FUNCTION FIELDS AND GEOMETRIC INTEGRALITY

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1. Function fields

If R is any integral domain, we have of course its field of fractions $K = K(R)$. In particular, if R is finitely generated over a field k, then we write $k(R)$ for K and call it the **function field** of R.

As we have already seen, when k is not algebraically closed, one must keep in mind two different objects: the algebra R over k and the "geometric algebra" $\overline{R} = R \otimes_k \overline{k}$. Especially, when given an integral k-algbra, it is vital to know whether $\overline{R}$ is integral, since the integral affine $\overline{k}$-algebras are precisely the coordinate rings of affine algebraic varieties in the classical sense (i.e., the ring of all polynomial functions on an irreducible closed subset of affine n-space over $\overline{k}$). It is natural to ask how geometric integrality relates to the function field $k(R)$. Let us first record a simple result to make sure we are all on the same page:

Proposition 1. *The integral k-algebra R is geometrically integral iff the $\overline{k}$-algebra $k(R) \otimes_k \overline{k}$ is a field, and in this case it is the fraction field of $\overline{R}$: $\overline{k}(\overline{R}) = k[R] \otimes \overline{k}$.*

Exercise: Prove it.

Thus the question can be rephrased as one of pure field theory: given a field K (here $K = k(R)$) which is finitely generated (in the sense of field extensions, not ring extensions) over a field k, give necessary and sufficient conditions for $K \otimes_k \overline{k}$ to be a field.

There is an easy necessary condition that we have seen before: if K contains a nontrivial algebraic extension l over k, then the nonintegral algebra $l \otimes_k \overline{k}$ will be a subalgebra of $K \otimes_k \overline{k}$, which of course means that $K \otimes_k \overline{k}$ itself cannot be integral. Thus we want, by definition, for k to be **algebraically closed in** K.[1]

Example: Consider $A = \mathbb{R}[x,y]/(x^2 + y^2)$, which is integral over $\mathbb{R}$ but not geometrically integral. In the fraction field of A, we have $0, \neq x, y$, $x^2 = -y^2$, so $-1 = \left(\frac{x}{y}\right)^2$, thus the fraction field contains $\mathbb{R}[\frac{x}{y}] \cong \mathbb{C}$.

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[1] Recall: for any field extension M/K, there is a unique maximal subextension L which is algebraic over K, called the algebraic closure of K in M. If M is itself algebraically closed, then so is L. Thus for instance the existence of an algebraic closure of $\mathbb{Q}$ can be deduced from the fundamental theorem of algebra, whereas the construction of an algebraic closure of an arbitrary field requires Zorn's Lemma.

Exercise: Consider $\mathbb{C}[t]$ as a finitely generated $\mathbb{R}$-algebra. Since $\mathbb{R}$ is not algebraically closed in $\mathbb{C}(t)$, the algebra is not gometrically integral. What is the structure of $\mathbb{C}[t] \otimes_{\mathbb{R}} \mathbb{C}$?

Here is a much trickier example that shows that in full generality the condition of k being algebraically closed in K is not enough:

Example: let $k = \mathbb{F}_p(a,b)$ be the two-variable rational function field over $\mathbb{F}_p$. Let $A_{/k}$ be the affine variety given by $k[x,y]/(ax^p + b - y^p)$, and let $K = k(A)$. Then:
(i) $P(x,y) = ab^p + x - y^p$ is irreducible in $k[x,y]$, so A is an integral domain.
(ii) k is algebraically closed in $\overline{K}$.
(iii) In the algebraic closure of k, we have unique elements α and β such that $\alpha^p = a$, $\beta^p = b$. Thus $P(x,y)$ factors over $\overline{k}$ as $(\alpha x + \beta - y)^p$, so $A \otimes_k \overline{k}$ is not integral.

To state the correct necessary and sufficient conditions, we briefly run through some field-theoretic terminology:

Let Ω/k be an extension of fields, and let K and L be k-subalgebras of Ω. Then K and L are said to be **linearly disjoint over k** if the natural algebra homomorphism $K \otimes_k L \rightarrow \Omega$, $a \otimes b \mapsto ab$, is injective. Here K and L are not required to be fields, but this is unimportant, because it can be shown that K and L are linearly disjoint iff their fraction fields are.

Exercise: a) Suppose K, L are linearly disjoint field extensions of k. Show that $K \cap L = k$.
b) Suppose K and L are both finite over k and at least one is Galois. Show that linear disjointness is equivalent to $K \cap L = k$.
c) Suppose K and L are conjugate finite separable[2] extensions of k – i.e., they have a common Galois closure M and there exists $\sigma \in \mathrm{Aut}(M/k)$ such that $\sigma(K) = L$. Show that K and L are **not** linearly disjoint over k, even though we may well have $K \cap L = k$ (give an example).

Exercise: Let K and L be field extensions of k, and suppose that K/k is algebraic. Show that K and L are linearly disjoint over k iff $K \otimes_k L$ is a field. (Hint: first do the case where K/k is finite.)

Let K/k be an arbitrary extension of fields. A **separating transcendence basis** $\mathcal{B} = \{x_i\}_{i \in I}$ for K/k is a subset of K which is algebraically independent over k, and such that $K/k(\mathcal{B})$ is a separable algebraic field extension. An extension K/k is **separably generated** if it admits a separating transcendence basis, i.e., if it can be decomposed into a purely transcendental extension followed by a separable algebraic extension. (Recall that a transcendence basis is the same thing with the condition of separability dropped, and that every field extension admits a transcendence basis. From this it follows that in characteristic 0 all field extensions are separably generated.) For example, an inseparable algebraic extension is not

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[2] This hypothesis is here to make it easier for you to understand why $K \cap L = k$ need not imply linear disjointess, even in characteristic 0. By all means think about the nonseparable case if you like.

separably generated.

Let k be a field of positive characteristic and $\overline{k}$ an algebraic closure. Then $k^{p^{-1}}$ denotes the subextension of $\overline{k}$ obtained by adjoining all pth roots of elements of k. In other words, if $F : x \mapsto x^p$ is the Frobenius automorphism of $\overline{k}$, then $k^{p^{-1}} = F^{-1}(k)$. For $a \in \mathbb{Z}^+$, we define $k^{p^{-a}}$ in a similar way: i.e., either by adjoining all p^ath roots of k, or as the preimage of k under F^a. Finally, we define $k^{p^{-\infty}} = \bigcup_{a \in \mathbb{Z}^+} k^{p^{-a}}$. Notice that $k^{p^{-\infty}}$ is the maximal purely inseparable subextension of $\overline{k}/k$ and at the same time the smallest subextension on which the Frobenius map is surjective, so in particular is a perfect field; it is often called the **perfection** (or **perfect closure**) of k.

Theorem 2. *(Main theorem on separability) Let K/k be an extension of fields, with* $\operatorname{char}(k) = p > 0$*. TFAE:*
(i) Every finitely generated subextension of K/k is separably generated.
(ii) K and $k^{p^{-1}}$ are linearly disjoint over k.
(iii) K and $k^{p^{-\infty}}$ are linearly disjoint over k.
(iv) For all field extensions l/k, $K \otimes_k l$ is reduced.

Corollary 3. *a) If k is perfect, every extension K/k is separable.*
b) Let K/k be any field extension. Then $K \otimes_k \overline{k}$ is a field iff k is algebraically closed in K and K/k is separable.

In field theory and related fields (e.g. inverse Galois theory), an extension K/k satisfying part b) is called **regular**. (This usage is not compatible with our notion of regularity.) It is also common to call a finitely generated field extension K/k a "function field" only if it is regular, i.e., only if the associated variety is geometrically integral. Since we already have the term "geometrically integral", we will not complicate matters by using either of these terms.