” by the formalism of locally ringed spaces. But it is right, and it is easier.

Informally speaking, one defines a class of locally ringed spaces to which we require a certain category of locally ringed spaces be locally isomorphic is called a class of **model spaces**. It follows from the definition that every locally ringed space of a certain class is obtained from a family of model spaces via the glueing construction.

Finally, a scheme is what we get by taking as the model spaces the locally ringed spaces associated to commutative rings. Let R be any ring. Remark 1 of the previous section applies to $X = \text{Spec } R$ endowed with the Zariski topology. Namely a base for the topology is given by sets $U(f) = \text{Spec } R \setminus V(f)$ for any $f \in R$. In other words, $U(f)$ is the set of all prime ideals which **do not** contain f . It is standard to check that this is a base for the Zariski topology, and moreover $U(f) \cap U(g) = U(fg)$, so it is closed under finite intersections. Let us now define a sheaf \mathcal{O}_X by $\mathcal{O}_X : U(f) \mapsto R_f$, i.e., the localization of R at the multiplicative set $\{f^n \mid n \in \mathbb{N}\}$. If $U(f) \supset U(g)$, then you can check that $f \mid g$, so that the natural map $R \rightarrow R_g$ factors through R_f . After checking some further stuff, we do indeed get a unique sheaf of rings on X in this way. Moreover, the stalk at a point $\mathfrak{p} \in \text{Spec } R$ is nothing else than the local ring $R_{\mathfrak{p}}$. Thus $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ is a locally ringed space.

Definition: An **affine scheme** is a locally ringed space isomorphic to $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ for a commutative ring R . A **scheme** is a locally ringed space which is locally isomorphic to an affine scheme, i.e., a lrs which for every $x \in X$ there exists an open set U containing x such that (U, \mathcal{O}_U) is isomorphic to an affine scheme.

Example: For any field k , we define \mathbb{A}_k^n to be the affine scheme $\text{Spec } R_n$, where $R_n = k[t_1, \dots, t_n]$. You should convince yourself that giving a quotient algebra

A of R_n gives a morphism of locally ringed spaces which on topological spaces is the inclusion of the closed subspace $\text{Spec } A$ into $\text{Spec } R_n$. Moreover, the canonical map $k \hookrightarrow R_n$ dualizes to give a morphism $\mathbb{A}_{/k}^n \rightarrow \text{Spec } k$. In fact we did not use anywhere that k is a field: the same works with k replaced by any commutative ring A , and we get affine n -space over A , which has a map down to $\text{Spec } A$.

Remark: As a matter of taste, I often prefer to write A instead of $\text{Spec } A$ for no good reason other than brevity. It shouldn't be confusing (I hope) because A is not itself a locally ringed space; in context it should be immediate to tell when $\text{Spec } A$ is meant.

If S is any scheme, then an **S-scheme** is a scheme X together with a morphism of schemes (this is just a morphism of lrs's whose source and target are schemes) $X \rightarrow S$. However, we often write instead X/S . If X and Y are S -schemes, a morphism of S -schemes is a morphism $X \rightarrow Y$ over S , i.e., such that the composite map $X \rightarrow Y \rightarrow S$ is the same as the given map $X \rightarrow S$. Especially, when $S = \text{Spec } k$ we have the category of $\text{Spec } k$ -schemes – but, as above, I will refer to it as the category of k -schemes instead. Note that an affine k -scheme is an arbitrary k -algebra⁴. Similarly, for any ring A we have the category of A -schemes, where $S = \text{Spec } A$.

Exercise: Show that for every scheme X there is a unique morphism of schemes $X \rightarrow \text{Spec } \mathbb{Z}$. (Start with the affine case, of course, and globalize.) Thus every scheme is a \mathbb{Z} -scheme, which is the globalization of the fact that any commutative ring is a \mathbb{Z} -algebra.

Exercise: For any scheme S , construct affine n -space \mathbb{A}_S^n by glueing affine n -spaces over an affine covering of S .

Definition: For a scheme S , an **affine S-scheme** is a scheme which is isomorphic to a closed subscheme of \mathbb{A}_S^n .

Definition: A morphism $X \rightarrow Y$ of schemes is **affine** if the preimage of every affine open subscheme is affine.

Exercise: How does this relate to Maxim's definition, that an affine morphism of S -schemes should factor through \mathbb{A}_S^n for some n ? (Some additional hypotheses seem to be needed for these to be equivalent.)

Exercise: If $S = \text{Spec } A$ is affine, show that any affine S -scheme $X = \text{Spec } R$, where R is a quotient of $A[t_1, \dots, t_n]$.

Example: Let k be an arbitrary field. We will construct the projective line \mathbb{P}^1 over k by glueing together two copies of the affine line \mathbb{A}^1 . The open subsets are $U_1 = U_2 = \mathbb{A}^1 \setminus \{0\}$. The coordinate ring is $k[t, t^{-1}]$, and we glue the coordinate rings via the isomorphism $t \mapsto t^{-1}$ from $k[t, t^{-1}]$ to $k[t, t^{-1}]$. Note that to glue along affine subschemes it's enough to give isomorphisms (satisfying the glueing

⁴Note: so not necessarily a finitely generated k -algebra, or what we have been calling "affine k -algebras." Sorry about that!

conditions) of the corresponding commutative rings.

Again, we didn't need k to be a field; the same works for any commutative ring A , giving us the projective line over A . Moreover, by glueing over an affine covering we can define \mathbb{P}^1 over any scheme S .

Exercise: By suitably glueing together $n + 1$ copies of affine n -space over A , construct \mathbb{P}^n_A , projective n -space over the ring A . (Suggestion: You can find this relatively early on in Liu's book.)

You should be at least dimly aware that there is another construction of projective space which is often extremely useful: just as affine space is constructed out of the prime ideals of any ring, given a **graded** commutative ring $B = \bigoplus_{d \in \mathbb{N}} B_d$ one can construct a lrs whose underlying space $Proj B$ is the set of homogeneous prime ideals of B which do not contain the **irrelevant ideal** $B^+ = \bigoplus_{d > 0} B_d$. The details of the projectivization process may be found in Liu, pp. 50-53.

Definition: A **projective S-scheme** is a scheme which is isomorphic to a closed subscheme of \mathbb{P}^n_S . When $S = \text{Spec } A$ is affine, there is a purely algebraic definition in terms of homogeneous ideals in the polynomial ring $A[T_0, \dots, T_n]$.

Definition: A morphism $f : X \rightarrow \text{Spec } A$ is **projective** if it factors into a closed immersion $X \rightarrow \mathbb{P}^n_A$ followed by the canonical morphism $\mathbb{P}^n_A \rightarrow \text{Spec } A$. (Note that we do not define what it means for a general morphism to be projective. This is more complicated and we will not need it.)

Definition: A morphism $X \rightarrow Y$ of schemes is **of finite type** if for each affine open subscheme $V = \text{Spec } R$ of Y , $f^{-1}(V)$ has a finite affine covering $\bigcup \text{Spec } S_i$, where each S_i/R is finitely generated as an algebra.

Definition: A morphism $X \rightarrow Y$ of schemes is **finite** if it is affine, and for each affine open $V = \text{Spec } R$ of Y , $f^{-1}(V) = \text{Spec } S$, where S is finitely generated as an R -module.

3. REDUCED AND INTEGRAL SCHEMES

Please read §2.4 in Qing Liu's book!

4. DIMENSION OF SCHEMES

Please read §2.5 in Qing Liu's book!

5. MORPHISMS FROM A FIELD TO A SCHEME

Let K be a field and X be any scheme. We will determine $\text{Hom}(\text{Spec } K, X)$, the set of all homomorphisms of schemes from $\text{Spec } K$ to X . First, we need a map of topological spaces; since $\text{Spec } K$ is a one point space, such maps are in bijective correspondence with points x of X . After a bit of unwinding of the definitions, one sees that a map $f^\# : X \rightarrow f_*(\text{Spec } K)$ inducing a local homomorphism on stalks is precisely given by a field homomorphism $\iota : k(x) \rightarrow K$, i.e., a map from the residue field of x into K . Thus it is not at all true that homomorphisms simply correspond

to points of x : $k(x)$ need not embed in K at all, and if it does it may do so in infinitely many different ways.

For example, $\text{Hom}(\text{Spec } k, \text{Spec } k) = \text{Aut}(k)$, which can be any group whatsoever. In particular $\text{Hom}(\text{Spec } \mathbb{C}, \text{Spec } \mathbb{C})$ is an enormous group, of cardinality $2^{2^{\aleph_0}}$.

Suppose on the other hand that we are working in the category of schemes over a field k . Then $X(K)$ can be identified with pairs $(x \in X, \iota \in \text{Hom}_k(k(x), K))$. In particular, if $K = k$, then $X(k)$ is the set of closed points with residue field k , as in the classical case.

On the other hand, any integral scheme X has a **generic point**, namely a canonical morphism from $\text{Spec } K \rightarrow X$, where $K = K(X)$ is the field of rational functions on X .

Exercise: For any field K , show that $\text{Hom}(\text{Spec } K, X) = \text{Hom}(\text{Spec } K, X^{\text{red}})$, where X^{red} is the reduced subscheme of X .

Remark: More generally, if X is an S -scheme, then a **section** of X over S is an element of $\text{Hom}_S(S, X)$.

6. THE FUNCTOR OF POINTS

Let X be any S -scheme. We then get a contravariant functor from the category of S -schemes to the category of sets, by

$$X(Y) := \text{Hom}_S(Y, X).$$

This is called the **functor of points** of X . Note that this generalizes the notion of points in the previous section.

In classical algebraic geometry, a variety V is determined by its k -valued points: this is true if $k = \bar{k}$ and V is finite type and **reduced**. However, by the exercise at the end of the previous section, this cannot be true if X is not reduced, because the K -valued points “do not see” the nilpotent elements in the structure sheaf.

However, the following is true (and indeed is a piece of general nonsense):

Theorem 1. (*Yoneda Lemma*) For S -schemes X and Y , TFAE:

- (i) $X \cong_S Y$.
- (ii) $\text{Hom}_S(-, X) \cong \text{Hom}_S(-, Y)$ are isomorphic functors.

In fact one can do a little better: if $\text{Hom}_S(-, X)$ and $\text{Hom}_S(-, Y)$ agree as functors restricted to affine S -schemes, then $X \cong_S Y$. Because of this, it is often easiest to define a scheme via its associated “functor of points.”

Example: We define the scheme $\mathbb{A}_{\mathbb{Z}}^n$ via the functor on \mathbb{Z} -schemes (i.e., schemes) $\text{Spec } R \mapsto R^n$.

Example: We define the scheme GL_N over \mathbb{Z} via the functor $\text{Spec } R \mapsto GL_N(R)$.

Note that for both of these examples the functor lands not just in the category of sets but in the category of groups. In such a way one can define a **group scheme**.

Warning: It does not work to define projective n -space over \mathbb{Z} functorially using homogeneous coordinates, since indeed for an arbitrary ring R we need not have $\mathbb{P}^n(R) = R^{n+1} \setminus \{0\}/R^\times$.