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Construction of symmetric orthogonal bases of wavelets and tight wavelet frames with integer dilation factor

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Abstract

Our goal is to present a systematic algorithm for constructing (anti)symmetric tight wavelet frames and orthonormal wavelet bases generated by a given refinable function with an integer dilation factor $d \geqslant 2$. Special attention is paid to the issues of the minimality of a number of framelet generators and the size of generator supports. In particular, our algorithm allows to reduce the computational costs approximately by a factor 2.

1. Introduction

Let d be an integer number $d \geqslant 2$. Recall that if a system of functions $S = \{d^{k/2}\psi_l(d^k x - n)\}_{k,n\in\mathbb{Z}}$, $l = 1, 2, \dots, d-1$, generated by the functions $\psi_l \in \mathbb{L}^2(\mathbb{R})$, $\|\psi_l\| = 1$, constitutes an orthonormal basis in $\mathbb{L}^2(\mathbb{R})$, this system is called a wavelet basis with the dilation factor d. Denoting by W_l^k, $l = 1, 2, \dots, d-1$, the linear span of the functions of S with a fixed k, we have the representation

$$\mathbb{L}^2(\mathbb{R}) = \overline{\bigoplus_{\substack{j\in\mathbb{Z}\\ 0<l<d}} W_l^j}.$$

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S. Mallat [8] suggested (for $d = 2$) a less general but more useful in applications approach. His construction is based on *Multiresolution Analysis* (MRA) generated by a so-called *scaling* (or *refinable*) function, i.e., a function φ satisfying the refinement equation

$$\varphi\left(\frac{x}{d}\right) = \sqrt{d}\sum_{n\in\mathbb{Z}} h_n\varphi(x-n) \quad \text{or} \quad \hat{\varphi}(d\omega) = m_0(\omega)\hat{\varphi}(\omega) \quad \text{(in Fourier domain)},$$

where $m_0(\omega) = (1/\sqrt{d})\sum_{n\in\mathbb{Z}} h_n \mathrm{e}^{-in\omega}$. The 2π-periodic function m_0 is called *a symbol* of the refinable function φ and its Fourier coefficients h_k are called *a mask*. If h_n is a finite sequence, m_0 becomes a trigonometric polynomial and φ is a compactly supported function. In what follows, we are interested in the case when m_0 is a polynomial.

Once $\varphi \in \mathbb{L}^2(\mathbb{R})$, an MRA of $\mathbb{L}^2(\mathbb{R})$ is introduced as a nested sequence of the spaces $V^k := \operatorname{span}\{d^{k/2}\varphi(d^k x - n)\}$,

$$\cdots \subset V^k \subset V^{k+1} \subset \cdots \subset \mathbb{L}^2(\mathbb{R}).$$

Under quite mild assumptions on m_0 we have $\bigcap V^n = \{0\}$ and $\overline{\bigcup V^n} = \mathbb{L}^2(\mathbb{R})$. Assuming also that the system $\varphi(x-n)$ is orthonormal, we define the wavelet spaces W^k as an orthogonal complement of V^k to V^{k+1}. It is known that W^0 may be generated by some functions (wavelets) $\{\psi_l\}_{l=1}^{d-1} \subset V^1$ with orthonormal integer translates as a linear span of these translates (see [7] for an explicit algorithm). Of course, these ψ_l satisfy the equations $\hat{\psi}_l(d\omega) = (1/\sqrt{d})m_l(\omega)\hat{\varphi}(\omega)$ with some 2π-periodic functions m_l.

The MRA-based approach allows to reduce the problem of constructing new wavelet bases to finding the symbols $\{m_l\}$ satisfying the very simple relation

$$\mathbf{M}(\omega)\mathbf{M}^*(\omega) = I, \tag{1}$$

where

$$\mathbf{M}(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \dots & m_{d-1}(\omega) \\ m_0\left(\omega + \frac{2\pi}{d}\right) & m_1\left(\omega + \frac{2\pi}{d}\right) & \dots & m_{d-1}\left(\omega + \frac{2\pi}{d}\right) \\ \dots & \dots & \dots & \dots \\ m_0\left(\omega + \frac{2\pi(d-1)}{d}\right) & m_1\left(\omega + \frac{2\pi(d-1)}{d}\right) & \dots & m_{d-1}\left(\omega + \frac{2\pi(d-1)}{d}\right) \end{pmatrix}. \tag{2}$$

High computational efficiency is the main attractive feature of compactly supported MRA-based wavelets. Unfortunately, not all solutions to (1) lead to orthonormal wavelet bases. However, any solution to (1) generates a tight frame (see W. Lawton [6] and M. Bownik [1]).

Recall that a set of elements $S = \{f_n\}$ of a Hilbert space H is called a tight frame if for any $f \in H$ the Parseval identity

$$\|f\|^2 = \sum \left|\langle f, f_n\rangle\right|^2$$

is valid. It means that a tight frame is a natural extension of the notion of orthonormal bases onto linearly dependant (redundant) systems.

Since tight frames are not linearly independent any more, the number of frame generators ψ_l does not need to be restricted by $d-1$. In this case, the matrix $\mathbf{M}(\omega)$ becomes rectangular. A. Ron and Z. Shen [11] proved that even if the matrix $\mathbf{M}(\omega)$ is rectangular the condition (1) is still sufficient for the system $\{\psi_l\}$ to generate a tight frame. Finding wavelets for a given refinable function is one of the most important problems in wavelet theory. At the same time, in terms of Eq. (1), this problem is equivalent to finding a unitary matrix with a given column. Therefore, A. Ron and Z. Shen called Eq. (1) *Unitary Extension Principle*.

In what follows we are interested only in solutions of Eq. (1) by means of the extension of the given first column to a unitary matrix. The property of a wavelet system to be an orthonormal basis depends only on m_0, i.e., only on the first column of $\mathbf{M}(\omega)$. So we do not distinguish the cases of orthonormal bases and tight frames.

While the wavelet construction with a square matrix $\mathbf{M}(\omega)$ may lead to wavelet frames, we can reach the real flexibility of the representation only if, in the MRA based construction, a few extra wavelets are permitted.

C. Chui and W. He [2] and A. Petukhov [9], independently found explicit systematic algorithms for solving matrix equation (1) for the dyadic case ($d = 2$).

The (anti)symmetry (the evenness or the oddness) of systems plays a very important and sometimes crucial role in applications. Even if we leave aside double reduction of the computational costs of symmetric systems, the “linear phase” property and the opportunity not to increase the dimension of data for encoding finite data vectors with symmetric wavelets (and in some sense with framelet systems) cannot be compensated by any other properties. Unfortunately, for the dyadic case, only the Haar orthonormal basis has an antisymmetric generator. C. Chui and J. Lian [3] showed that for the dilation factor 3, the choice of symmetric orthonormal bases becomes much wider. In particular, such wavelets can combine the symmetry and many vanishing moments. B. Han [4] considered the case of (anti)symmetric wavelets for $d = 4$. However, as was mentioned in [4], if a scaling function is symmetric about the origin, even for $d = 4$ the systematic algorithm for finding symmetric solutions to (1) is unknown.

The case of wavelet frames with a rectangular matrix $\mathbf{M}(\omega)$ is much more flexible. For dyadic framelets, C. Chui and W. He [2] proved that, having a symmetric scaling function, the problem of the symmetric extension may be solved with 3 framelets. A. Petukhov [10] found a criterion for the existence of 2 (anti)symmetric framelets. In [2] and [10], explicit algorithms for the matrix extension were presented.

Recently, M.J. Lai and J. Stöckler [5] found a quite general and very simple algorithm for the extension (including the symmetric one) to a rectangular unitary matrix. Their algorithm is applicable even for multivariate framelets. However, that algorithm leads to one extra framelet above the amount necessary for the extension.

This study was inspired by the article of W. Lawton, S.L. Lee, and Z. Shen [7], where the authors found a simple computationally efficient algorithm for the matrix extension. However, a systematic approach to the symmetry issue was not considered in [7]. The goal of this paper is to extend the results of [2] and [10] to an arbitrary integer dilation factor $d \geqslant 2$. The algorithm presented below is in fact the version of the algorithm from [7], with certain adaptation for the symmetric case.

The structure of the paper is as follows. In Section 2 we construct an algorithm implementing the extension of the unit column consisting of (anti)symmetric Laurent polynomials to a paraunitary matrix with (anti)symmetric components. In Sections 3 and 4, we show how the cases of square and rectangular matrices $\mathbf{M}(\omega)$ can be reduced to the problem with symmetric Laurent polynomials considered in Section 2.

2. Extension for matrices with symmetric components

Let $\mathcal{P}$ be the space of Laurent polynomials of the form

$$p(z) = \sum_{n=m}^{M} p_n z^n, \quad m, M \in \mathbb{Z},\ m \leqslant M, \tag{3}$$

with real coefficients p_n. In what follows, assuming that $p_m p_M \neq 0$, we use the notation $\mathrm{Mdeg}(p) := M$, $\mathrm{mdeg}(p) := m$, $M(p) := p_M$, $m(p) := p_m$, $\deg(p) := M - m$. In particular, the degree of a polynomial p is equal to 0 if and only if p is a monomial, i.e., $p(z) = cz^k$, $k \in \mathbb{Z}$. In what follows, we apply the functionals listed above to polynomial vectors and matrices. For example, if $P(z)$ is a polynomial $k \times n$ matrix, then

$$\deg(P) := \max\bigl\{\deg(P_{i,j}) \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n\bigr\},$$
$$\mathrm{mdeg}(P) := \min\bigl\{\mathrm{mdeg}(P_{i,j}) \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n\bigr\},$$
$$\mathrm{Mdeg}(P) := \max\bigl\{\mathrm{Mdeg}(P_{i,j}) \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n\bigr\},$$
$$m(P) := \{m_{i,j}\} = \bigl\{(P_{i,j})_{\mathrm{mdeg}(P)}\bigr\},$$
$$M(P) := \{M_{i,j}\} = \bigl\{(P_{i,j})_{\mathrm{Mdeg}(P)}\bigr\},$$

where, here and below, the operation $(p)_k$ means the kth Laurent coefficient of the polynomial p.

Now we introduce the subsets of Laurent polynomials with coefficients either symmetric or antisymmetric up to translation

$$\mathcal{S} = \bigcup_{k\in\mathbb{Z}} \bigl\{\bigl\{p(z) \in \mathcal{P} \mid p_n = p_{k-n},\ n \in \mathbb{Z}\bigr\} \cup \bigl\{p(z) \in \mathcal{P} \mid p_n = -p_{k-n},\ n \in \mathbb{Z}\bigr\}\bigr\}.$$

Obviously, the case of even k corresponds to the symmetry about *whole point* $k/2$, whereas odd k leads to *half-point* symmetry. So according to this classification, we can partition $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_w \cup \mathcal{S}_h$.

At the same time we also use another classification of the symmetry type. Since symmetric or antisymmetric sequences p_n are correspondingly *even* or *odd* functions of indices (up to appropriate translation), we introduce for them the notations $\mathcal{S}_e$ and $\mathcal{S}_o$. Besides, we use combinations of these two classifications. For example, $\mathcal{S}_{w,e} := \mathcal{S}_w \cap \mathcal{S}_e$. To fix the center of symmetry we will use an addition superscript k. For example,

$$\mathcal{S}^k_{w,o} := \bigl\{p(z) \in \mathcal{P} \mid p_n = -p_{k-n},\ n \in \mathbb{Z}\bigr\}.$$

Of course, the subscripts "w"can be combined only with an even superscript k, whereas the "h" can be combined only with an odd k.

In what follows, for $p(z) = \sum p_n z^n \in \mathcal{P}$ we denote by $\mathrm{rev}_k\, p$ the polynomial $q(z) = \sum q_n z^n$ with the coefficients $q_n = p_{k-n}$.

We define the operation $s_1 + s_2$, where $s_i \in \{w, h\}$ (or $s_i \in \{e, o\}$) as the binary "exclusive or," assuming $w = 0$, $h = 1$ (or $e = 0$, $o = 1$), i.e.,

$$s_1 + s_2 = \begin{cases} 0, & s_1 = s_2, \\ 1, & s_1 \neq s_2. \end{cases}$$

For (anti)symmetric polynomials the following simple properties take place.

Proposition 1. (1) *If* $p \in \mathcal{S}^{k_1}_{s_1,t_1}$ *and* $q \in \mathcal{S}^{k_2}_{s_2,t_2}$, *then* $pq \in \mathcal{S}^{k_1+k_2}_{s_1+s_2,t_1+t_2}$;
(2) (a) $p, q \in \mathcal{P}$ *and* $q = \mathrm{rev}_k\, p$ *hold if and only if* $(p+q) \in \mathcal{S}^k_e$ *and* $(p-q) \in \mathcal{S}^k_o$,
(b) $p, q \in \mathcal{P}$ *and* $q = -\mathrm{rev}_k\, p$ *hold if and only if* $(p+q) \in \mathcal{S}^k_o$ *and* $(p-q) \in \mathcal{S}^k_e$.

Let $\mathbf{p}(z)$ be a d-dimensional vector with components $\{p^i\}_{i=1}^d \subset \mathcal{S}$ satisfying the equality

$$\bigl\|\mathbf{p}(z)\bigr\|^2 := \langle \mathbf{p}, \mathbf{p}\rangle \equiv 1, \tag{4}$$

where the inner product of d-dimensional polynomial vectors $\mathbf{p}(z)$ and $\mathbf{q}(z)$ is defined as

$$\langle \mathbf{p}, \mathbf{q}\rangle := \sum_{i=1}^{d} p^i(z) q^i(1/z).$$

A matrix $P(z)$ that is unitary with respect to this inner product, i.e., $P(z)P(1/z)^{\mathrm{T}} = I$, is called *paraunitary*. Our objective is to prove the existence of a paraunitary polynomial matrix $P(z)$ with components from $\mathcal{S}$ whose first column coincides with $\mathbf{p}$.

Theorem 1. *Let $\mathbf{p}(z)$ be a unit vector with d components from $\mathcal{S}$. Then there exists a paraunitary extension $P(z)$ generated by $\mathbf{p}(z)$ with components from $\mathcal{S}$. Moreover, the choice of the components such that the degrees of the ith row of the matrix $P(z)$ does not exceed* $\deg(\mathbf{p}_i)$ (*in particular,* $\deg P(z) = \deg \mathbf{p}(z)$) *is possible.*

Proof. We give a constructive proof leading to the simple systematic algorithm for the paraunitary extension.

We start with a general case when the vector $\mathbf{p}$ contains components from all sets $\mathcal{S}_{h,e}$, $\mathcal{S}_{w,e}$, $\mathcal{S}_{w,o}$, $\mathcal{S}_{h,o}$. Thus, up to rearrangement, the vector $\mathbf{p}$ can be represented in the block form

$$\mathbf{p}^{\mathrm{T}} = \big(\mathbf{p}_{h,e}^{\mathrm{T}}\ \mathbf{p}_{w,e}^{\mathrm{T}}\ \mathbf{p}_{w,o}^{\mathrm{T}}\ \mathbf{p}_{h,o}^{\mathrm{T}}\big)^{\mathrm{T}},$$

where the dimensions of its components are $n_{h,e}, n_{w,e}, n_{w,o}, n_{h,o} > 0$, with $n_{h,e} + n_{w,e} + n_{w,o} + n_{h,o} = d$.

Without loss of generality, we suppose that all components belong to either $\mathcal{S}_w^0$ or $\mathcal{S}_h^1$. Otherwise, we have to multiply the components of $\mathbf{p}$ by appropriate monomials.

Our additional assumption is

$$\deg(\mathbf{p}) = \max\big(\deg(\mathbf{p}_{w,e}), \deg(\mathbf{p}_{w,o})\big) > \max\big(\deg(\mathbf{p}_{h,e}), \deg(\mathbf{p}_{h,o})\big). \tag{5}$$

The method for the reduction of the converse inequality will be considered later. At the same time, the equality sign is impossible due to different parity of the compared values. In particular, for the assumption above we have $\deg(\mathbf{p}) =: 2N > 0$ (the case $N = 0$ is trivial).

Following [7], we will construct a paraunitary matrix $A(z)$ with (anti)symmetric components such that $\mathbf{p}^{\mathrm{T}}(z)A(z)$ is a vector with zero components except for the first one which is equal to 1. Then $A(1/z)$ is the required matrix. We represent the matrix A as a product of paraunitary matrices of low degree.

First, we introduce the real vectors $\vec{u}_1 := m(\mathbf{p}_{w,e}) = M(\mathbf{p}_{w,e})$ and $\vec{u}_2 := -m(\mathbf{p}_{w,o}) = M(\mathbf{p}_{w,o})$. We also assume that the first component of $\vec{u}_1$ and the last component of $\vec{u}_2$ are not zeros. Otherwise, we just rearrange the components of $\mathbf{p}_{w,e}$ and $\mathbf{p}_{w,o}$. Because of (4), taking into account that $m(\mathbf{p}_{h,e}) = \vec{0}$, $m(\mathbf{p}_{h,o}) = \vec{0}$, we have

$$\begin{aligned} 0 = \big\langle m(\mathbf{p}), M(\mathbf{p})\big\rangle &= \big\langle m(\mathbf{p}_{h,e}), M(\mathbf{p}_{h,e})\big\rangle + \big\langle m(\mathbf{p}_{w,e}), M(\mathbf{p}_{w,e})\big\rangle + \big\langle m(\mathbf{p}_{w,o}), M(\mathbf{p}_{w,o})\big\rangle \\ &\quad + \big\langle m(\mathbf{p}_{h,o}), M(\mathbf{p}_{h,o})\big\rangle \\ &= \big\langle \vec{0}, M(\mathbf{p}_{h,e})\big\rangle + \langle \vec{u}_1, \vec{u}_1\rangle + \langle -\vec{u}_2, \vec{u}_2\rangle + \big\langle \vec{0}, M(\mathbf{p}_{h,o})\big\rangle = \|\vec{u}_1\|^2 - \|\vec{u}_2\|^2. \end{aligned}$$

It means that both vectors $\vec{u}_1$ and $\vec{u}_2$ are not degenerated and have equal norms, $a := \|\vec{u}_1\| = \|\vec{u}_2\|$. We denote by $\vec{\tilde{u}}_1 := \vec{u}_1/a$ and $\vec{\tilde{u}}_2 := \vec{u}_2/a$ the normalized versions of those vectors.

Now we introduce matrices which will serve us as building blocks,

$$A_1^1 = \left(\begin{array}{c|c|c|c|c|c} I_{n_{h,e}} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{(1+z)}{2}\vec{\tilde{u}}_1 & U_1 & 0 & \frac{(1-z)}{2}\vec{\tilde{u}}_1 & 0 \\ \hline 0 & \frac{(1-z)}{2}\vec{\tilde{u}}_2 & 0 & U_2 & \frac{(1+z)}{2}\vec{\tilde{u}}_2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{n_{h,o}} \end{array}\right),$$

where the dimensions of the rectangular matrices U_1 and U_2 are $n_{w,e} \times (n_{w,e} - 1)$ and $n_{w,o} \times (n_{w,o} - 1)$, respectively. The columns of those matrices extend the vectors $\vec{\tilde{u}}_1$ and $\vec{\tilde{u}}_2$ to orthonormal bases of the spaces $\mathbb{R}^{n_{w,e}}$ and $\mathbb{R}^{n_{w,o}}$. In addition, if for some i we have $\deg(\mathbf{p}_i) < \deg(\mathbf{p})$, we require that the ith column and row of the matrix A_1^1 has 1 on the main diagonal and 0 at the remaining coordinates. It is easy to verify that the matrix A_1^1 is paraunitary. Moreover, components of the vector

$$\left(\mathbf{p}^1(z)\right)^{\mathrm{T}} := \mathbf{p}^{\mathrm{T}}(z)A_1^1 \tag{6}$$

are symmetric and $\deg(\mathbf{p}^1) = 2N - 1$. Indeed, the operator A_1^1 does not change components of $\mathbf{p}_i$ for $i = 1, \ldots, n_{h,e}$ and $i = d - n_{h,o} + 1, \ldots, d$. The components $\mathbf{p}_i^1$ for $i = n_{h,e} + 2, \ldots, d - n_{h,o} - 1$ inherit the type of symmetry. In particular, all of them belong to $\mathcal{S}_w^0$ whereas their degree cannot exceed $2N - 2$ because of $\vec{u}_1^{\mathrm{T}} U_1 = \vec{0}^{\mathrm{T}}$ and $\vec{u}_2^{\mathrm{T}} U_2 = \vec{0}^{\mathrm{T}}$. As for the elements with indexes $n_{h,e} + 1$ and $d - n_{h,o}$, by item (1) of Proposition 1, we have

$$\mathbf{p}^1_{n_{h,e}+1}(z) = \frac{1}{2}\left(\left\langle (1+z)\mathbf{p}_{w,e}(z), \vec{\tilde{u}}_1\right\rangle + \left\langle (1-z)\mathbf{p}_{w,o}(z), \vec{\tilde{u}}_2\right\rangle\right) \in \mathcal{S}^1_{h,e}, \tag{7}$$

$$\mathbf{p}^1_{d-n_{h,o}}(z) = \frac{1}{2}\left(\left\langle (1-z)\mathbf{p}_{w,e}(z), \vec{\tilde{u}}_1\right\rangle + \left\langle (1+z)\mathbf{p}_{w,o}(z), \vec{\tilde{u}}_2\right\rangle\right) \in \mathcal{S}^1_{h,o}. \tag{8}$$

It is easy to prove that

$$\max\left\{\deg\left(p^1_{n_{h,e}+1}\right), \deg\left(p^1_{d-n_{h,o}}\right)\right\} = 2N - 1. \tag{9}$$

Indeed,

$$\begin{aligned} \left(\mathbf{p}^1_{n_{h,e}+1}\right)_{-N} = \left(\mathbf{p}^1_{n_{h,e}+1}\right)_{N+1} &= \frac{1}{2}\left(\left\langle m(\mathbf{p}_{w,e}), \vec{\tilde{u}}_1\right\rangle + \left\langle m(\mathbf{p}_{w,o}), \vec{\tilde{u}}_2\right\rangle\right) \\ &= \frac{1}{2a}\left(\langle \vec{u}_1, \vec{u}_1\rangle + \langle -\vec{u}_2, \vec{u}_2\rangle\right) = \frac{1}{2a}\left(a^2 - a^2\right) = 0. \end{aligned}$$

For the same reasons, $(\mathbf{p}^1_{d-n_{h,o}})_{-N} = (\mathbf{p}^1_{d-n_{h,o}})_{N+1} = 0$. Thus, we proved

$$\max\left\{\deg\left(p^1_{n_{h,e}+1}\right), \deg\left(p^1_{d-n_{h,o}}\right)\right\} \leqslant 2N - 1.$$

Now we prove that the strict inequality is impossible. By (7) and (8),

$$\begin{aligned} \left(\mathbf{p}^1_{n_{h,e}+1}\right)_{-N+1} = \left(\mathbf{p}^1_{n_{h,e}+1}\right)_{N} &= \frac{1}{2}\left(\left\langle (\mathbf{p}_{w,e})_{-N} + (\mathbf{p}_{w,e})_{-N+1}, \vec{\tilde{u}}_1\right\rangle + \left\langle -(\mathbf{p}_{w,o})_{-N} + (\mathbf{p}_{w,o})_{-N+1}, \vec{\tilde{u}}_2\right\rangle\right) \\ &= \frac{1}{2}\left(2a + \left\langle (\mathbf{p}_{w,e})_{-N+1}, \vec{\tilde{u}}_1\right\rangle + \left\langle (\mathbf{p}_{w,o})_{-N+1}, \vec{\tilde{u}}_2\right\rangle\right) =: a + b, \end{aligned}$$

and analogously

$$-\left(\mathbf{p}^1_{d-n_{h,o}}\right)_{-N+1} = \left(\mathbf{p}^1_{d-n_{h,o}}\right)_{N} = a - b.$$

Since $a = \|\vec{u}_1\| > 0$, then $a \pm b$ cannot be equal to 0 simultaneously. Hence, (9) is proved.

Summarizing what was said above, we can claim that the transform (6) decreases the degree of the vector $\mathbf{p}(z)$ exactly by 1, preserves the total symmetry but the $(n_{h,e}+1)$th and the $(d-n_{h,o})$th components now belong to $\mathcal{S}^1_{h,e}$ and $\mathcal{S}^1_{h,o}$ (not to $\mathcal{S}^0_{w,e}$ and $\mathcal{S}^0_{w,o}$). At the same time all the other components of degree $< 2N$ stay unchanged.

The next step becomes almost obvious. We introduce the matrix

$$A_1^2 = \left(\begin{array}{c|c|c|c|c|c} V_1 & \frac{(1+z^{-1})}{2}\tilde{\vec{v}}_1 & 0 & 0 & \frac{(1-z^{-1})}{2}\tilde{\vec{v}}_1 & 0 \\ \hline 0 & 0 & I_{n_{w,e}-1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n_{w,o}-1} & 0 & 0 \\ \hline 0 & \frac{(1-z^{-1})}{2}\tilde{\vec{v}}_2 & 0 & 0 & \frac{(1+z^{-1})}{2}\tilde{\vec{v}}_2 & V_2 \end{array}\right), \tag{10}$$

where

$$\tilde{\vec{v}}_1 := \frac{m(\mathbf{p}^1_{h,e})}{a_1} = \frac{M(\mathbf{p}^1_{h,e})}{a_1}, \qquad \tilde{\vec{v}}_2 := -\frac{m(\mathbf{p}^1_{h,o})}{a_1} = \frac{M(\mathbf{p}^1_{h,o})}{a_1},$$
$$a_1 := \|m(\mathbf{p}^1_{h,e})\| = \|M(\mathbf{p}^1_{h,e})\| = \|m(\mathbf{p}^1_{h,o})\| = \|M(\mathbf{p}^1_{h,o})\|, \qquad \tilde{\vec{v}}_1 \in \mathbb{R}^{n_{h,e}+1}, \qquad \tilde{\vec{v}}_2 \in \mathbb{R}^{n_{h,o}+1}.$$

The dimensions of the rectangular matrices V_1 and V_2 are $(n_{h,e}+1)\times n_{h,e}$ and $(n_{h,o}+1)\times n_{h,o}$, respectively. The columns of those matrices extend the vectors $\tilde{\vec{v}}_1$ and $\tilde{\vec{v}}_2$ to orthonormal bases of the spaces $\mathbb{R}^{n_{h,e}+1}$ and $\mathbb{R}^{n_{h,o}+1}$. And again, if for some i (except for $i = n_{h,e}+1$ and $i = d - n_{h,o}$) we have $\deg(\mathbf{p}^1_i) < \deg(\mathbf{p}^1)$, we require that the ith column and row of the matrix A_1^2 has 1 on the main diagonal and 0 at the remaining coordinates. Then, by repeating almost literally the previous reasonings, we arrive at the conclusion that the vector

$$\left(\mathbf{p}^2(z)\right)^{\mathrm{T}} := \left(\mathbf{p}^1(z)\right)^{\mathrm{T}} A_1^2(z) = \mathbf{p}^{\mathrm{T}}(z)A_1^1(z)A_1^2(z) =: \mathbf{p}^{\mathrm{T}}(z)A_1(z), \quad \deg\left(A_1(z)\right) = 2,$$

inherits all properties of the vector $\mathbf{p}$, including the types of symmetry and locations of all components. At the same time,

$$\deg\left(\mathbf{p}^2\right) = \deg\left(\mathbf{p}^2_{w,e}\right) = \deg\left(\mathbf{p}^2_{w,o}\right) = 2N-2$$

and components of $\mathbf{p}$ of degree less than $2N-1$ stay unchanged.

Thus, it is clear that by applying the procedure described above N times, we get the vector $(\mathbf{p}^{2N})^{\mathrm{T}} := \mathbf{p}^{\mathrm{T}}A_1A_2\ldots A_N =: \mathbf{p}^{\mathrm{T}}B$ of degree 0, i.e., not depending on z. Moreover, $\mathbf{p}^{2N}_{h,e} = \vec{0}$, $\mathbf{p}^{2N}_{h,o} = \vec{0}$, $\mathbf{p}^{2N}_{w,o} = \vec{0}$. In addition, requiring for each step k that the ith column and row of the matrix A_k has 1 on the main diagonal and 0s at the remaining coordinates for i satisfying the condition $\deg(\mathbf{p}_i) \leqslant 2(N-k)$, we can control the degree of all rows of the matrix $A(z)$ such that the degree of its ith row does not exceed $\deg(\mathbf{p}_i)$.

Now we examine the properties of the matrix B. By direct computation, it is easy to verify that for any $i = 1,\ldots,N$ we have the block structure of the form

$$\left(\begin{array}{c|c|c|c} B_{1,1}\in\mathcal{S}^0_{w,e} & B_{1,2}\in\mathcal{S}^{-1}_{h,e} & B_{1,3}\in\mathcal{S}^{-1}_{h,o} & B_{1,4}\in\mathcal{S}^0_{w,o} \\ \hline B_{2,1}\in\mathcal{S}^1_{h,e} & B_{2,2}\in\mathcal{S}^0_{w,e} & B_{2,3}\in\mathcal{S}^0_{w,o} & B_{2,4}\in\mathcal{S}^1_{h,o} \\ \hline B_{3,1}\in\mathcal{S}^1_{h,o} & B_{3,2}\in\mathcal{S}^0_{w,o} & B_{3,3}\in\mathcal{S}^0_{w,e} & B_{3,4}\in\mathcal{S}^1_{h,e} \\ \hline B_{4,1}\in\mathcal{S}^0_{w,o} & B_{4,2}\in\mathcal{S}^{-1}_{h,o} & B_{4,3}\in\mathcal{S}^{-1}_{h,e} & B_{4,4}\in\mathcal{S}^0_{w,e} \end{array}\right), \tag{11}$$

where sizes of the blocks are determined by the dimension of the diagonal blocks $B_{i,i}$ which are equal to $k_i \times k_i$, $i = 1, \dots, 4$, where $k_1 = n_{h,e}$, $k_2 = n_{w,e}$, $k_3 = n_{w,o}$, $k_4 = n_{h,o}$. We denote by $\mathcal{A}(k_1, k_2, k_3, k_4)$ a family of matrices of the form (11).

Proposition 2. *If $B_1, B_2 \in \mathcal{A}(l_1, l_2, l_3, l_4)$, $l_1, l_2, l_3, l_4 \geqslant 0$, then $B_1 B_2 \in \mathcal{A}(l_1, l_2, l_3, l_4)$.*

The statement of Proposition 2 is verified by direct computation.

Recall that $\mathbf{p}^{2N}$ still has nonzero component $\mathbf{p}^{2N}_{w,e}$. Now it remains to multiply $B(z)$ by a matrix $A_{N+1}(z)$ of the class $\mathcal{A}(k_1, k_2, k_3, k_4)$ which differs from the identity matrix only in the block $B_{2,2}$. If we choose $B_{2,2}$ as the extension of the vector $\mathbf{p}^{2N}_{w,e}$ to an orthogonal matrix, then the vector $\mathbf{p}^{2N+1} := \mathbf{p}^{2N} A_{N+1}$ differs from zero only at one coordinate $((\mathbf{p}^{2N+1})_{n_{h,e}+1} = 1)$. Note that multiplication of $B(z)$ by A_{N+1} does not increase the degree of the matrix $A(z)$ nor the degree of its rows.

Thus, the matrix $A(z^{-1}) = \prod_{i=1}^{N+1} A_i(z^{-1})$ is a required extension of the vector $\mathbf{p}(z)$.

The case when the converse of inequality (5) takes place can be easily reduced to the considered one. Indeed, applying a transform

$$A_0(z) := \left(\begin{array}{c|c|c|c|c|c} V_1 & \frac{(1+z^{-1})}{2}\vec{\tilde{v}}_1 & 0 & 0 & \frac{(1-z^{-1})}{2}\vec{\tilde{v}}_1 & 0 \\ \hline 0 & 0 & I_{n_{w,e}} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n_{w,0}} & 0 & 0 \\ \hline 0 & \frac{(1-z^{-1})}{2}\vec{\tilde{v}}_2 & 0 & 0 & \frac{(1+z^{-1})}{2}\vec{\tilde{v}}_2 & V_2 \end{array}\right),$$

which looks like (10) but has different size of blocks, to the vector $\mathbf{p}(z)$, we have a new vector $\mathbf{p}^1(z)$ satisfying (5). Following the described procedure, we can find the matrix $A(z) \in \mathcal{A}(n_{h,e} - 1, n_{w,e} + 1, n_{w,o} + 1, n_{h,o} - 1)$ extending the vector $\mathbf{p}^1(z)$ to a paraunitary matrix. In this case, the matrix $A_0(z)A(z)$ does not belong to any class $\mathcal{A}(k_1, k_2, k_3, k_4)$. However, it has a structure of the form (11), where the dimensions of the diagonal blocks are equal to $n_{h,e} \times (n_{h,e} - 1)$, $n_{w,e} \times (n_{w,e} + 1)$, $n_{w,o} \times (n_{w,o} + 1)$, and $n_{h,o} \times (n_{h,o} - 1)$, respectively.

The algorithm described above does not work for the cases when either (a) $n_{h,e} = n_{h,o} = 0$ or (b) $n_{w,e} = n_{w,o} = 0$. We consider them separately.

First, we consider case (a). We suppose that $N := -\operatorname{mdeg}(\mathbf{p}) = \operatorname{Mdeg}(\mathbf{p}) > 0$ (the case $N = 0$ is trivial). We note that in this case $n_{w,e} n_{w,o} > 0$. Otherwise, condition (4) is violated. Indeed, if for the definiteness $n_{w,o} = 0$, then $(\langle \mathbf{p}(z), \mathbf{p}(1/z)\rangle)_{2N} = \|(\mathbf{p})_N\|^2 \neq 0$.

Applying the operator

$$A_1 = \left(\begin{array}{c|c|c|c} \frac{z^{-1}+z}{2}\vec{\tilde{u}}_1 & U_1 & 0 & \frac{z^{-1}-z}{2}\vec{\tilde{u}}_1 \\ \hline \frac{z^{-1}-z}{2}\vec{\tilde{u}}_2 & 0 & U_2 & \frac{z^{-1}+z}{2}\vec{\tilde{u}}_2 \end{array}\right),$$

where $\vec{\tilde{u}}_i$ and U_i are defined as above, to $\mathbf{p}^{\mathrm{T}}$, we get the vector $(\mathbf{p}^1)^{\mathrm{T}} := \mathbf{p}^{\mathrm{T}} A_1$ with components belonging to the same type of symmetry as the components of $\mathbf{p}$. It is easy to see that $A_1 \in \mathcal{A}(0, n_{w,e}, n_{w,o}, 0)$. At the same time, $\deg(\mathbf{p}^1) \leqslant \deg(\mathbf{p}) - 2$. Indeed,

$$\left(\mathbf{p}^1_1\right)_{N+1} = \frac{1}{2}\left(\langle \vec{\tilde{u}}_1, \vec{u}_1\rangle - \langle \vec{\tilde{u}}_2, \vec{u}_2\rangle\right) = \frac{1}{2a}\left(\|\vec{u}_1\|^2 - \|\vec{u}_2\|^2\right) = 0,$$

where $a = \|\vec{u}_1\| = \|\vec{u}_2\|$. Let us show that $(\mathbf{p}^1_1)_N = 0$. We introduce the vectors $\vec{w}_1 := (\mathbf{p}_{w,e})_{-N+1}$ and $\vec{w}_2 := -(\mathbf{p}_{w,o})_{-N+1}$. Due to (4),

$$0 = \big(\langle \mathbf{p}, \mathbf{p}\rangle\big)_{2N-1} = \big\langle (\mathbf{p})_N, (\mathbf{p})_{-N+1}\big\rangle + \big\langle (\mathbf{p})_{N-1}, (\mathbf{p})_{-N}\big\rangle$$
$$= \langle \vec{u}_1, \vec{w}_1\rangle + \langle \vec{u}_2, -\vec{w}_2\rangle + \langle \vec{w}_1, \vec{u}_1\rangle + \langle \vec{w}_2, -\vec{u}_2\rangle = 2\big(\langle \vec{u}_1, \vec{w}_1\rangle - \langle \vec{u}_2, \vec{w}_2\rangle\big).$$

Thus, $\langle \vec{u}_1, \vec{w}_1\rangle = \langle \vec{u}_2, \vec{w}_2\rangle$ and

$$\big(\mathbf{p}_1^1\big)_N = \frac{1}{2}\big(\langle \vec{\tilde{u}}_1, \vec{w}_1\rangle - \langle \vec{\tilde{u}}_2, \vec{w}_2\rangle\big) = 0.$$

Estimating the degree of the polynomial $\mathbf{p}^1_{n_{w,e}+n_{w,o}}$ may be conducted in an analogous way. Because of the choice of U_1 and U_2, the estimate for the remaining components of the vector $\mathbf{p}^1$ is obvious. The paraunitarity of the matrix A_1 can be verified by direct computation.

Thus, by applying the described procedure N times, we find a sequence of paraunitary matrices of the class $\mathcal{A}(0, n_{w,e}, n_{w,o}, 0)$ such that for $B := \prod_{i=1}^N A_i$ and $(\mathbf{p}^N)^{\mathrm{T}} := \mathbf{p}^{\mathrm{T}} B$, we have $\deg(\mathbf{p}^N) = 0$, $\mathbf{p}^N_{w,o} = 0$. It remains to define

$$A(z) = B(z)\begin{pmatrix} U & 0 \\ 0 & I_{n_{w,o}} \end{pmatrix},$$

where U is an extension of the vector $\mathbf{p}^N_{w,e}$ to an orthogonal matrix.

The case (b) is quite similar. Let $N := -\operatorname{mdeg}(\mathbf{p}) = \operatorname{Mdeg}(\mathbf{p}) - 1 > 0$. Applying the algorithm used for the previous case, after N steps we obtain a polynomial vector $\tilde{\mathbf{p}}^N$, $\deg(\tilde{\mathbf{p}}^N) = 1$. It remains to multiply it by the paraunitary matrix

$$A_{N+1} = \left(\begin{array}{c|c|c|c} \frac{z^{-1}+1}{2}\,\vec{\tilde{v}}_1 & V_1 & 0 & \frac{z^{-1}-1}{2}\,\vec{\tilde{v}}_1 \\ \hline \frac{z^{-1}-1}{2}\,\vec{\tilde{v}}_2 & 0 & V_2 & \frac{z^{-1}+1}{2}\,\vec{\tilde{v}}_2 \end{array}\right),$$

where $\vec{\tilde{v}}_i$ and V_i are defined as above. We introduce the paraunitary matrix

$$A(z) := \prod_{i=1}^{N+1} A_i(z).$$

Then the matrix $A(1/z)$ gives us the required extension with the first column coinciding with the vector $\mathbf{p}(z)$. □

3. Symmetric wavelets and framelets with $d-1$ generators

We show that the problem of constructing symmetric wavelets and framelets can be easily reduced to the symmetric matrix extension considered in Section 2.

Let $m_0(\omega)$ be a polynomial symbol of a scaling function with the dilation factor d satisfying the condition

$$\sum_{k=0}^{d-1} \big|m_0(\omega + 2\pi k/d)\big|^2 = 1.$$

It is known that such symbols allow to construct either an orthonormal wavelet basis (if the shifts $\varphi(x-k)$ constitute orthonormal basis of its span) or a tight wavelet frame with $d-1$ generators.

By substitution $z = \mathrm{e}^{-i\omega}$, we have $h(z) = h(\mathrm{e}^{-i\omega}) := m_0(\omega)$. In what follows, we use the so-called polyphase representation of the polynomial $h(z)$

$$h(z) = \frac{1}{\sqrt{d}} \sum_{k=n_1}^{n_2} z^k h_k^1(z^d), \tag{12}$$

where $h_k^1(z)$ are Laurent polynomials, $n_2 - n_1 = d - 1$. It is well known that in this case

$$\sum_{k=n_1}^{n_2} h_k^1(z) h_k^1(1/z) = 1. \tag{13}$$

Our choice of n_1 and n_2 depends on the parity of d and on the type of symmetry:

(1) $n_2 = -n_1 = (d-1)/2$ if d is odd, $h \in \mathcal{S}_w$;
(2) $n_2 = -n_1 = (d-1)/2$ if d is odd, $h \in \mathcal{S}_h$;
(3) $n_2 + 1 = -n_1 = d/2$ if d is even, $h \in \mathcal{S}_w$;
(4) $n_2 = -n_1 + 1 = d/2$ if d is even, $h \in \mathcal{S}_h$.

However, such a choice is not mandatory. We make it just for definiteness. It is easy to see that for this choice the set of the polynomials $\{h_k^1\}_{k=n_1}^{n^2}$ consists of symmetric polynomials (from the sets $\mathcal{S}_{w,e}^0$ or $\mathcal{S}_{h,e}^1$) and of pairs of mutually reverse polynomials $p = \mathrm{rev}_0\, q$. By Proposition 1, this implies

$$\frac{p+q}{\sqrt{2}} \in \mathcal{S}_{w,e}^0, \qquad \frac{p-q}{\sqrt{2}} \in \mathcal{S}_{w,o}^0. \tag{14}$$

This transform is unitary, and by applying it to mutually reverse pairs of the vector $(h_{n_1}^1, \dots, h_{n_2}^1)$, we have a new vector $\mathbf{p}(z)$, $\{p^i(z)\}_{i=1}^d \subset \mathcal{S}$, satisfying (4). Now we can find a paraunitary extension of the vector $\mathbf{p}(z)$ guaranteed by Theorem 1 with the follow-up transform inverse to (14).

In order to verify that, we really have (anti)symmetric wavelets, we need to check how the transform inverse to (14) acts on the matrix of the paraunitary extension. We again consider four cases of symmetry of a refinable function.

Let $d = 2N + 1$ and $h \in \mathcal{S}_w$. Then obviously, the vector $\mathbf{p}(z)$ consists of $N + 1$ components from $\mathcal{S}_{w,e}$ and N components from $\mathcal{S}_{w,o}$. Thus, we have the unitary symmetric extension

$$E(\mathbf{p}) = \left(\begin{array}{c|c} B_{1,1} \in \mathcal{S}_{w,e}^0 & B_{1,2} \in \mathcal{S}_{w,o}^0 \\ \hline B_{2,1} \in \mathcal{S}_{w,o}^0 & B_{2,2} \in \mathcal{S}_{w,e}^0 \end{array}\right), \tag{15}$$

where the dimensions of the blocks $B_{1,1}$ and $B_{2,2}$ are equal to $(N+1) \times (N+1)$ and $N \times N$. Therefore, we have $(N + 1)$ symmetric generators (including the scaling function) symmetric about the origin and N odd generators antisymmetric about the origin.

Let $d = 2N + 1$ and $h \in \mathcal{S}_h$. Then the vector $\mathbf{p}(z)$ consists of N components from $\mathcal{S}_{w,e}$, N components from $\mathcal{S}_{w,o}$ and 1 component from $\mathcal{S}_{h,e}$. We note that the degree of the component from $\mathcal{S}_{h,e}$ cannot exceed the maximum of the degrees of the components from $\mathcal{S}_w$. Otherwise, property (4) fails. Thus,

$$E(\mathbf{p}) = \left(\begin{array}{c|c|c} B_{1,1} \in \mathcal{S}_{w,e}^0 & B_{1,2} \in \mathcal{S}_{h,e}^1 & B_{1,3} \in \mathcal{S}_{h,o}^1 \\ \hline B_{2,1} \in \mathcal{S}_{h,e}^{-1} & B_{2,2} \in \mathcal{S}_{w,e}^0 & B_{2,3} \in \mathcal{S}_{w,o}^0 \\ \hline B_{3,1} \in \mathcal{S}_{h,o}^{-1} & B_{3,2} \in \mathcal{S}_{w,o}^0 & B_{3,3} \in \mathcal{S}_{w,e}^0 \end{array}\right), \tag{16}$$

where the dimensions of the blocks $B_{1,1}$, $B_{2,2}$, and $B_{3,3}$ are equal to 1×1, $N \times N$, and $N \times N$. Therefore, the first columns of the matrix $E(\mathbf{p})$ gives an even generator with the mask symmetric about N, the columns with numbers from 2 to $N+1$ give symmetric generators (including the scaling function) with masks symmetric about $1/2$, and the remaining N columns give odd generators with masks antisymmetric about $1/2$.

Let $d = 2N$ and $h \in \mathcal{S}_w$. Then the vector $\mathbf{p}(z)$ consists of N components from $\mathcal{S}_{w,e}$, $N-1$ components from $\mathcal{S}_{w,o}$ and 1 component from $\mathcal{S}_{h,e}$. We note again that the degree of the component from $\mathcal{S}_{h,e}$ cannot exceed the maximum of the degrees of the components from $\mathcal{S}_w$. Otherwise, property (4) fails. Thus, $E(\mathbf{p})$ has the form (16), where the dimensions of the blocks $B_{1,1}$, $B_{2,2}$, and $B_{3,3}$ are equal to 1×1, $N \times N$, and $(N-1) \times (N-1)$. Therefore, the first column of the matrix $E(\mathbf{p})$ gives an even generator with coefficients symmetric about N, the columns with numbers from 2 to $N+1$ give symmetric generators (including the scaling function) symmetric about the origin, and the remaining $N-1$ columns give odd generators with masks antisymmetric about the origin.

Let $d = 2N$ and $h \in \mathcal{S}_h$. Then the vector $\mathbf{p}(z)$ consists of N components from $\mathcal{S}_{w,e}$ and N components from $\mathcal{S}_{w,o}$. Thus, $E(\mathbf{p})$ has the form (15), where the dimensions of the blocks are equal to $N \times N$. Therefore, we have N even and N odd generators with masks (anti)symmetric about $1/2$.

The explicit algorithm given above proves the existence of (anti)symmetric wavelets associated with any symmetric scaling function. However, generally speaking, it does not provide us with wavelets whose algorithms of decomposition and reconstruction have minimal computational complexity. Indeed, if we implement the decomposition algorithm in a direct form as convolutions with masks (multiplication by a wavelet symbol), then the amount of necessary arithmetic operations is determined by the degrees of the polyphase components of the symbol. At the same time, if $\operatorname{mdeg}(h_k^1) \neq -\operatorname{Mdeg}(h_k^1)$, the transform (14) and our choice of n_1 and n_2 in (12) may lead to the vector $\mathbf{p}$ with components whose degrees exceed the degrees of the corresponding components of the vector $\mathbf{h}^1$. As a result, some of the components of the vectors $\mathbf{h}^k$, $k > 1$, may have a higher degree than the same components of $\mathbf{h}^1$. To avoid this situation we can introduce a preliminary transform consisting in multiplication of the vector $\mathbf{h}^1$ by an appropriate diagonal matrix with monomials of the form z^{k_i}, $i = 1, 2, \dots d$, on the diagonal. The integer numbers k_i are chosen so that the reciprocal pairs p and q intended for the transform (14) have either $\operatorname{mdeg}(p) = \operatorname{mdeg}(q) = -\operatorname{Mdeg}(p) = -\operatorname{Mdeg}(q)$ for even $\deg(p) = \deg(q)$ or $\operatorname{mdeg}(p) = \operatorname{mdeg}(q) = -\operatorname{Mdeg}(p) + 1 = -\operatorname{Mdeg}(q) + 1$ for odd $\deg(p) = \deg(q)$. It is clear that the sum of the degrees of the monomials corresponding to any reciprocal pair of polyphase components is equal to 0 or 1. So for that choice of the components the vector $\mathbf{p}$ belongs either to $\mathcal{S}_w^0$ or to $\mathcal{S}_h^1$. Thus, Theorem 1 can be applied to the vector $\mathbf{p}$. It is easy to verify that the extension algorithm described in the proof of Theorem 1 generates (anti)symmetric wavelets.

We have to note that, varying degrees of the monomials in the diagonal matrix, not only computational complexity can be reduced but systems of wavelets with different types of symmetry for the same scaling function can be constructed. For $\mathbf{h} \in \mathcal{S}_{w,e}^0$ (anti)symmetric wavelets with masks (anti)symmetric about $d/2$ may substitute some of the wavelets whose masks are (anti)symmetric about the origin. Likewise, for $\mathbf{h} \in \mathcal{S}_{h,e}^1$, (anti)symmetric wavelets with masks (anti)symmetric about $(d-1)/2$ may substitute some of the wavelets whose masks are (anti)symmetric about $1/2$.

It should also be noted that the given factorization allows to reduce the computational cost approximately by a factor 2.

4. Symmetric framelets with d generators

Suppose we have a scaling function φ with a polynomial symbol (12) satisfying the condition

$$0 \leqslant \sum_{k=n_1^1}^{n_2^1} h_k^1(z) h_k^1(1/z) \leqslant 1, \quad |z| = 1, \tag{17}$$

which is milder than (13). Then it is well known that condition (17) is necessary and sufficient for the existence of a UEP-based framelet system consisting of d framelet generators $\{\psi^i\}_{i=2}^{d+\Delta}$, $\Delta \geqslant 1$, $\Delta \in \mathbb{N}$, associated with the scaling function $\psi_1 := \varphi$. We introduce the symbols of framelets ψ^i as

$$h^i(z) = \frac{1}{\sqrt{d}} \sum_{k=n_1}^{n_2} z^k h_k^i(z^d).$$

Then condition (1) can be re-written in the form

$$\mathcal{M}(z)\mathcal{M}^*(z) = I,$$

where

$$\mathcal{M}(z) = \begin{pmatrix} h_1^1(z) & h_2^1(z) & \dots & h_{d+\Delta}^1(z) \\ h_1^2(z) & h_2^2(z) & \dots & h_{d+\Delta}^2(z) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^d(z) & h_2^d(z) & \dots & h_{d+\Delta}^d(z) \end{pmatrix}.$$

Thus, the problem of constructing framelets associated with a scaling function can be reduced to finding a rectangular paraunitary matrix with the given first column.

It is clear that the bigger Δ the more freedom we have for choosing the rectangular paraunitary matrix. In [9] and [2], it was proved (for $d = 2$) that the problem of polynomial unitary extension can be always solved for $\Delta = 1$, i.e., for two framelets.

As was mentioned in Introduction, the case of symmetric scaling functions is more complicated. For symmetric scaling functions and $d = 2$, Chui and He [2] proved that (anti)symmetric framelets can be constructed for $\Delta = 2$ (3 framelets). In [10], a criterion for the existence of 2 (anti)symmetric framelets ($\Delta = 1$) was found and the method for finding the framelets with minimal support was given.

The goal of this section is to design an algorithm for the symmetric extension for $\Delta = 2$ ($d + 1$ framelets) and to find a condition when d (anti)symmetric framelets are possible.

Recall that we consider the case when condition (17) (not (13)) takes place. Thus, the polynomial

$$H_0(z) := 1 - \sum_{k=n_1^1}^{n_2^1} h_k^1(z) h_k^1(1/z) \geqslant 0, \quad |z| = 1,$$

is symmetric and, by the Riesz lemma, there exists a Laurent polynomial $h_0(z)$ with real coefficients satisfying the condition $H_0(z) = h_0(z)h_0(1/z)$. Without loss of generality, we assume that either $\mathrm{mdeg}(h_0(z)) = -\mathrm{Mdeg}(h_0(z))$ or $\mathrm{mdeg}(h_0(z)) = -\mathrm{Mdeg}(h_0(z)) + 1$ (the evenness of $\deg(h_0)$ defines which of the two cases takes place).

We extend the vector $\mathbf{h} := (h_1^1, \ldots, h_d^1)^{\mathrm{T}}$ by two polynomials

$$h_{d+1}^1(z) = \frac{h_0(z)}{\sqrt{2}}, \qquad h_{d+2}^1(z) = \frac{h_0(1/z)}{\sqrt{2}}, \tag{18}$$

to the vector $\tilde{\mathbf{h}} := (h_1, \ldots, h_{d+2})^{\mathrm{T}}$. Obviously, the extended vector satisfies (13). Thus, we reduce the problem of the extension to a rectangular paraunitary matrix to the problem solved in Section 3. Applying the algorithms from Section 3, we can extend the vector $\tilde{\mathbf{h}}$ to a paraunitary square matrix with components corresponding to symmetric wavelets (or framelets) with $d+1$ generators and the dilation factor $d+2$ with the follow-up rejection of two last components of the columns of the extended square matrix. Note that the problem of the existence of a scaling function for the symbol $\tilde{\mathbf{h}}$ is irrelevant since the extension of $\mathbf{h}$ to $\tilde{\mathbf{h}}$ appears as an intermediate step. Once the extension of the matrix is found, we do not need it any more.

As we mentioned above, if $d = 2$, for a symmetric scaling function, sometimes two (anti)symmetric framelets exist. In [10], we proved that two (anti)symmetric framelets exist if and only if the roots of the polynomial $H_0(z)$ have even multiplicity.

It turns out that the same condition is sufficient for the existence of $d+1$ framelets. Indeed, the even multiplicity of the roots of the polynomial $H_0 \in \mathcal{S}_{w,e}^0$ implies that the polynomial h_0 can be chosen to be (anti)symmetric. It remains to extend the vector $\mathbf{h}$ by one element $h_{d+1}^1(z) = h_0(z)$ and to apply the procedure described in Section 3 to the resulting vector $\tilde{\mathbf{h}}$. Thus, the sufficiency is proved.

We guess that the evenness of the roots is also necessary for the existence of $d+1$ (anti)symmetric framelets. However, we could not find the proof of that fact.

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