FIBER PRODUCTS; SEPARATED AND PROPER MORPHISMS

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1. Fiber products and base change

A very important tool across geometry is the fibered product. Here the basic statement is simple: given two S-schemes X and Y, the fiber product $X \times_S Y$ exists in the category of schemes. Namely, this is an S-scheme, endowed with S-morphisms π_X, π_Y to X and Y, satisfying the following universal mapping property: given any S-scheme Z and S-morphisms $f : Z \rightarrow X$, $g : Z \rightarrow Y$, there exists a unique morphism $f \times_S g : Z \rightarrow X \times_S Y$ making the usual diagram (see e.g. Liu, p. 79) commutative.

As usual, the first thing to do is consider the affine case with all the arrows reversed. In this case, the object which makes the diagram commute is nothing other than the tensor product $R \otimes_A S$. The general case is, as usual, reduced to the affine case by glueing (and, as is often the case, this requires some nontrivial work).

Most often we take fiber products over an affine scheme $\operatorname{Spec} A$, and we will abuse notation by writing $X \times_A Y$ rather than $X \times_{\operatorname{Spec} A} Y$. In particular, for any field (or indeed any ring) k we have

$$\mathbb{A}^m_k \times_k \mathbb{A}^n_k \cong \mathbb{A}^{m+n}_k,$$

this being the geometric analogue of the usual statement about tensor products of polynomial rings.

A useful fact, which follows immediately from the universal property, is:

$$(X \times_S Y)(S) = X(S) \times Y(S).$$

If X is an S-scheme and S' is another S-scheme, then we can form the fiber product $X \times_S S'$. This is still an S-scheme, but it is also an S'-scheme via the second projection map. This process – preciesly, of starting with a scheme over S taking the fiber product with a second scheme S' over S, and regarding the fiber product as an S'-scheme, is called **base change**. We denote the base change by $X_{/S'}$. In the case of a field extension $k \hookrightarrow l$ and a k-scheme X, the base change $X_{/l}$ is a scheme over l. When X is an affine k-variety, this recovers our previous process of "base extension" by tensorization from k to l. Note that in general a base change from a $\operatorname{Spec} R$ scheme to a $\operatorname{Spec} S$ involves a homomorphism $R \rightarrow S$ which need not be injective, hence we use the term "base change" rather than "base extension."

An extremely important application of base change is to the notion of **fibers** of a morphism. Let $f : X \rightarrow Y$ be a morphism of schemes, and $y \in Y$ be any point (closed or otherwise). We want to define a scheme X_y, the **fiber of f over y** in a way which generalizes the usual idea of fibers of a map of sets. The key to this is to use the canonical morphism $\operatorname{Spec} k(y) \hookrightarrow Y$. Then we can define

$$X_y := X \times_Y \operatorname{Spec} k(y),$$

which is a scheme over the field $\operatorname{Spec} k(y)$.

So for instance, if Y is any scheme and $y \in Y$, the fiber of $\mathbb{P}^n_Y$ over y is the usual projective space over the residue field $k(y)$.

It is not obvious that this fiber product is compatible with fibers in the topological sense, but it is true:

Proposition 1. *(Liu 4.1.16) Let $f : X \to Y$ be a morphism of schemes and $y \in Y$. Then the projection map $\pi : X_y = X \times_Y \operatorname{Spec} k \to X$ induces a homeomorphism of X_y onto $f^{-1}(y)$.*

Therefore we can view a morphism $f : X \to Y$ as a **family** of scheme structures over fields on the topological fibers of the morphism. This is of crucial importance in algebraic geometry.

This brings us to a very important definition. A property of morphisms of schemes is said to be **stable under base change** if for any morphism $X \to Y$ satisfying that property, all base changes $X \times_Y Y' \to Y'$ also have that property. A property is said to be **faithfully preserved by base change** if also its negation is stable under base change: that is, $X \to Y$ has that property if and only if every base change $X \to Y$ has that property.

Needless to say, many important properties are stable, or faithfully preserved, only under base changes of a particular kind. We will see examples soon enough.

Proposition 2. *(Liu, 3.1.23, 3.2.4) a) Open immersions and closed immersions are each stable under base change.*
b) Finite morphisms are stable under base change.
c) Finite type morphisms are stable under base change.
d) Affine and projective morphisms are stable under base change.

Example: (Liu, p. 84): Let m be a nonzero integer. Put $f : X = \operatorname{Spec} \mathbb{Z}[x,y]/(xy - m) \to \operatorname{Spec} \mathbb{Z}$. The fiber over the generic point is $\mathbb{Q}[x,y]/(xy - m)$, which is isomorphic to $\mathbb{G}_m$ over $\mathbb{Q}$ (i.e., to the projective line with two points removed). Any closed point of $\operatorname{Spec} \mathbb{Z}$ corresponds to a prime ideal (p), and the base change is just the spectrum of $\mathbb{Z}[x,y]/(xy - m) \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[x,y]/(xy - m)$. If m is not zero modulo p, again this curve is isomorphic to $\mathbb{G}_m$ over $\mathbb{F}_p$. If $p \mid m$ then the fiber is just $\mathbb{F}_p[x,y]/(xy)$, i.e., two lines crossing. So unless $m = \pm 1$ there will be at least one singular fiber. On the other hand, a more subtle question is whether X itself is integral, normal or regular. Evidently it is integral: the polynomial $xy - m$ is irreducible. We can see it is normality by applying, as Maxim did, Serre's criterion: the singular locus has codimension 2 and the scheme is a local complete intersection. Exactly what this means and why this is true is something we will come back to later. (I want to be clearer on it myself, so don't worry; we will not skip it!)

Exercise: Show that $\operatorname{Spec} \mathbb{Z}[x,y]/(xy - m)$ is regular iff m is squarefree.

Example: Let $X = \operatorname{Spec} \mathbb{Z}[x]/(17x)$. Then the fiber over the generic point is simply a single point $\operatorname{Spec} \mathbb{Q}$, and similarly the fiber over $\mathbb{F}_p$ is just $\operatorname{Spec} \mathbb{F}_p$ for all $p \neq 17$. On the other hand, the fiber over $p = 17$ is the entire affine line $\mathbb{A}^1$. Thus the fibers fail to be equidimensional, and have a "discontinuity" at $p = 17$, whereas we would like the fibers of our "family" to be, in some vague geometric/topological sense, "continuous." It turns out that the problem here is that X is not **flat** over $\mathbb{Z}$.

Example: Let $X = \operatorname{Spec} \mathbb{Z}[x]/(17x - 1)$. This time X is integral, and the fiber over any point except $p = 17$ is a single point. However, the fiber over $p = 17$ is empty! This time the problem is a failure of **properness**.

2. Applications to algebraic varieties

At this point we can revisit the results about base extension we established (or at least stated) for affine varieties over a field k and generalize them to all varieties over k. We just state the results from Liu, but note that he gives complete proofs.

Proposition 3. *(Liu, §3.2.2) Let $X_{/k}$ be an algebraic variety, and let l/k be an algebraic extension.*
a) $\dim X_{/l} = \dim X$.
b) If X is reduced and l/k is separable, then $X_{/l}$ is reduced.
c) If l/k is purely inseparable, then the projection $X_l \to X$ is a homeomorphism.
d) If $\operatorname{char}(k) = p > 0$, put $l := k^{p^{-\infty}}$ be the perfect closure of k. Then X is geometrically reduced iff $X_{/l}$ is reduced.
e) Let $l = k^{\mathrm{sep}}$, the separable algebraic closure of k. Then X is geometrically connected (resp. geometrically irreducible) iff $X_{/l}$ is geometrically connected (resp. geometrically irreducible).
f) The variety X is geometrically integral iff $K(X)$ and $\overline{k}$ are linearly disjoint over k. In this case we have $K(X_{\overline{k}}) = K(X) \otimes_k \overline{k}$.
g) X is geometrically irreducible iff $K(X) \cap k^{\mathrm{sep}} = k$.
h) X is geometrically reduced iff $K(X)$ is a finite separable extension of a purely transcendental extension $k(t_1, \ldots, t_d)$ of k.

Exercise: Let $f : X \to Y$ be a morphism (i.e., continuous map) of sober topological spaces, with generic points η_X and η_Y. Show that the following are equivalent:
(i) $f(\eta_X) = \eta_Y$.
(ii) $f(X)$ is dense in Y.
A morphism satisfying these equivalent conditions is called **dominant**.

Exercise: Let $f : X \to Y$ be a morphism of integral schemes, and let η_X, η_Y be the generic points of X and Y respective. Show that TFAE:
(i) f is dominant.
(ii) f induces an extension $K(Y) \hookrightarrow K(X)$.[1]

By definition, a dominant morphism between integral schemes is **birational** if the induced map $K(Y) \hookrightarrow K(X)$ is an isomorphism.

[1] I could be more precise, but I think it is instructive to figure out exactly how this works for yourself.

Exercise: Let X and Y be k-varieties (or, more generally, finite type schemes over any base S). Show that TFAE:
(i) There exists an dense open subset U of X and a birational morphism $f : U \to Y$.
(ii) There exist dense open subsets U of X and V of Y such that the open subschemes $(U, \mathcal{O}_U)$ and $(V, \mathcal{O}_Y)$ are isomorphic k-schemes.[2]

Exercise (Liu, 3.2.7): Let k be a field with algebraic closure $\overline{k}$. Let $\overline{X}$, $\overline{Y}$ be varieties over $\overline{k}$, and $\overline{f} : \overline{X} \to \overline{Y}$ a morphism. Show that everything can be defined over a finite subextension l: that is, there exists a finite extension l/k, varieties X, Y defined over l and a morphism $f : X \to Y$ defined over l such that after base change we have isomorphisms $\iota_X : X_{\overline{k}} \cong \overline{X}$, $\iota_Y : Y_{\overline{k}} \cong \overline{Y}$

$$\overline{f} = \iota_Y \circ .f_{\overline{k}} \circ \iota_X^{-1}.$$

Can the finite extension l always be taken to be normal? separable?

Exercise (Liu, 3.2.9): Let X, Y be varieties over a field k, with X geometrically reduced. Suppose $f, g : X \to Y$ are morphisms which induce the same map $X(\overline{k}) \to Y(\overline{k})$ on $\overline{k}$-valued points. Show that $f = g$. (Suggestion: a plan of attack is given in Liu's book.) Does the conclusion hold if X is not assumed to be geometrically reduced?

Exercise (Liu, 3.2.10): Let k be a field and l/k be a finite Galois extension with Galois group G. Let X be an irreducible variety over k.
a) Show that G acts transitively on the irreducible components of X_l. Deduce that $X_{\overline{k}}$ is equidimensional (i.e., all irreducible components have the same dimension.)
b) Suppose X is connected, and show G acts transitively on the connected components of X_l.

Exercise: Let X be a variety over a field k.
a) Suppose X is connected and $X(k) \neq \emptyset$. Show that X is geometrically connected.
b) Deduce that the following conditions on a group variety $G_{/k}$ are equivalent:
(i) G is integral.
(ii) G is geometrically integral.
(iii) G is reduced and conected.
c) Does part a) hold if "connected" is replaced by "irreducible"?

Exercise: Show that for each of the following properties "P" of algebraic varieties over a(ny) field k, being "geometrically P" implies being "absolutely P": (i) connectedness, (ii) reducedness, (iii) irreducibility. (Suggestion: consult Exercise 3.2.14 of Liu's book.)

2.1. **Separated morphisms.**

As we have seen, scheme theory incorporates topology in that every scheme has an underlying topological space. This perspective was slow in coming historically. Although the Zariski topology on a complex algebraic variety is very natural one (Galois connections were around at the time!), when compared to the classical topology on a complex manifold it is bizarrely coarse. In particular the underlying

[2]This is a corrected version of Exercise 3.2.6 in Liu.

space of a scheme is only a Hausdorff space if the space is zero-dimensional, which is for most purposes a fairly trivial case. But the Hausdorff property of topological spaces is ubiquitously useful in topology and geometry, so it is natural to have some "geometric analogue" of the Hausdorff property, which is called **separatedness**.

We begin by remarking that Liu – or at least, his French to English translator R. Erné– uses the word "separated" also for topological spaces where we anglophones would use "Hausdorff". This is a somewhat unfortunate usage. In my own secret notes on general topology, I call a space "separated" if for any distinct points p, q the subsets $\{p\}$ and $\{q\}$ are separated from each other in the sense that each is disjoint from the closure of the other. Equivalently all singleton sets are closed, a.k.a. T_1. In the context of I-adic topologies in commutative algebra, one often uses the word "separated" to mean precisely Hausdorff – usually in the phrase "complete and separated for the I-adic topology" – but in the context of topological groups (or all uniformizable spaces; equivalently, all spaces satisfying the $T_{3.5}$ separation axiom), separated and Hausdorff are equivalent. This is sort of ironic because as we have seen the Zariski topology restricted to the set of closed points is a T_1 topology, but then this T_1 property is generally lost in the entire Zariski topology. (I should now say something about **sober spaces**, but I'll spare you.)

Proposition 4. *Let X and Y be topological spaces, and $f, g : X \to Y$ be continuous maps between them. Let $S = \{x \in X \mid f(x) = g(x)\}$. Then, if Y is Hausdorff, S is closed. In particular, two continuous maps into a Hausdorff space which agree on a dense subspace must be equal.*

I leave the proof to you, as well as the task of investigating to what extent the converse is true.

The geometric definition of separated will be, as is the fashion of the time, a **relative** definition; that is, we will not (just) define what it means for a scheme to be separated but rather what it means for a morphism $X \to S$ of schemes to be separated. (But then a scheme itself can be called separated if it is separated as a $\mathbb{Z}$-scheme, and a variety over a field will be separated if the natural map $V \to \operatorname{Spec} k$ is separated.) Whatever the definition is, it will render true the following result:

Proposition 5. *(Liu, Prop. 3.11, p. 102) Let S be a scheme, X a reduced S-scheme and Y a separated S-scheme. Consider two morphisms $f, g : X \to Y$. Suppose that there is a dense open set U [so any nonempty open set, if X is irreducible] such that $f|_U = g|_U$. Then $f = g$.*

Exercise: Give an example to show that reducedness of X is necessary here.

Anyway, to get the right definition, we recall from topology that a space X is Hausdorff iff for the diagonal map $\Delta : X \to X \times X$, we have $\Delta(X)$ is closed.

Definition: A morphism $X \to S$ is separated if the diagonal morphism $\Delta : X \to X \times_S X$ is a closed immersion. (You should stop and check that the universal property of the fiber product gives rise to a diagonal morphism!)

Example: Let R be any ring. If we take two copies of the affine line $\mathbb{A}^1_R$ and glue the open subsets $\mathbb{G}_m$ (i.e., $\mathbb{A}^1 \setminus \{0\}$) to each other *identically* – rather than via the "half twist" $t \mapsto \frac{1}{t}$ that results in the projective line – then we end up with "the line with two origins", which can be seen not to be separated. Notice that this is directly analagous to a similar construction of a non-Hausdorff manifold in classical geometry.

Some useful facts about separated morphisms are collected in Proposition 3.3.9 of Liu's book (and elsewhere in the section). In particular: affine and projective morphisms are separated, open and closed immersions are separated), compositions of separated morphisms are separated, and separatedness is stable under base change.

This last property is very convenient: it means that if we required varieties over a field to be separated as well as of finite type (as is much more standard), then it shouldn't make our life any different: still any finite type separated m morphism of schemes $f : X \to Y$ gives rise to a family of algebraic varieties.

Exercise: Let R be an integral domain with fraction field k, and let X be an R-scheme.
a) Explain why there is a canonical morphism $X(R) \to X(k)$ from R-valued points to k-valued points.
b) Suppose R is integrally closed (or "normal") and $X \to \operatorname{Spec} R$ is separated. Show that the above morphism is injective. (Hint: first do the case of R a DVR.)
c)* Find an example where the map $X(R) \to X(k)$ is not injective. Can this happen when $X \to \operatorname{Spec} R$ is separated?

3. Proper morphisms

Another consequence of the failure of Zariski topologies to be Hausdorff is that compactness does not work the way it should. It is well known that a compact Hausdorff space has many more nice properties than a space which merely satisfies the open covering definition of compactness (and this is the motivation behind Bourbaki's neologism[3] "quasi-compact"). For instance, a compact Hausdorff space is necessarily normal and thus satisfies the Baire category theorem.

Moreover, if X is quasi-compact and Y is Hausdorff, then any continuous map $f : X \to Y$ is necessarily closed. Indeed, if S is a closed subset of X then S is quasi-compact, and then $f(S)$ is quasi-compact in the Hausdorff space Y and is therefore closed. This does not hold without the Hausdorff assumption: for instance an open immersion of Noetherian schemes is usually not a closed map.

Note also that requiring separatedness does not fix this: the open immersion from $\mathbb{G}_m$ to $\mathbb{A}^1$ is a map from separated, quasi-compact $\mathbb{Z}$-schemes which is not closed. We have to work a little harder.

Here it turns out that it is better to consider the apparently more difficult problem of how to define compactness in a relative sense.

[3]If the term "neologism" applies to a word that is about 70 years old...

3.1. **Analytic topologies.**

Let k be a topological field, which we will assume to be locally compact and nondiscrete. There is a complete classification of such fields – first there is the complex numbers, which alone among all such fields is algebraically closed. Next there is the real numbers, for which the topological behavior of algebraic varieties is still "classical" but is surprisingly complicated compared to the complex case. The remaining locally compact nondiscrete fields are precisely the fraction fields of complete discrete valuation rings (CDVRs) with finite residue field. They include: $\mathbb{Q}_p$ and all of its finite extensions, and $\mathbb{F}_p((t))$ and all of its finite extensions (all of which turn out to be isomorphic to $\mathbb{F}_q((t))$ for some prime power q).

If $X_{/k}$ is a variety in our sense, then the set $X(k)$ of k-points can be endowed with a new topology, which is finer than the Zariski topology. We call this the **analytic topology** on $X(k)$. The task of a precise definition I leave to you, but roughly:

Step 1: We endow $\mathbb{A}^n(k) = k^n$ with the natural topology of the product of n copies of the analytic topology on k.

Step 2: If $X = \operatorname{Spec} R$ is an affine k-variety, we endow it with the topology it inherits as a subset of affine n-space.

Step 3: A general k-variety is obtained via glueing, and this glueing data equally well allows us to glue to get an analytic topology.

Exercise X.X:
a) For any k-variety X, $X(k)$ is locally compact.
b) If $X_{/k}$ is separated, $X(k)$ is Hausdorff.
c) If $X = \mathbb{C}$ and X is connected in the algebraic sense, then $X(\mathbb{C})$ is connected in the analytic topology.
d) IF $X = \mathbb{R}$, then X may be Zariski connected even though $X(\mathbb{R})$ is disconnected. However, in all caess $X(\mathbb{R})$ has finitely many connected components. If k is non-Archimedean, then $X(k)$ is totally disconnected in the analytic topology.
e) If $X_{/k}$ is any projective variety, $X(k)$ is compact.

A key question is then to what extent the converse to part e) holds. A bit of thought shows that unless $k = \mathbb{C}$, then, because k is not algebraically closed, we cannot expect the point set $X(k)$ to tell us enough about the algebraic variety $X_{/k}$. For instance, $X(k)$ might be empty! So really the question is: when $k = \mathbb{C}$, what is a purely geometric property on a variety X which is equivalent to compactness of $X(\mathbb{C})$ in the analytic topology?

Again it turns out that the correct idea can be found in general topology. Namely, a continuous map $f : X \to Y$ of topological spaces is **proper** if the preimage of any compact subspace is compact. Thus compactness of a space X can be recast "functorially" as saying that the map from X to a single point $\bullet$ is proper. In general, properness is a useful geometric property which is indeed a relative version of compactness.

Here is a true fact about proper maps: suppose that X and Y are Hausdorff spaces and Y is locally compact (which again, we think of varieties as always being). Then a map $f : X \to Y$ is proper iff it is **universally closed**: for any topological space Z, the "base change" $f_Z : X \times Z \to Y \times Z$ is a closed map.

Finally, we have achieved sufficient motivation to make the following definition palatable:

Definition: A morphism $f : X \to Y$ of schemes is **proper** if it is of finite type, separated and universally closed.

An especially important special case is when $Y = \operatorname{Spec} k$. Then a variety $f : X \to \operatorname{Spec} k$ is called **complete** if the structural morphism is proper.

The most basic – and unfortunately nontrivial – fact that we need to have in order to accept this as a plausible definition is the following:

Proposition 6. *A projective morphism is proper.*

In particular, any projective variety is a complete variety. It turns out that completeness is the answer to the above question about analytic topologies:

Theorem 7. *For a variety $X_{/\mathbb{C}}$, $X(\mathbb{C})$ is compact in the analytic topology iff X is a complete variety, i.e., $X \to \operatorname{Spec} \mathbb{C}$ is proper.*

This is a bit more trouble than it is worth to prove, so we omit it entirely.

Related to this is the much more delicate question of whether and when a complete k-variety must be projective. Here is some information about this:
(i) A quasi-projective variety is complete iff it is projective.
(ii) An algebraic curve is complete iff it is projective.
(iii) A group variety is quasi-projective. In particular, a complete group variety is necessarily projective.
(iv) There is a nice algebraic surface over $\mathbb{C}$ which is complete but not projective.
(v) There exists a projective variety $V_{/\mathbb{C}}$ and an algebraic action of $G = \mathbb{Z}/2\mathbb{Z}$ on V such that the quotient variety V/G is complete but not projective.
(vi) In a quasi-projective variety, any finite set of closed points is contained in an affine open subvariety. This need not hold for general varieties.

The following result gives deeper information on the connection between proper and projective morphisms.

Theorem 8. *(Chow's Lemma) Let $f : X \to Y$ be a proper morphism, with Y a Noetherian scheme. Then there exists a projective morphism $g : X' \to Y$ and a birational morphism $p : X' \to X$ such that $g = p \circ f$.*

A **compactification** of a variety $V_{/k}$ is a birational morphism $\iota : V \hookrightarrow \overline{V}$, where $\overline{V}$ is a complete variety.

Exercise X.X: Show that every algebraic variety admits a compactification. (Hint:

use Noether normalization.)

Centuries of geometric intuition reveal that it is fruitful to think about incomplete varieties from this extrinsic perspective, i.e., in terms of a(t least one) compactification with some points removed. This works out very nicely for algebraic curves: any algebraic curve is either affine or projective, and on an affine curve the number of points one must add in order to get a compactification is an invariant of the curve. It is beyond the scope of these notes to explain all the various ways in which completeness is useful for geometry. This shows up in intersection theory, going all the way back to the slogan "parallel lines meet at infinity", it shows up in the acyclicity of coherent sheaves on affine varieties, and the need for more complicated cohomology theories – e.g. "cohomology with compact supports" – for noncompact manifolds.

We record a technical proposition from Liu's book.

Proposition 9. *Let $X \to \operatorname{Spec} A$ be a proper scheme.*
a) Then $\mathcal{O}_X(X)$ is integral over A.
b) If $A = k$ is a field and X is reduced then $\mathcal{O}_X(X)$ is a finite dimensional k-vector space. If moreover X is connected, then $\mathcal{O}_X(X)$ is a finite field extension of k. If X is moreover geometrically connected (resp. geometrically integral) then $\mathcal{O}_X(X)$ is purely inseparable over k (resp. $\mathcal{O}_X(X) = k$).

3.2. **Valuative criterion for properness.**

Theorem 10. *Let $f : X \to Y$ be a finite type morphism of locally Noetherian schemes. TFAE:*
(i) f is proper.
(ii) Let R be a DVR with fraction field K and $\operatorname{Spec} R \to Y$ be a morphism of schemes such that the restriction to $\operatorname{Spec} K \to Y$ factors through X. Then the natural map

$$\operatorname{Hom}_Y(R, X) \to \operatorname{Hom}_Y(K, X)$$

is a bijection.

Application: $Y = \operatorname{Spec} R$. Then the theorem asserts that if $X \to SpecR$ is proper, then the canonical map $X(R) \to X(K)$ is a bijection. Notice that for a projective variety we get this result by clearing denominators; the valuative criterion assures us that properness still gives this important property.

This implies a result mentioned at the beginning of the course:

Theorem 11. *Let k be a field, X a regular k-variety and Y a complete k-variety. Let $f : X \to Y$ be a rational map. Then the indeterminacy locus of f has codimension at least 2 in X. In particular, if X is a curve, then every rational map extends uniquely to a morphism.*

Corollary 12. *A complete nonsingular curve C over a field k is necessarily projective.*

Proof: Applying Chow's Lemma, we get a projective curve $\tilde{C}$ and a birational morphism $p : \tilde{C} \to C$. Thus the inverse map p^{-1} is a birational map from the nonsingular variety C into the complete variety $\tilde{C}$, hence p^{-1} is everywhere defined. So p and p^{-1} are both isomorphisms, and hence C is already projective.