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LETTER TO THE EDITOR

Explicit Construction of Framelets [1]

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Abstract—We study *tight wavelet frames* associated with given refinable functions which are obtained with the *unitary extension principles*. All possible solutions of the corresponding matrix equations are found. It is proved that the problem of the extension may be always solved with two framelets. In particular, if symbols of the refinable functions are polynomials (rational functions), then the corresponding framelets with polynomial (rational) symbols can be found.

Key Words: tight frames; multiresolution analysis; wavelets.

1. INTRODUCTION

The main goal of our paper [3] is to present an explicit construction of an arbitrary *wavelet frames* generated by a refinable function. After submission this paper to Applied and Computational Harmonic Analysis we received information that the editorial portfolio already contains the paper by C. Chui and W. He [3] that contains similar results.

In this paper, we shall consider only functions of one variable in the space $\mathbb{L}^2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

As usual, $\hat{f}(\omega)$ denotes the Fourier transform of $f(x) \in \mathbb{L}^2(\mathbb{R})$, defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx.$$

Suppose a real-valued function $\varphi \in \mathbb{L}^2(\mathbb{R})$ satisfies the following conditions:

(a) $\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega)$, where m_0 is essentially bounded 2π-periodic function;
(b) $\lim_{\omega\to 0}\hat{\varphi}(\omega) = (2\pi)^{-1/2}$;

then the function φ is called *refinable* or *scaling*, m_0 is called a *symbol* of φ, and the relation in item (a) is called a *refinement equation*.

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[3] This paper is a reduced version of preprint [7].

In spite of the fact that in most practically important cases the refinement function can be easily reconstructed by its symbol, the problem of existence of a scaling function satisfying a refinement equation with the given symbol is not completely solved. Here we shall not discuss the problem of recovering the function φ by its symbol. So in what follows the notion of a refinable function is basic for us and a symbol is only an attribute of a refinable function.

Every refinable function generates a *multiresolution analysis* (MRA) of the space $\mathbb{L}^2(\mathbb{R})$, i.e., a nested sequence

$$\cdots \subset V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^j \subset \cdots$$

of closed linear subspaces of $\mathbb{L}^2(\mathbb{R})$ such that

(a) $\bigcap_{j\in\mathbb{Z}} V^j = \{0\}$;
(b) $\overline{\bigcup_{j\in\mathbb{Z}} V^j} = \mathbb{L}^2(\mathbb{R})$;
(c) $f(x) \in V^j \Leftrightarrow f(2x) \in V^{j+1}$.

To obtain the MRA we just have to take as above V^j the closure of the linear span of the functions $\{\varphi(2^j x - n)\}_{n\in\mathbb{Z}}$. Fulfillment of items (a) and (b) for the spaces V^j was proved in [1]. Property (c) is evident.

The most popular approach to the design of orthogonal and biorthogonal wavelets is based on construction of MRA of the space $\mathbb{L}^2(\mathbb{R})$, generated with a given refinable function. Mallat [6] showed that if the system $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ constitutes a Riesz basis of the space V^0, then there exists a refinable function $\phi \in V^0$ with a symbol m_ϕ such that the functions $\{\phi(x-n)\}_{n\in\mathbb{Z}}$ form an orthonormal basis of V^0. If we denote by W^j the orthogonal complement of the space V^j in the space V^{j+1}, then the function ψ (which is called a *wavelet*), defined by the relation

$$\hat{\psi}(2\omega) := m_\psi(\omega)\hat{\phi}(\omega),$$

where $m_\psi(\omega) = \overline{e^{i\omega} m_\phi(\omega+\pi)}$, generates orthonormal basis $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ of the space W^0. Thus, the system

$$\left\{2^{k/2}\psi(2^k x - n)\right\}_{n,k\in\mathbb{Z}} \tag{1}$$

constitutes an orthonormal basis of the space $\mathbb{L}^2(\mathbb{R})$.

We see that if we have a refinable function, generating a Riesz basis, then we have explicit formulae for the wavelets, associated with this function. It gives a simple method for constructing wavelets. Generally speaking, any orthonormal basis of $\mathbb{L}^2(\mathbb{R})$ of the form (1) is called a wavelet system. However, wavelet construction based on multiresolution has an advantage from the point of view effectiveness of computational algorithms, because it leads to a pyramidal scheme of wavelet decomposition and reconstruction (see, for example, [4]).

It is well known that the problem of finding orthonormal wavelet bases, generated by a scaling function, can be reduced to solving the matrix equation

$$M(\omega)M^*(\omega) = I, \tag{2}$$

where

$$M(\omega)=\begin{pmatrix} m_0(\omega) & m_1(\omega)\\ m_0(\omega+\pi) & m_1(\omega+\pi)\end{pmatrix},$$

and $m_0(\omega)$, $m_1(\omega)$ are essentially bounded functions $m_0(-\omega)=\overline{m_0(\omega)}$; i.e., Fourier series of these functions have real coefficients. It is known (see [4]) that for any scaling function $\varphi(x)$ and the associated wavelet $\psi(x)$, generating an orthogonal wavelet basis, the corresponding symbols $m_0(\omega)$, $m_1(\omega)$ satisfy (2). Any refinable function φ, whose symbol m_0 is solution to (2), generates a tight frame (see [5] for the case when m_0 is polynomial, the general case was proved in [2]).

We cannot independently look for the functions m_0 and m_1. In fact, usually we find a solution of the equation

$$|m_0(\omega)|^2+|m_0(\omega+\pi)|^2=1, \tag{3}$$

and then all possible functions m_1 can be represented in the form

$$m_1(\omega)=\alpha(\omega)\overline{e^{i\omega}m_0(\omega+\pi)}, \tag{4}$$

where $\alpha(\omega)$ is an arbitrary π-periodic function, satisfying $|\alpha(\omega)|=1$, $\alpha(-\omega)=\overline{\alpha(\omega)}$.

Now suppose we have an arbitrary refinable function $\varphi(\omega)$ with the symbol m_0 which does not satisfy (3). Then the set $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ does not constitute an orthonormal basis of V^0. If this set forms a Riesz basis, then we can use orthogonalization, proposed in [6]. However, in this case, when the function φ has a compact support, this property fails for the orthogonalized basis. This argues for construction other systems keeping compactness of support. It will be shown in Section 4 that tight frame of wavelets leads to one of the possible compactly supported systems.

We note that sometimes the orthogonalization can be conducted even if our set is not a Riesz basis. The simplest example gives a refinable function

$$\varphi(x)=\begin{cases}1/2, & |x|\le 1;\\ 0, & |x|>1;\end{cases}$$

with the symbol $m_0(\omega)=\cos 2\omega$. In this case the MRA coincides with the Haar MRA. Thus, the function

$$\varphi(x)=\begin{cases}1, & 0\le x\le 1;\\ 0, & x>1 \text{ or } x<0;\end{cases}$$

is the natural orthogonalization.

Nevertheless, it is easy to design a refinable function such that its MRA does not allow orthogonalization. Indeed, let us introduce a refinable function $\varphi(x)=\sin\pi ax/\pi x$, where $0<a<1$. It generates the space V^0 which consists of functions of $\mathbb{L}^2(\mathbb{R})$ with Fourier transform supported on $[-a\pi, a\pi]$. Thus, for any function $f\in V^0$ the function $\sum_{k\in\mathbb{Z}}|\hat{f}(\omega+2k\pi)|^2$ vanishes on the set $[-\pi,\pi]\backslash[a\pi,a\pi]$. Hence, its integer translates do not form an orthonormal bases (see [4]). In this case the traditional procedure of constructing an orthonormal wavelet basis cannot be applied. We note that by the same reason even a biorthogonal pair with this MRA cannot be constructed.

In the case when the symbol m_0 of a refinable function φ does not satisfy (3) we cannot construct an orthonormal bases of V^1 of the form $\{\varphi(x-k), \psi(x-k)\}$. However, we can hope that there exists a collection of several framelets $\psi^1, \psi^2, \dots, \psi^n \in V^1$, satisfying the following conditions:

(1) functions $\{\{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}}\}_{l=1}^n$, where $\psi^l_{j,k}(x) = 2^{j/2}\psi_l(2^j x - k)$, form a tight frame of the space $\mathbb{L}^2(\mathbb{R})$;

(2) for any $f \in \mathbb{L}^2(\mathbb{R})$, algorithms of decomposition and reconstruction the recurrent formulae

$$\begin{aligned} \langle \varphi_{j,k}, f\rangle = c_{j,l} &= \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{h}_{k-2l}, \\ & \qquad\qquad 1 \le q \le n, \\ \langle \varphi^g_{j,k}, f\rangle = d^q_{j,l} &= \sum_{k\in\mathbb{Z}} c_{j+1,k}\bar{g}^q_{k-2l}, \end{aligned} \tag{5}$$

and

$$c_{j+1,l} = \sum_{k\in\mathbb{Z}} c_{j,k}h_{l-k} + \sum_{q=1}^{n}\sum_{k\in\mathbb{Z}} d^q_{j,k}g^q_{l-k}, \tag{6}$$

where h_k, g^q_k are coefficients of the expansions $m_0(\omega) = 2^{-1/2}\sum_{k\in\mathbb{Z}} h_k e^{-ik\omega}$ and $m_q(\omega) = 2^{-1/2}\sum_{k\in\mathbb{Z}} g^q_k e^{-ik\omega}$, $q = 1, \dots, n$ take place.

The goal of Section 2 is to show that this problem can be solved with at most two framelets and to present explicit formulae for symbols of the framelets. In Sections 3 and 4 we prove that in the case when $m_0(\omega)$ is either a rational function or a polynomial we can choose $m_1(\omega)$, $m_2(\omega)$ as rational functions or polynomials respectively.

2. GENERAL FRAMELETS

Let φ be a refinable function with a symbol m_0, $\hat{\psi}^k(\omega) = m_k(\omega/2)\hat{\phi}(\omega/2) \in V^1$, where each symbol m_k is a 2π-periodic and essentially bounded function for $k = 1, 2, \dots, n$. It is well known that for constructing practically important tight frames the matrix

$$\mathcal{M}(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \dots & m_n(\omega) \\ m_0(\omega+\pi) & m_1(\omega+\pi) & \dots & m_n(\omega+\pi) \end{pmatrix},$$

plays an important role.

It is easy to see that the equality

$$\mathcal{M}(\omega)\mathcal{M}^*(\omega) = I \tag{7}$$

is equivalent to (5) and (6).

It turns out that (7) also implies the tightness of the corresponding frame.

THEOREM 2.1. *If* (7) *holds, then the functions* $\{\psi^k\}_{k=1}^n$ *generate a tight frame of* $\mathbb{L}^2(\mathbb{R})$.

Remark. For $n = 1$ this theorem was proved in [5] for polynomial symbols and in [2] for the general case. For an arbitrary n it was proved in [8] under some additional decay assumption for $\hat{\phi}$ and in [3] for an arbitrary polynomial symbol. In [8] Theorem 2.1 was called the *unitary extension principle*.

We split the proof of Theorem 2.1 into several lemmas.

LEMMA 2.1. *Let the symbols* $\{m_k\}_{k=0}^n$ *satisfy* (7). *Then for any* ω

$$|m_l(\omega)|^2 + |m_l(\omega+\pi)|^2 \le 1, \qquad l = 0, 1, \ldots, n. \tag{8}$$

Proof. Obviously, without lose of generality it suffices to prove inequality (8) only for $l = 0$. Let us rewrite relation (7) in the form

$$\mathbb{M}(\omega) := \mathcal{M}_\psi(\omega)\mathcal{M}_\psi^*(\omega) = \begin{pmatrix} 1 - |m_0(\omega)|^2 & -m_0(\omega)\overline{m_0(\omega+\pi)} \\ -\overline{m_0(\omega)}m_0(\omega+\pi) & 1 - |m_0(\omega+\pi)|^2 \end{pmatrix}, \tag{9}$$

where

$$\mathcal{M}_\psi(\omega) = \begin{pmatrix} m_1(\omega) & m_2(\omega) & \ldots & m_n(\omega) \\ m_1(\omega_\pi) & m_2(\omega+\pi) & \ldots & m_n(\omega_\pi) \end{pmatrix}.$$

The Hermitian matrix $\mathbb{M}(\omega)$ has eigenvalues

$$\lambda_1(\omega) \equiv 1, \qquad \lambda_2(\omega) = 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2.$$

By definition (9), $\mathbb{M}(\omega)$ is a positive definite matrix. Hence, $\lambda_2(\omega) \ge \mathbf{0}$, which is (8) for $l = 0$. ■

LEMMA 2.2. *If* $\Phi \in \mathbb{L}^2(\mathbb{R})$ *is a refinable function with a symbol* $m(\omega)$ *that satisfies the condition*

$$|m(\omega)|^2 + |m(\omega+\pi)|^2 \le 1 \text{ a.e.}, \tag{10}$$

then $S_j := \sum_{k\in\mathbb{Z}} |\langle f, \Phi_{j,k}\rangle|^2 < \infty$ *for any function* $f \in \mathbb{L}^2(\mathbb{R})$ *and*

$$(i) \quad \lim_{j\to\infty} S_j = \|f\|^2; \qquad (ii) \quad \lim_{j\to-\infty} S_j = 0,$$

where $\Phi_{j,k} = 2^{j/2}\Phi(2^j x - k)$.

Proof. First, we prove that

$$\sum_{k\in\mathbb{Z}} |\hat{\Phi}(x + 2\pi k)|^2 \le \frac{1}{2\pi}. \tag{11}$$

We note that due to (10) and the continuity $\hat{\Phi}(\omega)$ at $\omega = 0$ we have $|\hat{\Phi}(\omega)| \le (2\pi)^{-1/2}$ a.e. Thus, for any positive $l \in \mathbb{Z}$ we obtain

$$\sum_{k=-2^l}^{2^l-1} |\hat{\Phi}(\omega + 2\pi k)|^2 = \sum_{k=-2^l}^{2^l-1} \prod_{n=1}^{l+1} |m(2^{-n}(\omega + 2\pi k))|^2 |\hat{\Phi}(2^{-l-1}(\omega + 2\pi k))|^2$$

$$\le \frac{1}{2\pi} \sum_{k=-2^l}^{2^l-1} \prod_{n=1}^{l+1} |m(2^{-n}(\omega + 2\pi k))|^2$$

$$\leq \frac{1}{2\pi}\sum_{k=0}^{2^l-1}\left(\prod_{n=1}^{l+1}|m(2^{-n}(\omega+2\pi k))|^2 + \prod_{n=1}^{l+1}|m(2^{-n}(\omega+2\pi(k-2^l)))|^2\right)$$

$$\leq \frac{1}{2\pi}\sum_{k=0}^{2^l-1}\prod_{n=1}^{l}|m(2^{-n}(\omega+2\pi k))|^2$$

$$\leq \frac{1}{2\pi}\sum_{k=0}^{2^{l-1}-1}\left(\prod_{n=1}^{l}|m(2^{-n}(\omega+2\pi k))|^2 + \prod_{n=1}^{l}|m(2^{-n}(\omega+2\pi(k+2^{l-1})))|^2\right)$$

$$\leq \frac{1}{2\pi}\sum_{k=0}^{2^{l-1}-1}\prod_{n=1}^{l-1}|m(2^{-n}(\omega+2\pi k))|^2 \leq \cdots \leq \frac{1}{2\pi}.$$

Applying the Plancherel and Parseval formulae, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\langle f,\Phi_{j,k}\rangle|^2 &= 2\pi 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\infty}\hat f(\omega)\overline{\hat\Phi(2^{-j}\omega)}e^{i2^{-j}\omega k}\,d\omega\right|^2\\ &= 2\pi 2^{-j}\sum_{k\in\mathbb{Z}}\left|\int_{-\pi 2^j}^{\pi 2^j}\left(\sum_{n\in\mathbb{Z}}\hat f(\omega+2\pi 2^j n)\overline{\hat\Phi(2^{-j}(\omega+2\pi 2^j n))}\right)\right.\\ &\qquad\left.\times\, e^{i2^{-j}\omega k}\,d\omega\right|^2\\ &= (2\pi)^2\int_{-\pi 2^j}^{\pi 2^j}\left|\sum_{n\in\mathbb{Z}}\left(\hat f(\omega+2\pi 2^j n)\overline{\hat\Phi(2^{-j}(\omega+2\pi 2^j n))}\right)\right|^2 d\omega\\ &= (2\pi\|F_j\|)^2, \end{aligned}\tag{12}$$

where $F_j(\omega)=\sum_{n\in\mathbb{Z}}(\hat f(\omega+2\pi 2^j n)\overline{\hat\Phi(2^{-j}(\omega+2\pi 2^j n))})$. Let us introduce the following sequences of functions

$$\hat g_j(\omega)=\begin{cases}\hat f(\omega), & |\omega|<2^j\pi;\\ 0, & |\omega|\geq 2^j\pi;\end{cases},\qquad h_j=f-g_j,\ \ j=0,1,2,\ldots,$$

$$G_j(\omega)=\sum_{n\in\mathbb{Z}}\left(\hat g_j(\omega+2\pi 2^j n)\overline{\hat\Phi(2^{-j}(\omega+2\pi 2^j n))}\right),$$

$$H_j(\omega)=\sum_{n\in\mathbb{Z}}\left(\hat h_j(\omega+2\pi 2^j n)\overline{\hat\Phi(2^{-j}(\omega+2\pi 2^j n))}\right).$$

It is clear that, on the one hand. $\|G_j\|\to(2\pi)^{-1/2}\|f\|$ as $j\to\infty$. On the other hand, in view of (11),

$$\|H_j\|^2 = \int_{-\pi 2^j}^{\pi 2^j} \left| \sum_{n\in\mathbb{Z}} \left(\hat{h}_j(\omega + 2\pi 2^j n) \overline{\hat{\Phi}(2^{-j}(\omega + 2\pi 2^j n))} \right) \right|^2 d\omega$$

$$\leq \int_{-\pi 2^j}^{\pi 2^j} \sum_{n\in\mathbb{Z}} \left| \hat{h}_j(\omega + 2\pi 2^j n) \right|^2 \sum_{n\in\mathbb{Z}} |\Phi(2^{-j}\omega + 2\pi n)|^2 \, d\omega$$

$$\leq \frac{1}{2\pi} \int_{-\pi 2^j}^{\pi 2^j} \sum_{n\in\mathbb{Z}} \left| \hat{h}_j(\omega + 2\pi 2^j n) \right|^2 d\omega = \frac{1}{2\pi} \|\hat{h}_j\|^2 \to 0, \qquad \text{as } j \to \infty. \tag{13}$$

Thus, since

$$\|G_j\| - \|H_j\| \leq \|F_j\| = \|G_j + H_j\| \leq \|G_j\| + \|H_j\|,$$

it follows from (12) and (13) that

$$\sum_{k\in\mathbb{Z}} |\langle f, \Phi_{j,k}\rangle|^2 = (2\pi \|F_j\|)^2 \to 2\pi \|\hat{f}\|^2 = \|f\|^2, \qquad \text{as } j \to +\infty.$$

Thus, relation (i) is proved.

Now we shall prove (ii). Let us denote χ_R the characteristic function of a segment $[-R, R]$ and by f_R the function $f\chi_R$. We fix an arbitrary $\varepsilon > 0$ and choose $R > 0$ such that $\|f \cdot (1 - \chi_R)\| < \varepsilon$.

Since

$$\sum_{k\in\mathbb{Z}} |\langle f, \Phi_{j,k}\rangle|^2 \leq 2 \sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + 2 \sum_{k\in\mathbb{Z}} |\langle f - f_R, \Phi_{j,k}\rangle|^2$$

$$\leq 2 \sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + \|f - f_R\|/\pi \leq 2 \sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 + \varepsilon/\pi,$$

we need only to prove that

$$\lim_{j\to-\infty} \sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 = 0.$$

If we assume that $2^j R \leq 1/2$, then the last relation follows from the chain of inequalities

$$\sum_{k\in\mathbb{Z}} |\langle f_R, \Phi_{j,k}\rangle|^2 = \sum_{k\in\mathbb{Z}} \left(\int_{|x|\leq R} f(x) \Phi_{j,k}(x) \, dx \right)^2$$

$$\leq \|f\|^2 \sum_{k\in\mathbb{Z}} \int_{|x|\leq R} \Phi_{j,k}^2(x) \, dx = \|f\|^2 \sum_{k\in\mathbb{Z}} \int_{|x+k|\leq 2^j R} \Phi^2(x) \, dx$$

$$= \|f\|^2 \int_{\cup_{k\in\mathbb{Z}}[-2^j R+k,\ 2^j R+k]} \Phi^2(x) \, dx \to 0 \qquad \text{as } j \to -\infty. \quad \blacksquare$$

LEMMA 2.3. *If* (7) *holds, then for any* $f \in \mathbb{L}^2(\mathbb{R})$ *and* $J \in \mathbb{Z}$

$$\sum_{k=1}^{n} \sum_{j,l\in\mathbb{Z}} |\langle f, \psi_{j,l}^k\rangle|^2 = \sum_{l\in\mathbb{Z}} |\langle f, \phi_{J,l}\rangle|^2 + \sum_{k=1}^{n} \sum_{j\geq J} \sum_{l\in\mathbb{Z}} |\langle f, \psi_{j,l}^k\rangle|^2 < \infty.$$

Proof. It follows from (7) that

$$|m_0(\omega)|^2 + |m_1(\omega)|^2 + \cdots + |m_k(\omega)|^2 = 1,$$

$$m_0(\omega)\overline{m_0(\omega+\pi)} + m_1(\omega)\overline{m_1(\omega+\pi)} + \cdots + m_n(\omega)\overline{m_n(\omega+\pi)} = 0.$$

So, introducing the notation

$$\Delta_1(\omega) := \sum_{l\in\mathbb{Z}} \hat{f}(\omega + 2\pi 2^{L+1} l)\overline{\hat{\phi}(2^{-L-1}\omega + 2\pi l)},$$

$$\Delta_2(\omega) := \sum_{l\in\mathbb{Z}} \hat{f}(\omega + 2\pi 2^{L+1} l + 2\pi 2^L)\overline{\hat{\phi}(2^{-L-1}\omega + 2\pi l + \pi)},$$

we have, by analogy with (12), for any $L \in \mathbb{Z}$

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} |\langle f, \varphi_{L,l}\rangle|^2 + \sum_{k=1}^{n}\sum_{l\in\mathbb{Z}} |\langle f, \psi^k_{L,l}\rangle|^2 \\
&= (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^L l)\overline{\hat{\phi}(2^{-L}(\omega + 2\pi 2^L l))} \Big) \right|^2 d\omega \\
&\quad + (2\pi)^2 \sum_{k=1}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^L l)\overline{\hat{\phi}^k(2^{-L}(\omega + 2\pi 2^L l))} \Big) \right|^2 d\omega \\
&= (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^L l) \right. \\
&\qquad\qquad \left. \times\, \overline{m_k(2^{-L-1}(\omega + 2\pi 2^L l))\hat{\phi}(2^{-L-1}(\omega + 2\pi 2^L l))} \Big) \right|^2 d\omega \\
&= (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \Delta_1(\omega)\overline{m_k(2^{-L-1}\omega)} \right|^2 d\omega \\
&\quad + (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \left| \Delta_2(\omega)\overline{m_k(2^{-L-1}\omega + \pi)} \right|^2 d\omega \\
&\quad + (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \Delta_1(\omega)\overline{m_k(2^{-L-1}\omega)\Delta_2(\omega)} m_k(2^{-L-1}\omega + \pi)\, d\omega \\
&\quad + (2\pi)^2 \sum_{k=0}^{n} \int_{-\pi 2^L}^{\pi 2^L} \Delta_2(\omega)\overline{m_k(2^{-L-1}\omega + \pi)\Delta_1(\omega)} m_k(2^{-L-1}\omega)\, d\omega \\
&= (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^{L+1} l)\overline{\hat{\phi}(2^{-L-1}\omega + 2\pi l)} \Big) \right|^2 d\omega \\
&\quad + (2\pi)^2 \int_{-\pi 2^L}^{\pi 2^L} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^{L+1} l + 2\pi 2^L)\overline{\hat{\phi}(2^{-L-1}\omega + 2\pi l + \pi)} \Big) \right|^2 d\omega \\
&= (2\pi)^2 \int_{-\pi 2^{L+1}}^{\pi 2^{L+1}} \left| \sum_{l\in\mathbb{Z}} \Big(\hat{f}(\omega + 2\pi 2^{L+1} l)\overline{\hat{\phi}(2^{-L-1}\omega + 2\pi l)} \Big) \right|^2 d\omega \\
&= \sum_{l\in\mathbb{Z}} |\langle f, \varphi_{L+1,l}\rangle|^2 < \infty.
\end{aligned}$$

Using Lemma 2.2 we obtain of Lemma 2.3. ∎

Now Theorem 2.1 is an easy consequence of Lemmas 2.1–2.3.

Thus, the problem of constructing tight frames, generated by a refinable function, can be reduced to finding m_k, that satisfy (7). Now we shall describe all possible solutions to (7).

Let the symbol m_0 satisfy (10). Unit eigenvectors of the matrix $\mathbb{M}(\omega)$ can be represented in the form

$$\vec{\mathrm{v}}_1(\omega) = \begin{pmatrix} \overline{\left(\frac{e^{i\omega} m_0(\omega+\pi)}{B(\omega)}\right)} \\ -\overline{\left(\frac{e^{i\omega} m_0(\omega)}{B(\omega)}\right)} \end{pmatrix}, \qquad \vec{\mathrm{v}}_2(\omega) = \begin{pmatrix} \frac{m_0(\omega)}{B(\omega)} \\ \frac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix}, \qquad B(\omega) \neq 0,$$

where $B(\omega)$ is an arbitrary π-periodic measurable functions, satisfying $|B(\omega)|^2 = |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$ a.e. For definiteness, we can take here the positive root of the right-hand expression. For those ω when $m_0(\omega) = m_0(\omega+\pi) = 0$ the matrix $\mathbb{M}(\omega)$ becomes the identity matrix. So any non-zero vector is its eigenvector. In this case we put $\vec{\mathrm{v}}_1(\omega) = (1,0)^T$, $\vec{\mathrm{v}}_2(\omega) = (0,1)^T$.

Thus, we have

$$\mathbb{M}(\omega) = P(\omega)\Lambda(\omega)P^*(\omega), \tag{14}$$

where

$$P(\omega) = \begin{pmatrix} \overline{\left(\frac{e^{i\omega} m_0(\omega+\pi)}{B(\omega)}\right)} & \frac{m_0(\omega)}{B(\omega)} \\ -\overline{\left(\frac{e^{i\omega} m_0(\omega)}{B(\omega)}\right)} & \frac{m_0(\omega+\pi)}{B(\omega)} \end{pmatrix},$$

$$\Lambda(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2 \end{pmatrix}.$$

We note that eigenvectors are determined up to multiplication by a scalar function of absolute value 1 a.e. We have chosen the normalization convenient for further consideration.

THEOREM 2.2. *Let a 2π-periodic function $m_0(\omega)$ satisfy* (10). *Then there exists a pair of 2π-periodic measurable functions m_1, m_2 which satisfy* (7) *for $n = 2$. Any solution of* (7) *can be represented in the form of the first row of the matrix*

$$\widetilde{\mathcal{M}}(\omega) = P(\omega)\sqrt{\Lambda(\omega)}\,Q(\omega),$$

where $Q(\omega)$ is an arbitrary unitary (a.e.) matrix with π-periodic measurable components.

Proof. The matrix $\mathcal{M}_\psi$ can be represented in the form of its singular decomposition

$$\mathcal{M}_\psi(\omega) = \mathcal{P}(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega),$$

where $\mathcal{P}$, $\mathcal{Q}$ are unitary matrices, $\mathcal{D}(\omega)$ is a nonnegative diagonal matrix. These representations may differ by multiplication of columns of the matrix $\mathcal{P}$ by functions $\alpha_1(\omega)$, $\alpha_2(\omega)$, $|\alpha_1(\omega)| = |\alpha_2(\omega)| \equiv 1$ and simultaneous multiplication of rows of the matrix $\mathcal{Q}$ by $\alpha_1^{-1}(\omega)$ and $\alpha_2^{-1}(\omega)$. Thus, in view of (9) and (14) without loss of generality we can suppose $\mathcal{P} \equiv P$, $\mathcal{D} \equiv \sqrt{\Lambda}$.

Let us prove that we can take any a.e. unitary matrix with π-periodic elements as above, with $Q(\omega) = \mathcal{Q}(\omega)$. In fact, our choice is restricted to such matrices.

For any 2×2 matrix Z, we denote by Z^R the matrix with the transposed rows. On the one hand we have

$$\mathcal{M}_\psi(\omega+\pi) = P(\omega+\pi)\mathcal{D}(\omega+\pi)\mathcal{Q}(\omega+\pi) = P^R(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega+\pi),$$

and on the other hand, we have

$$\mathcal{M}_\psi^R = (P(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega))^R = P^R(\omega)\mathcal{D}(\omega)\mathcal{Q}(\omega).$$

Since $\mathcal{M}_\psi^R(\omega) = \mathcal{M}_\psi(\omega+\pi)$, it means that $\mathcal{Q}(\omega+\pi) = \mathcal{Q}(\omega)$ at least for those ω and $\omega+\pi$ for which $\lambda_2(\omega) = \lambda_2(\omega+\pi) \neq 0$. If $\lambda_2(\omega) = \lambda_2(\omega+\pi) = 0$, then $\mathcal{M}_\psi(\omega)$ does not depend on the choice of the second row of the matrix $\mathcal{Q}$, so that we can take an arbitrary value of $\mathcal{Q}(\omega+\pi)$ and $\mathcal{Q}(\omega)$. In particular, we can assume $\mathcal{Q}(\omega+\pi) = \mathcal{Q}(\omega)$. ■

Remark. To describe all possible solutions to (7) for an arbitrary n, we have to take an arbitrary $n \times n$ unitary matrix Q with π-periodic elements and a $2 \times n$ matrix D' which is extension of the matrix $\sqrt{\Lambda}$ by mean of filling all new columns with zeros.

3. FRAMELETS WITH RATIONAL SYMBOLS

For numerical implementation, framelets with rational and polynomial symbols are the most suitable. Under the assumptions of Section 2 we require additionally that $m_0(\omega)$ is a rational 2π-periodic function with real coefficients; i.e., m_0 is a ratio of trigonometric polynomials with real coefficients. It is well known that in spite of the fact that such functions have infinitely many nonzero Fourier coefficients, implementation of numerical algorithms for this case can be economically designed with, so-called, recursive filters.

The only difference in the case of a rational symbol and the general case is that we have to extract the square root more carefully. If $m_0(\omega)$ is a rational function, then $\mathcal{B}(\omega) = |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$ and $\mathcal{A}(\omega) = 1 - |m_0(\omega)|^2 - |m_0(\omega+\pi)|^2$ are rational nonnegative functions. So according to Riesz lemma, we can take such rational π-periodic functions $A(\omega)$ and $B(\omega)$ that $|A(\omega)|^2 = \mathcal{A}(\omega)$, $|B(\omega)|^2 = \mathcal{B}(\omega)$. Thus, we have proved the following statement.

THEOREM 3.1. *Let a 2π-periodic rational function $m_0(\omega)$ satisfy* (10). *Then there exists a pair of 2π-periodic rational functions m_1, m_2 which satisfy* (7). *Any such rational solution to* (7) *can be represented in the form of the first row of the matrix*

$$\widetilde{\mathcal{M}}(\omega) = P(\omega)D(\omega)Q(\omega), \tag{15}$$

where $Q(\omega)$ is an arbitrary unitary rational matrix with π-periodic rational components,

$$D(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & A(\omega) \end{pmatrix}.$$

4. FRAMELETS WITH POLYNOMIAL SYMBOLS

The subject of this section is framelets generated by compactly supported refinable functions with polynomial symbols. They are the most simple from the point of view of numerical implementation. Our main goal is to prove the existence of compactly supported framelets for this case.

Here the degree of the trigonometric polynomial $\sum_{j=l}^{k} a_k e^{ijx}$, where $a_l \neq 0$ and $a_k \neq 0$, is defined to be $k - l$.

We denote by $\mathcal{L}$ a set of all Laurent polynomials with real coefficients, and by $\mathcal{L}_n$ a set of Laurent polynomials with real coefficients of degree at most n; i.e.,

$$\mathcal{L}_n := \left\{ \sum_{j=l}^{k} a_j z^j \;\middle|\; l, k \in \mathbb{Z};\ a_j \in \mathbb{R};\ 0 \le k - l \le n \right\}.$$

THEOREM 4.1. *Let a trigonometric polynomial $m_0(\omega)$ of degree n satisfy* (10). *Then there exists a pair of trigonometric polynomials m_1, m_2 of degree at most n which satisfies* (7).

Remark 4.2. In [3] this theorem was proved by different method without consideration of the polynomial degree, although a close investigation of the proof reveals that the degree n is also guaranteed in [3].

Proof. In fact, we cannot exert control over the choice of the matrices $P(\omega)$ and $D(\omega)$ in (15). So we need to choose a unitary rational π-periodic matrix $Q(\omega)$ such that $\mathcal{M}_\psi(\omega)$ consists of trigonometric polynomials.

Let us use the change of variable $z = e^{i\omega}$ in (15). In what follows we consider the Laurent polynomials $h(e^{i\omega}) = m_0(\omega)$, $b(e^{2i\omega}) = B(\omega)$, $a(e^{2i\omega}) = A(\omega)$.

After the change of variable, the matrix $P(\omega)$ becomes

$$H(z) = \begin{pmatrix} \frac{\frac{1}{z}h(-\frac{1}{z})}{b(1/z^2)} & \frac{h(z)}{b(z^2)} \\ -\frac{\frac{1}{z}h(\frac{1}{z})}{b(1/z^2)} & \frac{h(-z)}{b(z^2)} \end{pmatrix}.$$

We put the last representation of the matrix $H(z)$ through the procedure of reduction. If the three polynomials $h(z)$, $h(-z)$, $b(z^2)$ are divisible by $z - z_0$, we cancel the corresponding fractions in the first and second column of $H(z)$ by $1/z^2 - z_0^2$ and $/z^2 - z_0^2$, respectively. After all possible cancellations we obtain the same matrix $H'(z) = H(z)$ but its elements are expressed in terms of new functions $h'(z)$ and $b'(z)$. It is clear that $b'(z^2)b'(1/z'^2) = h'(z)h'(1/z) + h'(-z)h'(-1/z)$ and the numerators of the matrix $H(z)$ do not vanish simultaneously. Indeed, the determinant of $H(z)$ is equal to $1/z$. If for some z_0 we have $h(z_0) = h(-z_0) = h(1/z_0) = h(-1/z_0) = 0$, then either $b(z_0^2) = 0$ or $b(1/z_0^2) = 0$. It means that the reduction of $H(z)$ can be continued. We note that because the coefficients of $h(z)$ and $b(z)$ are real, the polynomials $h'(z)$ and $b'(z)$ also have real coefficients.

Furthermore, the elements $q_{11}(z^2)$, $q_{12}(z^2)$, $q_{21}(z^2)$, $q_{22}(z^2)$ of the matrix $Q(\omega)$ satisfy the relations

$$q_{22}(z) = q_{11}(1/z)z^N, \qquad q_{12}(z) = -q_{21}(1/z)z^N, \qquad N \in \mathbb{Z}.$$

Here, without loss of generality, we may suppose $N = 0$, because any other choice leads to the integer shift of one of the basic framelets.

To reduce poles of the matrix $H'(z)$ after multiplication by $Q(\omega)$, we suppose that

$$q_{11}(z) = \frac{g_1(z)}{b'(z)}, \qquad q_{21}(z) = \frac{g_2(z)}{b'(1/z)},$$

where g_1, g_2 are Laurent polynomials.

Let $\mathcal{R} = \{\pm z_1^{\pm 1}, \pm z_2^{\pm 1}, \dots, \pm z_n^{\pm 1}\}$ be the set of all different roots of the polynomial $b'(z^2)b'(1/z^2)$. We denote by k_j the multiplicity of the root z_j. It is clear that all four roots $\pm z_j^{\pm 1}$ have the same multiplicity. Therefore degree of the polynomial $b'(z^2)b'(1/z^2)$ is equal to $4\sum k_j = 4k$, where k is degree of polynomial b'.

To prove the theorem we need to find polynomials g_1, g_2 which satisfy equations

$$\frac{1}{z}h'\left(-\frac{1}{z}\right)g_1(z^2) + a(z^2)h'(z)g_2(z^2) = b'(z^2)b'(1/z^2)f_1(z); \tag{16}$$

$$-\frac{1}{z}h'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) + a(z^2)h'(z)g_1\left(\frac{1}{z^2}\right) = b'(z^2)b'(1/z^2)f_2(z); \tag{17}$$

$$-\frac{1}{z}h'\left(\frac{1}{z}\right)g_1(z^2) + a(z^2)h'(-z)g_2(z^2) = b'(z^2)b'(1/z^2)f_3(z); \tag{18}$$

$$\frac{1}{z}h'\left(\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) + a(z^2)h'(-z)g_1\left(\frac{1}{z^2}\right) = b'(z^2)b'(1/z^2)f_4(z), \tag{19}$$

where $f_1, f_2, f_3, f_4 \in \mathcal{L}$. Moreover, for the matrix $Q(\omega)$ to be unitary, we also require

$$g_1(z)g_1(1/z) + g_2(z)g_2(1/z) = b'(z)b'(1/z). \tag{20}$$

Now we leave aside Eq. (20) and prove the existence of polynomials $g_1, g_2 \in \mathcal{L}_k$, satisfying Eqs. (16)–(19). Let us fix the lowest and highest powers of the polynomials g_1 and g_2 and suppose that their degree is equal to k. Thus, we have $2k + 2$ unknown coefficients.

First we show that there exist polynomials g_1 and g_2, satisfying Eqs. (16)–(19) at points of the set $\mathcal{R}$. As it usually is in the case of a root $\tilde{z}$ of multiplicity k, we require that not only Eqs. (16)–(19) are satisfied, but also their derivatives of orders $1, 2, \dots, \tilde{k} - 1$ are satisfied.

Equations (16)–(19) give us $16k$ homogeneous linear equations for $2k + 2$ unknown coefficients of the polynomials g_1 and g_2. We shall prove that at most $2k$ of them are linearly independent. We conduct the proof of this fact in three steps. Each of these steps is based on the following lemma.

LEMMA 4.1. *Let $a_1(z)$, $a_2(z)$, $a_3(z)$, $a_4(z)$, $b_1(z)$, $b_2(z)$, $c_1(z)$, $c_2(z)$ be Laurent polynomials, $|a_1(z_0)|^2 + |a_2(z_0)|^2 \neq 0$, and l a positive integer. If*

$$a_1(z)b_1(z) + a_2(z)b_2(z) = (z - z_0)^l c_1(z); \tag{21}$$

$$a_1(z)a_4(z) - a_2(z)a_3(z) = (z - z_0)^l c_2(z), \tag{22}$$

then we have

$$a_3(z)b_1(z) + a_4(z)b_2(z) = (z - z_0)^l c(z), \tag{23}$$

where $c(z) \in \mathcal{L}$. ■

Proof. Let us assume for definiteness that $a_1(z_0) \neq 0$. We substitute b_1 from (21) and a_4 from (22) into (23) to get

$$\begin{aligned} &a_3(z)b_1(z) + a_4(z)b_2(z) \\ &\quad = a_3(z)\frac{(z - z_0)^l c_1(z) - a_2(z)b_2(z)}{a_1(z)} + b_2(z)\frac{(z - z_0)^l c_2(z) + a_2(z)a_3(z)}{a_1(z)} \\ &\quad = (z - z_0)^l \frac{a_3(z)c_1(z) + b_2(z)c_2(z)}{a_1(z)} =: (z - z_0)^l c(z). \end{aligned}$$

In the first step we prove that for every $\tilde{z} \in \mathcal{R}$, only one equation of the pairs $\{(16), (18)\}$ and $\{(17), (19)\}$ should be retained. Indeed, on the one hand,

$$\det \begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ -\frac{1}{z}h'\left(\frac{1}{z}\right) & a(z^2)h'(-z) \end{vmatrix} = \frac{1}{z}a(z^2)b'(z^2)b'(1/z^2) = (z - \tilde{z})^{\tilde{k}} c_1(z), \qquad c_1(z) \in \mathcal{L}.$$

On the other hand, since $a(z)a(1/z) = 1 - b(z)b(1/z)$, then $a(\tilde{z}^2) \neq 0$ for any $\tilde{z} \in \mathcal{R}$. Hence, the last matrix has at least one nonzero element at point $\tilde{z}$. We assume for definiteness that the first row contains some nonzero element. Then by Lemma 4.1, if g_1 and g_2 satisfy (16) at the point $\tilde{z}$ with multiplicity $\tilde{k}$, they also satisfy (18) at least with the same multiplicity. So at the point $\tilde{z}$, we can exclude Eq. (18) from consideration. In the same manner we eliminate one of Eqs. (17) and (19).

In the second step we eliminate equations, corresponding to the roots $\tilde{z}$ and $1/\tilde{z}$. Now for two roots $\tilde{z}$ and $1/\tilde{z}$ we have $4\tilde{k}$ equations. It turns out that at most $2\tilde{k}$ of them are linearly independent. We show that we can keep only equations of the form (16) and (18). Indeed, let us assume that in the previous step we kept Eq. (16) for $\tilde{z} \in \mathcal{R}$ and Eq. (17) for $1/\tilde{z}$. Now we prove that the linear equations generated by (17) for $1/\tilde{z}$ can be omitted. We apply the change of variable $z \mapsto 1/z$ to (17). Then the left-hand side of (17) becomes

$$a(1/z^2)h'(1/z)g_1(z^2) - zh'(-z)g_2(z^2). \tag{24}$$

Since

$$\det \begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ a\left(\frac{1}{z^2}\right)h'\left(\frac{1}{z}\right) & -zh'(-z) \end{vmatrix} = b'(z^2)b'(1/z^2)(b''(z^2)b''(1/z^2)h'(z)h'(1/z) - 1),$$

where $b''(z) = b(z)/b'(z)$, is divisible by $(z - \tilde{z})^{\tilde{k}}$, expression (24) is also divisible by $(z - \tilde{z})^{\tilde{k}}$ and the left-hand side of (17) is divisible by $(z - 1/\tilde{z})^{\tilde{k}}$.

The dependence of the equations, generated by (19), is obtained by the same methods. Indeed, after the transform $z \mapsto 1/z$, the left-hand side of (19) is equal to

$$a(1/z^2)h'(-1/z)g_1(z^2) + zh'(z)g_2(z^2).$$

Since

$$\det\begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ a\left(\frac{1}{z^2}\right)h'(-1/z) & zh'(z) \end{vmatrix} = b(z^2)b(1/z^2)h'(z)h'(-1/z),$$

the left-hand side of (19) is divisible by $(z - 1/\tilde{z})^{\tilde{k}}$.

In the third step, we prove that the equations, corresponding $\tilde{z}$ and $-\tilde{z}$, are linear dependent.

Let us assume that we have chosen Eq. (16) for the both roots $\pm\tilde{z}$. After substitution $z \mapsto -z$, the right-hand side of (16) is transformed to

$$-\frac{1}{z}h'\left(\frac{1}{z}\right)g_1(z^2) + a(z^2)h'(-z)g_2(z^2).$$

Since

$$\det\begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ -\frac{1}{z}h'\left(\frac{1}{z}\right) & a(z^2)h'(-z) \end{vmatrix} = \frac{1}{z}a(z^2)b'(z^2)b'(z)(1/z^2)$$

is divisible by $(z - \tilde{z})^{\tilde{k}}$, the equations for $-\tilde{z}$ are linear dependent from these for $\tilde{z}$.

In the case, when we take Eq. (16) for $\tilde{z}$ and Eq. (18) for $-\tilde{z}$, the corresponding linear equations coincide.

Thus, we have proved the existence of a pair of polynomials $g_1, g_2 \in \mathcal{L}_n$, satisfying Eqs. (16)–(19) on all of $\mathcal{R}$. Although the polynomial $b'(z^2)b'(1/z^2)$ can have complex roots, it is easy to check that we can choose the polynomials g_1, g_2 with real coefficients. Indeed, if z_0 is a root of $b'(z^2)b'(1/z^2)$. then $\tilde{z}_0$ is also a root. Coefficients of the equations, corresponding to these roots, differ in complex conjugation. So we can consider real equations, corresponding to real and imaginary parts of the initial equations.

Thus, we have $2k$ homogeneous linear equations for $2k+2$ unknown values. Let us take any nondegenerate solution of the system. Now we prove that a pair of the polynomials g_1 and g_2 of degree at most k, satisfying (16)–(19) and the relation

$$g_1^2(1) + g_2^2(1) = b'^2(1), \tag{25}$$

satisfies also the equation

$$g_1(z)g_1(1/z) + g_2(z)g_2(1/z) = b'(z)b'(1/z). \tag{26}$$

Indeed, let us assume for definiteness that $\tilde{z} \in \mathcal{R}$ and $|h'(-1/\tilde{z})|^2 + |a(z^2)h'(z)|^2 \neq 0$. By (17), we have

$$\det\begin{vmatrix} \frac{1}{z}h'\left(-\frac{1}{z}\right) & a(z^2)h'(z) \\ g_1\left(\frac{1}{z^2}\right) & g_2\left(\frac{1}{z^2}\right) \end{vmatrix} = \frac{1}{z}h'\left(-\frac{1}{z}\right)g_2\left(\frac{1}{z^2}\right) - a(z^2)h'(z)g_1\left(\frac{1}{z^2}\right)$$
$$= -b'(z^2)b'(1/z^2)f_2(z).$$

Thus, by Lemma 4.1 and from (16), the expression $g_1(z)g_1(1/z) + g_2(z)g_2(1/z)$ is divisible by $(z - \tilde{z})^{\tilde{k}}$. This means that the polynomials in the left-hand and right-hand sides of (26) have $2k$ common zeros. It remains to normalize the left-hand side of the polynomial, according to (25). The normalization is impossible only in the case when $g_1(1) = g_2(1) = 0$. However, it implies that the left-hand side of (26) has $2k + 1$ zeros. It follows from this that $g_1(z)g_1(1/z) + g_2(z)g_2(1/z) \equiv 0$. Hence, $g_1(z) \equiv g_2(z) \equiv 0$. This contradicts the assumption that at least one of the polynomials g_1 and g_2 is nondegenerate. ■

We note that there are infinitely many solutions g_1 and g_2 satisfying (25). However, it is not difficult to prove that there is a unique solution with the initial conditions

$$g_1(1) = a, \qquad g_2(1) = b_1, \qquad a^2 + b^2 = (b'(1))^2. \tag{27}$$

Indeed, let us introduce two real linear independent vectors $\vec{r}$, $\vec{r}''$ of dimension $2k + 2$, composed of coefficients of the polynomials g_1', g_2' and g_1'', g_2'', satisfying (16)–(19). In this case, the vectors $(g_1'(1), g_2'(1))$ and $(g_1''(1), g_2''(1))$ are linear independent. It follows from the fact that in the linear dependent case we can obtain a nondegenerate solution g_1, g_2 of (16)–(19) for which $g_1(1) = g_2(1) = 0$. However, as we mentioned above, this is impossible. Hence, these are solutions with initial conditions (27).

Since the differences between any two solutions that satisfy (27) take on the value 0 at the point $z = 1$, they are equal to 0 identically. Hence, there is only one pair of polynomials g_1, g_2, satisfying (27).

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