Method of Virtual Components for Constructing Wavelet Frames

Ming-Jun Lai and Alexander Petukhov
Department of Mathematics,
University of Georgia, Athens, GA 30602

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Abstract

For a wavelet frame with polynomial symbols generated by an MRA with a given approximation order, we present the method for parametric description and construction of all dual analysis operators (filter banks) with the maximum number of vanishing moments whose symbols are polynomials. The similar problems related to the maximum frame approximation order and some other wavelet frame properties are considered.

1 Introduction

The main goal of our paper is to present an algorithm of a decomposition in wavelet frames (in particular, tight wavelet frames) allowing to provide maximum approximation order, staying within a framework of Mixed Extension Principle.

In the 80's, the interconnection between the rate of approximation and properties of scaling functions and wavelets was investigated in detail. In particular, it was known that approximation order of linear combinations of shifts of a scaling function is tightly connected with its ability to recover algebraic polynomials of a given degree. For orthogonal and bi-orthogonal wavelets the approximation order of a scaling functions is equivalent to the existence of wavelets with the given number of vanishing moments.

This is not the case for wavelet frames. Neither approximation order of a multiresolution used for the expansions nor this for dual multiresolution have a direct impact on a number of vanishing moments of framelets even for the case of tight wavelet frames. This phenomenon was analyzed in [4] and [7].

The general theory of wavelet frames was developed by A. Ron and Z. Shen [17], [18]. In particular, they found very handy tools for construction wavelet tight frames and bi-frames (for shortness, *framelets*) associated with a given scaling function. These tools were called the Unitary Extension Principle (UEP) and Mixed Extension Principle (MEP correspondingly. Those principles give the opportunity to reduce the problem of finding framelets to solving a matrix equation (generally speaking, with non-square matrices), where the first columns of the matrices are defined by the known scaling functions. The level of the interest to framelet systems increased after the papers by C. Chui and W. He [3] and A. Petukhov [14] (see also [13] for the full version), where two different algorithms for solving the mentioned equation were found. However, it became clear very soon (actually, it has been clear since papers [17], [18]) that there is a serious drawback within UEP. Leaving the detail discussion until Section 3, we describe now the qualitative essence of this drawback. For an overwhelming majority of scaling functions with a high approximation order, if framelet generators are obtained with UEP, a number of vanishing moments for at least one of framelets is equal to one. The high Multiresolution Analysis (MRA) approximation order means that smooth data can be well represented with linear combinations of shifts (integer translations) of a scaling function. In particular, it means that the residual has a small norm. Assuming the Riesz property of the wavelet basis, an analysis operator for computing the expansion coefficients defined by the dual wavelet basis is a projector on MRA-spaces. Thus, the analysis operator does provide the necessary approximation order, i.e., an approximation order of a bi-orthogonal basis (in terminology introduced in [7]) coincides with an approximation order of the MRA. Moreover, taking into account the smallness of the residual, the Riesz properties implies the smallness of the decomposition coefficients with respect to some norm in the space of sequences. Thus, for Riesz bases, approximation properties of MRAs (i.e., potential properties of *a synthesis operator*) allow to judge about coefficients of the decomposition (properties of *an analysis operator*).

For wavelet frames, the reasonings above are not valid any more. At the same time, small wavelet (framelet) coefficients for smooth functions guaranteed by a high order of vanishing moments are still very desirable. Thus, we need to satisfy some extra conditions to provide actual approximation order equal to the approximation order of the MRA. Otherwise, the low order of framelet vanishing moments necessarily leads to either the leak of the information from the low frequency components (a scaling function) to the high frequency (framelets) or just to unjustified increasing the norm of framelet coefficients. Since wavelet frames promised to be a very flexible tool for data representation, this drawback was very painful. While a few examples of compactly supported UEP-framelets combining a few vanishing moments and the symmetry were found by I. Selesnik [19], A. Petukhov [15] (2 vanishing moments), and Q. Jiang [9], I. Selesnick, A. Abdelnour [20] (3 vanishing moments), those results rather emphasized the lack of the flexibility. It should be mentioned that any examples of symmetric UEP-framelets with more than 3 vanishing moments have still not been found.

For fairness sake, we have to mention that the compactness of the representation is not a universal requirement. For some of applied problems like de-noising, where wavelet frames deserved a good reputation, the compactness does not play any role. At the same time, usually, it does not contradict directly to other requirements. For this reason, this problem is among the most attractive problems of the wavelet frame theory.

The original beautiful solution of this problem allowing to overcome the disbalance between MRA approximation order and a number of vanishing moments was found independently by C. Chui, W. He, J. Stoeckler [4] and I. Daubechies, B. Han,

A. Ron, Z. Shen [7]. Shortly, it was proved that if an MRA is generated by a compactly supported scaling function φ and provides approximation order N, then there exists another compactly supported generator $\tilde{\varphi}$ of the same MRA such that the UEP-framelets associated with $\tilde{\varphi}$ have N vanishing moments. This method for constructing the framelets was called the Oblique Extension Principle (OEP). In spite of easy realizability and effectiveness, OEP suffers from at least three drawbacks which in some applications may be critical. First of all, the OEP trick results in the increase of the support of a scaling function and framelets. Secondly, in spite of the compactness of the support the decomposition-reconstruction algorithms cannot be implemented any more with Finite Impulse Response (FIR) filters. The involving of recursive Infinite Impulse Response (IIR) filters or truncated filters becomes necessary. It may be restrictive for real time applications. In addition, application of filters with rational impulse response characteristic is even more problematic in multivariate case.

In this paper, we are going to present a new scheme of data representation with MEP-framelets which enables us to avoid all the problems listed above. The main idea of this scheme lies in the non-uniqueness of the decomposition in a given frame. We are going to parameterize and to use this uncertainty in the choice of the decomposition coefficients for fitting the analysis operator to given properties. Among these properties, the maximum number of vanishing moments and the approximation order provided by analysis operator play a crucial role. One of this paper objectives is to show that the construction of a dual wavelet frame with the maximum approximation order and the maximum number of vanishing moments defined by the approximation order of MRA is always possible.

The structure of the paper is as follows. In Sections 2 and 3, we introduce main notions and formulate fundamental results motivating this research as well as the research goals. In Section 4, we describe all possible one level frame expansions and linear analysis operators by means of a parametrization. Section 5 is the main part of this paper. All possible dual filter banks annihilating polynomials of the maximum degree will be described. In Section 6, we show how to solve a few problems of description of dual frames providing some natural representation properties. Among those properties maximization of frame the approximation order is considered. Sections 7 is devoted to constructing examples of univariate bi-frames with polynomial symbols with the maximum number of vanishing moments.

2 Notation

In this section we consider functions in the space $\mathbb{L}^2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx.$$

As usual we denote by $\hat{f}(\omega)$, $\omega \in \mathbb{R}$, Fourier transform of the function $f(x) \in \mathbb{L}^2(\mathbb{R})$,

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}\,dx.$$

It is well-known that the notion of Fourier transform can be extended to the space of tempered distribution. In particular, Fourier transform of a polynomial of the order n is a linear combination of δ-function at the origin and its derivatives up to the order n.

Let us recall that a family of elements $\mathcal{F} := \{f_k\}_{k\in\mathbb{Z}}$ of a Hilbert space $\mathcal{H}$, whose closure of the linear span coincides with $\mathcal{H}$, is called a Riesz basis if for any sequence $\{a_k\} \in \ell^2$

$$A_1 \sum_k |a_k|^2 \leq \| \sum_k a_k f_k \|^2 \leq B_1 \sum_k |a_k|^2,$$

the positive constants A_1 and B_1 are called Riesz constants. When $A_1 = B_1$ elements of the basis are orthogonal, i.e., after renormalization they constitutes an orthonormal basis of $\mathcal{H}$. It is well-known that for any Riesz basis there exists a unique dual Riesz basis satisfying the condition

$$\langle f_i, \tilde{f}_j \rangle = \delta_{i,j} := \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{1}$$

A pair of mutually dual bases is called *bi-orthogonal*.

R.J. Duffin and A.C. Shaeffer [8] gave the following definition of a frame. A family $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{H}$ is called a frame in $\mathcal{H}$ if for any $f \in \mathcal{H}$

$$A_2 \|f\|^2 \leq \sum_k |\langle f, f_k \rangle|^2 \leq B_2 \|f\|^2, \tag{2}$$

where optimal positive A_2 and B_2 are called frame constants. If $A_2 = B_2$, the frame is called *a tight frame*. The right inequality of (2) is called the Bessel property of the system $\mathcal{F}$. For any frame $\mathcal{F}$, there exists a dual frame (a bi-frame) satisfying the condition

$$f = \sum_k \langle f, f_k \rangle \tilde{f}_k = \sum_k \langle f, \tilde{f}_k \rangle f_k \tag{3}$$

and

$$\langle f, g \rangle = \sum_k \langle f, f_k \rangle \cdot \overline{\langle g, \tilde{f}_k \rangle} \tag{4}$$

for any f, $g \in \mathcal{H}$. In general, the dual frame is not unique.

Riesz bases and frames have a lot of common properties. The main reason of this similarity can be explained by the fact that any Riesz basis constitutes a frame whose bounds coincide with the Riesz constants. In particular, Riesz bi-orthogonal bases satisfy properties (3) and (4). Thus, the notion of a frame is just an extension of the notion of a basis to linear dependent systems.

However, the main issue of this paper is not about the similarity of the notions but rather about the distinctions which give an opportunity to set and solve the problems typical only for frames. Frames bring more flexibility for data representation due to their redundancy which appears, for example, in non-uniqueness of dual frames. Thus, fixing one tight frame for the decomposition, we can optimize either coefficients of the decomposition of an individual function or the linear operator (read a dual frame) for some class of functions. However, we have to emphasize that since the notion of a frame is wider than this of a basis, many of traditional methods employed for bases are not available for frames. First of all this remark relates to the property (actually definition) (1) of bi-orthogonal bases.

Now we introduce necessary notions related to univariate wavelet frames with an integer dilation factor $d \geq 2$.

The frame $\left\{\{\psi^l_{j,i}\}_{j,i\in\mathbb{Z}}\right\}_{l=1}^{n}$, where $\psi^l_{j,i}(x) = d^{j/2}\psi^l(d^j x - i)$, generated by translates and dilations of finite number of functions, is called *an affine or wavelet frame*. For applications, this definition of wavelet frames is very general. Mutiresolution-based wavelets the algorithms for decomposition and reconstruction of functions can be implemented in a pyramid form which is much more computationally efficient than just a regular inner products from (3).

Suppose a real-valued function $\varphi \in \mathbb{L}^2(\mathbb{R})$ satisfies the following conditions:

(a) $\hat{\varphi}(\omega) = m_0(d^{-1}\omega)\hat{\varphi}(d^{-1}\omega)$, where m_0 is an essentially bounded 2π-periodic function;

(b) $\lim_{\omega\to 0}\hat{\varphi}(\omega) = (2\pi)^{-1/2}$;

then the function φ is called *refinable* or *scaling*, m_0 is called *a symbol* of φ, and the relation in item (a) is called *a refinement equation.*

For any refinable function φ the linear spaces $V^j := \overline{\mathrm{span}}\,\{\varphi(d^j x - n)\}_{n\in\mathbb{Z}}\}$ constitute *a multiresolution analysis* (MRA) of the space $\mathbb{L}^2(\mathbb{R})$ (cf. [12]), i.e., a nested sequence

$$\cdots \subset V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^j \subset \ldots$$

of closed linear subspaces of $\mathbb{L}^2(\mathbb{R})$ such that

1. $\cap_{j\in\mathbb{Z}} V^j = \{0\}$;
2. $\overline{\cup_{j\in\mathbb{Z}} V^j} = \mathbb{L}^2(\mathbb{R})$;
3. $f(x) \in V^j \Leftrightarrow f(dx) \in V^{j+1}$.

Fulfillment of item (1) and (2) for the obtained spaces V^j was proved in [1]. Property (3) is evident. Note that we do not require that the functions $\{\varphi(d^j x - n)\}_{n\in\mathbb{Z}}$ constitute either a Riesz basis or a frame of the space V^0.

While this paper is about constructing wavelet frames in $\mathbb{L}^2(\mathbb{R})$, theoretical reasonings can be mostly repeated for multivariate case but computation is much more complicated.

When for some MRA $\{V^j\}$ the frame generators $\{\psi^l\}_{l=1}^r$ belong to V^1 the wavelet frame is call MRA-based. We assume that the MRA is generated by a function φ. Then the refinable function and framelets satisfy the refinement equations

$$\hat{\varphi}(\omega) = m_0(\omega/d)\hat{\varphi}(\omega/d), \quad \hat{\psi}^k(\omega) = m_k(\omega/d)\hat{\varphi}(\omega/d), \quad k = 1, 2, \ldots, r \tag{5}$$

where 2π-periodic essentially bounded functions m_k are called *the symbols* of φ and ψ^k,

$$m_0(\omega) := d^{-1/2}\sum_{l\in\mathbb{Z}} h_l e^{-il\omega}, \quad m_k(\omega) := d^{-1/2}\sum_{l\in\mathbb{Z}} g^k_l e^{-il\omega}, \quad k = 1, 2, \ldots, r. \tag{6}$$

We introduce the linear spaces

$$W^{k,l} := \overline{\operatorname{span}} \left\{ d^{k/2}\psi^l(d^k x - n) \right\}_{n,k\in\mathbb{Z}}, \; l = 1, \ldots, r;$$

$W^k := \oplus W^{k,l}$. Like in the case of wavelet bases, we call them wavelet (or framelet) spaces. However, unlike wavelet bases the last sum as well as the sum $V^{k+1} = V^k \oplus W^k$ does not have to be direct.

Let φ and $\tilde{\varphi}$ be scaling functions generating two MRAs. In what follows, we are interested in systems $\psi^1, \psi^2, \ldots, \psi^r \in V^1$, satisfying the following conditions:

1) the functions $\left\{ \{\psi^l_{j,k}\}_{j,k\in\mathbb{Z}} \right\}_{l=1}^r$ and $\left\{ \{\tilde{\psi}^l_{j,k}\}_{j,k\in\mathbb{Z}} \right\}_{l=1}^r$ form a bi-frame of the space $\mathbb{L}^2(\mathbb{R})$;

2) for algorithms of decomposition and reconstruction the recurrent formulae

$$\langle \tilde{\varphi}_{j,k}, f \rangle = c_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\tilde{h}_{k-dl}, \qquad \langle \tilde{\psi}^q_{j,k}, f \rangle = d^q_{j,l} = \sum_{k\in\mathbb{Z}} c_{j+1,k}\tilde{g}^q_{k-dl}, \quad 1 \le q \le r, \tag{7}$$

and

$$c_{j+1,l} = \sum_{k\in\mathbb{Z}} c_{j,k}h_{l-dk} + \sum_{s=1}^{r}\sum_{k\in\mathbb{Z}} d^s_{j,k}g^s_{l-dk}, \tag{8}$$

where h_k, $\tilde{h}_k$, g^s_k, $\tilde{g}^s_k$ are Fourier coefficients (sometimes they are called masks) of the symbols defined by (6).

Following [18], we call transform (7) *an analysis operator* and transform (8) *a synthesis operator*. We will use the same terms for the operators acting for a fixed j (one-level operators) as well as for global operators setting the correspondence $f \mapsto \left\{ \{d^l_{j,k}\}_{j,k\in\mathbb{Z}} \right\}_{l=1}^r \mapsto f$.

It can be proved that relations (7) and (8) take place if and only if

$$\mathcal{M}(\omega)\tilde{\mathcal{M}}^*(\omega) = I, \tag{9}$$

where

$$\mathcal{M}(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \ldots & m_r(\omega) \\ m_0(\omega + 2\pi/d) & m_1(\omega + 2\pi/d) & \ldots & m_r(\omega + 2\pi/d) \\ \ldots & \ldots & \ldots & \ldots \\ m_0(\omega + 2\pi(d-1)/d) & m_1(\omega + 2\pi(d-1)/d) & \ldots & m_r(\omega + 2\pi(d-1)/d) \end{pmatrix}$$

and the matrix $\tilde{\mathcal{M}}(\omega)$ is defined in the same way, using the symbols $\tilde{m}_k(\omega)$. In [17], the equation (9) was called *the Mixed Extension Principle* (MEP). In the same paper, the authors showed that under some (very mild) conditions on functions φ and $\tilde{\varphi}$ associated with the symbols m_0 and $\tilde{m}_0$ functions ψ^l and $\tilde{\psi}^l$ indeed generate a dual bi-frame system. MEP is a natural generalization of the corresponding extension principles which were actively used for constructing orthogonal and bi-orthogonal wavelet bases. The presented in "MEP" the word "extension" reflects the most popular method for constructing wavelet or framelet systems relying on (9) (see, for instance, [2], [3], [13]). In this method, two scaling functions (to be more precise their symbols m_0, $\tilde{m}_0$)

defining MRAs are considered as input data for the problem and symbols of framelets are looked for as solutions of (9). UEP is a special case of MEP when $\mathcal{M}(\omega) = \tilde{\mathcal{M}}(\omega)$. Both of them are a handy tool for constructing tight wavelet frames.

In what follows, we need two modifications of the matrix. We introduce the matrix $\mathbf{M}(z)$ whose elements are Laurent polynomials which is defined by the relation $\mathbf{M}(e^{i\omega}) := \mathcal{M}(\omega)$. The matrix $\mathbf{M}(z)$ is just the representation of the matrix $\mathcal{M}(\omega)$ in z-domain, $z \in \mathbb{C} \setminus 0$. This matrix can be generated by Laurent polynomials $H_k(e^{i\omega}) := m_k(\omega)$, $k = 0, 1, \ldots, r$. Both matrices $\mathcal{M}(\omega)$ and $\mathbf{M}(z)$ are called *modulation matrices.*

Let us introduce the $d \times d$ matrix of Discrete Fourier Transform $\mathcal{F} = (\mathcal{F}_{k,j})_{0 \le k,j < d}$, $\mathcal{F}_{k,j} = \exp\left(-\frac{2\pi kj}{d} i\right) \Big/ \sqrt{d}$; $j, k = 0, \ldots, d-1$, and the diagonal matrix $\mathcal{D}(\omega)$ with elements $\exp(-ik\omega)$, $k = 0, \ldots, d-1$, on the diagonal. It is well-known that the matrix $\mathcal{D}(\omega)\mathcal{F}\mathcal{M}(\omega)$ consists of $2\pi/d$-periodic components. So this matrix can be represented in another form $\mathbb{M}(e^{id\omega}) = \mathcal{D}(\omega)\mathcal{F}\mathcal{M}(\omega)$, where components of the matrix $\mathbb{M}(z)$ are Laurent polynomials. The matrix $\mathbb{M}(z)$ is called *a polyphase matrix.* The modulation matrix $\tilde{\mathbf{M}}(z)$ and the polyphase matrix $\tilde{\mathbb{M}}(z)$ are defined in the same way from the modulation matrix $\tilde{\mathcal{M}}(\omega)\}$. Because of the unitarity of the matrixes $\mathcal{F}$ and $\mathcal{D}(\omega)$, the equality (9) may be rewritten in the form

$$\mathbb{M}(z)\tilde{\mathbb{M}}^T(z^{-1}) = I. \tag{10}$$

In what follows, we keep using calligraphic, boldface, and "blackboard bold" fonts to distinct modulation matrices and vectors in ω and z and their polyphase forms correspondingly.

3 Objectives

Since 2000, MEP and UEP which have become a subject of intensive study. In [3] and [13], it was proved for the case $d = 2$, $r = 2$ that for any refinable function with a polynomial symbol satisfying the inequality

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 \le 1 \tag{11}$$

a solution satisfying UEP exists and gives a tight wavelet frame with two generators (framelets). Explicit algorithms for constructing the framelets were presented (cf. [10] for the multivarate case). In [10], inequality (11) was called the subQMF condition. Its special case when it turns into equality is a popular QMF (Quadratic Mirror Filter) condition which is used for constructing orthonormal wavelet bases and filter banks for subband filtration. More general case of two different scaling functions with an arbitrary dilation factor $d \ge 2$, $r = d$ was considered in [6] (for $d = r = 2$, see also [14]). The authors proved the existence of a polynomial solution to (9).

In classical theory of wavelet bases, the tightly connected notions of an MRA approximation order and wavelet vanishing moments play very important role. The MRA approximation order means the decaying rate of approximation of smooth enough functions by elements from V^j. Usually, the smoothness means the belonging to the Sobolev class W_2^n with the norm

$$\|f\|_{W_2^n} = \left(\int_{\mathbb{R}} |\hat{f}(\omega)|^2(1+\omega^2)^n \, d\omega\right)^{1/2}.$$

An MRA $\{V^j\}$ provides approximation order n if, for any $f \in W_2^n$,

$$\inf\{\|f-g\| : g \in V^j\} \le C d^{-nj}\|f\|_{W_2^n},$$

were C depends only on the MRA. In more general settings of shift invariant spaces, a criterion for a given approximation order was found in [1]. In the same paper, the authors showed that for a shift invariant space generated by a compactly supported function φ, $\hat{\varphi}(0) \neq 0$ their criterion is equivalent to the Strang-Fix condition [21]: $\hat{\varphi}^{(i)}(2\pi k) = 0$, $k \in \mathbb{Z} \setminus 0$, $i = 0, 1, \dots, n-1$. In turn, if $\varphi(0) \neq 0$, the Strang-Fix condition is equivalent to the maximum of the order of polynomials which can be reproduced in the closures of the spaces V^j in the topology of distributions. We denote these closures by $\mathbf{V}^j$. At last, we mention that a given approximation order can be guaranteed by a multiplicity of the factor $S(\omega) := 1 + e^{-i\omega} + \cdots + e^{-i(d-1)\omega}$ in the symbol $m_0(\omega)$ of the scaling function φ. In spite the fact that necessary and sufficient condition in terms of zeros of the polynomial $m_0(\omega)$ is formulated in more sophisticated form, for scaling functions of interest in applications, the assumption that $S^n(\omega)$ is a factor of $m_0(z)$ for MRAs reproducing polynomials of the order $n-1$ is quite usual. In particular, it can be justified by the known fact that for an MRA with the polynomial reproduction property, the generating scaling function with the mentioned factorization property exists. Moreover, such a function has the minimum size of the support.

The number of vanishing moments of a function ψ is defined as the maximum integer number n satisfying the relations

$$\int_{\mathbb{R}} x^{k-1}\psi(x)\, dx = 0, \quad k = 1, 2, \dots, n.$$

Unlike the MRA approximation order, the vanishing moments are interesting as an attribute of a dual system. Indeed, if framelets generating an analysis operator have vanishing moments of the order n, then components of the analysis operator computing framelet coefficients $d_{j,k}^l$ annihilate polynomials of the order $n-1$. It means that an analysis operator projects such polynomials to the space V^j. This guarantees that the rate of approximation of any $f \in W_2^n$ by its projections to V^j defined by the analysis operator is equal to n.

Within MRA framework, a high approximation order is a good start point allowing to hope that a smooth function f can be represented as a sum of a function f_V from V^j containing the most essential low frequency information about the function f and the function $f_W := f - f_V$ for which $\|f_W\| = O(d^{-jn})$ and it is assumed to be in wavelet space W^j. To realize this scheme the analysis operator (a dual system) may be required to satisfy some properties. In traditional settings for bi-orthogonal bases, it is well known that if V^0 and $\tilde{V}^0$ are generated by the functions φ and $\tilde{\varphi}$ whose shifts form Riesz bases and

$$\langle\varphi(\cdot-n),\tilde\varphi(\cdot-m)\rangle=\begin{cases}1, & n=m,\\ 0, & n\neq m,\end{cases},$$

the approximation order of the MRA generated by φ guarantees that the analysis operator (dual wavelet basis) generated by the function $\tilde\varphi$ realizes the maximum possible approximation order. In this case we say that a bi-orthogonal basis provides the given approximation order. As we saw, the approximation orders of MRA and bi-orthogonal wavelet basis induced by it necessarily coincide. Moreover, they are always less by 1 than a number of vanishing moments provided by wavelets of a dual basis. Therefore, there is no necessity to consider those options separately.

Unfortunately, even having a nice refinable functions providing a high approximation order, we cannot guarantee the high order of vanishing moments for framelet generators.

The essence of this effect is as follows. The MRA approximation order is defined by the multiplicity of the factor $S(\omega)$ of the polynomial m_0. At the same time, for UEP-based tight wavelet frames, the maximum number of vanishing moments of framelet generators is determined as half of the multiplicity of the root $\omega=0$ of the polynomial $1-m_0(\omega)$. If (11) turns into an equality (QMF condition), then the approximation order of MRA is automatically equal to the number of vanishing moments whereas for a supQMF symbol, the potential opportunity to approximate smooth function by elements of the space V^j with the precision $O(d^{-jn})$, generally speaking, cannot be reached because of the lack of n vanishing moments. Thus, for subQMF condition a number of vanishing moments of analysis operator depends rather on a scaling function than on the MRA generated by this function. For bi-frame settings, the upper estimate for a number of vanishing moments is determined the multiplicity of the root of $1-m_0(\omega)\tilde m_0(\omega)$ at the origin gives the of the analysis operator.

An elegant solution overcoming that problem was given in [4] and [7]. The desired flatness of the symbol at the origin was achieved by the correction of the scaling function. Instead of the original scaling function φ providing appropriate approximation order of MRA, a new function $\varphi^{\#}(x)=\sum t_k\varphi(x-k)$ is introduced. The symbol of the new scaling function can be represented in the form

$$m_0^{\#}(\omega)=m_0(\omega)\frac{\tau(2\omega)}{\tau(\omega)},\quad \tau(\omega):=\sum t_k e^{-ki\omega}.$$

Then, choosing an appropriate function τ (which can be chosen as a polynomial if m_0 is a polynomial), the appropriate flatness can be obtained. A similar construction in [6] allows to construct also bi-frames with many vanishing moments for pairs of MRAs generated by two arbitrary scaling functions. These constructions preserve MRAs as well as the compactness of the support of the scaling functions. Nevertheless, generally speaking, for a polynomial m_0 the function $m_0^{\#}$ is rational. Thus, the algorithms of decomposition and reconstruction (7) and (8) become infinite convolutions and cease to be local. In Section 5, we show how to overcome this drawback.

In [7], the authors introduced the notion of (bi-)frame approximation order. Let $f^j(x):=d^{j/2}\sum_{k\in\mathbb{Z}}c_{j,k}\varphi(d^jx-k)\in V^j$, where c_j is a sequence of the decomposition coefficients defined in (7). If for any $f \in W_2^n$ the estimate

$$\|f^j - f\| = O(d^{-jn}) \tag{12}$$

takes place, we say that the bi-frame has an approximation order n. The frame approximation order cannot exceed MRA $\{V^j\}$ approximation order. At the same time, their equality means that the frame analysis operator implements an expansion in a frame with the maximum rate of approximation available for the MRA.

In spite the fact that n vanishing moments of a dual frame gives a pass to estimate (12), this is not necessary for (12). Indeed, due to the redundancy, the smallness of the norm in (12) does not guarantee the smallness of the coefficients of the expansion. In Section 6, we, in particular, show how to construct analysis operators with polynomial modulation matrices providing a given approximation order.

There are two more properties of wavelet frames which always take place for biorthogonal wavelet bases but may be violated for frames:

- if $f \in V^j$, then $f^j = f$;
- if $f \in V^j$, then wavelet coefficients of f on the level j are equal to 0.

Generally speaking, for wavelet bi-frames, both of them may fail even for an analysis operator with the maximum number of vanishing moments. We note that the second property implies the first one. Obviously, these properties are strengthened versions of the bi-frame maximum approximation order and the maximum vanishing moment property correspondingly. In Section 6, we show how to construct dual frames satisfying each of these properties.

At the same time, we have to say that our approach does not give an answer to many natural questions related to properties of framelet coefficients. For instance, the method does not guarantee the belonging of the coefficients of the function to ℓ^2 (Bessel property). Moreover, dual framelets may be outside the class $\mathbb{L}^2(\mathbb{R})$ (staying, for a polynomial symbol $\tilde{m}_0$, compactly supported tempered distribution). All of these properties can be checked with the methods known in wavelet literature.

Actually, there is no surprise in this fact. Our goal is to obtain an analysis operator giving small coefficients for smooth functions. This property does not imply any smallness of the coefficients of arbitrary functions. Sometimes a good representation of smooth functions may be obtained at the expense of functions of a wider class.

It is known that MEP principle (9) gives just a necessary condition for a dual wavelet frame in the sense of $\mathbb{L}^2$-theory. At the same time, real world applications deal mainly with functions from some smoothness subspaces of $\mathbb{L}^2(\mathbb{R})$ like the Sobolev spaces. In such cases, an appropriate Sobolev norm (not necessarily $\mathbb{L}^2(\mathbb{R})$) is important for the analysis operator. Moreover, compactly supported wavelet frames an one-level decomposition operator has a finite $\mathbb{L}^2$-norm. The same is true for the finite-level decompositions which are required in applications. Thus, if we have a method for finding a good approximation of a function from some space V^j, i.e. we can find a coefficients $c_{j,k}$ such that

$$f(x) \approx d^{j/2} \sum_{k \in \mathbb{Z}} c_{j,k} \varphi(d^j x - k),$$

then the finite-level decomposition and reconstruction are well defined even if $\tilde{\varphi} \notin \mathbb{L}^2(\mathbb{R})$ and $c_j \notin \ell^2$. Therefore, any polynomial solution of (9) may be a subject of interest for some applied problems.

While all examples given in Section 7 form actual dual $\mathbb{L}^2$-wavelet frames, the general theory developed in Sections 4–6 relates to one-level decompositions or, in the terminology from Electrical Engineering, to filter banks with perfect reconstruction.

4 Method of Virtual Components for Parametrization of Expansions in Frames and Analysis Operators

We consider an arbitrary MRA generated by a compactly supported refinable function φ with a dilation factor $d \geq 2$ and a polynomial symbol m_0. We assume that that we have a set of framelets $\{\psi^k\}_{k=1}^r$, $r \geq d$, with polynomial symbols $\{m_k\}_{k=1}^r$ and the dual MRA generated by $\tilde{\varphi}$, $\{\tilde{\psi^k}\}_{k=1}^r$, $r \geq d$, with polynomial symbols $\{\tilde{m}_k\}_{k=0}^r$. However, the existing methods for constructing the framelets (cf. [3], [6] [7], [14], [10]) are not satisfactory for many reasons mentioned above.

Let us fix a frame generated by φ with modulation and polyphase matrices $\mathbf{M}(z)$ and $\mathbb{M}(z)$, and consider expansions in this system.

Taking into account that matrices in (10) are rectangular, the choice of $\tilde{\mathbb{M}}$ which defines expansion coefficients in the fixed frame is not unique. We are going to describe a simple convenient to use method to obtain all possible linear analysis operators $\tilde{\mathbb{M}}(z)$ satisfying (10). Moreover, this method allows to find operators $\tilde{\mathbb{M}}(z)$ providing the maximum number of vanishing moments coinciding with the approximation order of the MRA generated by φ and some variations of that property. In addition, we give a sketch how the same method allows to construct computationally efficient algorithms for the best approximations of an individual function in different metrics.

Let us assume that we are given arbitrary rectangular polynomial matrices $\mathbf{M}(z)$ and $\tilde{\mathbf{M}}(z)$ with the square extensions $\mathbf{M}_e(z)$ and $\tilde{\mathbf{M}}_e(z)$ satisfying the relation

$$\mathbf{M}_e(z)\tilde{\mathbf{M}}_e^T(1/z) = I. \tag{13}$$

Probably, the simplest constructive way to get those matrices is as follows. Introduce the polyphase matrix $\mathbb{M}(z)$ associated with $\mathbf{M}(z)$. Using the algorithm from [11], find the extension matrix $\mathbb{M}_e(z)$ with the determinant z^k, where k is an arbitrary integer number (for instance, $k = 0$). Assign $\tilde{\mathbb{M}}_e(z) := (\mathbb{M}_e(z^{-1}))^T$. Define the matrix $\tilde{\mathbb{M}}(z)$ as the first d rows of the matrix $\tilde{\mathbb{M}}_e(z)$. Define the matrices

$$\mathbb{G}(z) := \begin{pmatrix} G_{0,1}(z) & G_{1,1}(z) & \dots & G_{r,1}(z) \\ G_{0,2}(z) & G_{1,2}(z) & \dots & G_{r,2}(z) \\ \dots & \dots & \dots & \dots \\ G_{0,r+1-d}(z) & G_{1,r+1-d}(z) & \dots & G_{r,r+1-d}(z) \end{pmatrix},$$

and $\tilde{\mathbb{G}}(z)$ as the last $r+1-d$ rows of the matrices $\mathbb{M}_e(z)$ and $\tilde{\mathbb{M}}_e(z)$ correspondingly.

Thus, we have

$$\mathbf{M}_e(z) = \left[\begin{array}{c} \mathbf{M}(z) \\ \mathbb{G}(z^k) \end{array}\right], \quad \tilde{\mathbf{M}}_e(z) = \left[\begin{array}{c} \tilde{\mathbf{M}}(z) \\ \tilde{\mathbb{G}}(z^k) \end{array}\right], \quad \mathbb{M}_e(z) = \left[\begin{array}{c} \mathbb{M}(z) \\ \mathbb{G}(z) \end{array}\right], \quad \tilde{\mathbb{M}}_e(z) = \left[\begin{array}{c} \tilde{\mathbb{M}}(z) \\ \tilde{\mathbb{G}}(z) \end{array}\right].$$

We note that the existence of the extension $\mathbb{M}_e(z)$ follows from the fact that $\mathbb{M}(z)$ is a polyphase matrix of a wavelet frame, hence, its minors of the size $d \times d$ do not vanish simultaneously. Here and in what follows, we use the subscript e for the extensions of a given matrix or a vector.

Let $X(\omega) := \sum_j x_j e^{ij\omega}$ be a formal Fourier series whose coefficients are considered as an input data for the analysis operator. We note that, in what follows, all operations are defined for an arbitrary sequence $\{x_j\}$. So it does not have to belong to ℓ^2. In z-domain, the action of the decomposition operator can be represented in any of the forms

$$\vec{\mathbb{Y}}(z^d) = \vec{\mathbf{Y}}(z) = \tilde{\mathbf{M}}^T(z^{-1})\vec{\mathbf{X}}(z) = \tilde{\mathbb{M}}^T(z^{-d})\vec{\mathbb{X}}(z^d),$$

where

$$\vec{\mathbf{X}}(e^{i\omega}) = \left(\mathbf{X}_0(e^{i\omega}), \dots, \mathbf{X}_{d-1}(e^{i\omega})\right)^T := (X(\omega),\, X(\omega+2\pi/d), \dots, X(\omega+2\pi(d-1)/d))^T,$$

$\vec{\mathbf{Y}}(z)$ is an $(r+1)$-dimensional vector of the z-transform of coefficients of the expansion into the system of shifts of the functions φ, $\{\psi^k\}_{k=1}^r$. Note that components of $\vec{\mathbf{Y}}(z)$ depend only on z^d. Of course, the inverse operation

$$\vec{\mathbf{X}}(z) = \mathbf{M}(z)\vec{\mathbb{Y}}(z^d) \tag{14}$$

brings us back to the signal vector $\vec{\mathbf{X}}(z)$. However, as mentioned above, $\vec{\mathbb{Y}}(z^d)$ is not an unique vector satisfying (14). It is clear that a vector $\vec{\mathcal{Y}}(z^d)$ satisfies (14) if and only if the difference $\Delta\vec{\mathbb{Y}}(z^{-d}) := \vec{\mathbb{Y}}(z^{-d}) - \vec{\mathcal{Y}}(z^{-d})$ is orthogonal to rows of the matrix $\mathbf{M}(z)$ (or $\Delta\vec{\mathbb{Y}}(z^{-1})$ is orthogonal to rows of the matrix $\mathbb{M}(z)$). Thus, it is easy to see that the vector $\Delta\vec{\mathbb{Y}}(z)$ can be represented as a liner combination of the extension rows of the matrix $\tilde{\mathbb{M}}(z^{-1})$ introduced above. The coefficients of the linear combinations either can be considered as given in the beginning or can be found as an inner products of $\Delta\vec{\mathbb{Y}}(z)$ with the extension rows of the matrix $\mathbb{M}(z)$. We note that since the components of $\mathbb{M}(z)$ are Laurent polynomials, then the components of $\Delta\vec{\mathbb{Y}}(z)$ can be any formal Laurent series with an arbitrary (even not necessarily bounded) sequence of coefficients. If we denote those coefficients by $\mathbb{X}_d(z), \dots, \mathbb{X}_r(z)$ and extend the vector $\vec{\mathbb{X}}(z)$ with these coefficients up to the vector $\vec{\mathbb{X}}_e(z)$, we have a vector $\vec{\mathcal{Y}}(z) = \mathbb{M}_e^T(z)\vec{\mathbb{X}}_e(z)$ satisfying (14). Moreover, all sequences $\mathbb{X}_d(z), \dots, \mathbb{X}_r(z)$ generate vectors $\vec{\mathcal{Y}}(z)$ satisfying (14).

We will call this approach *the Virtual Components (VC) Method*.

Thus, we found the description all possible vectors $\vec{\mathcal{Y}}(z)$ in a parametric form through all possible extensions of the vector $\vec{\mathbb{X}}(z)$ up to the vector $\vec{\mathbb{X}}_e(z)$ of the dimension $r+1$. As it will be shown below, this parametrization is extremely convenient for optimization of decompositions in wavelet frames.

While that parametrization can be used for (non-linear) optimization of decompositions of individual functions, we put aside this problems for the future research and concentrate our attention on linear optimization which can be useful for function classes.

In what follows, we are interested in the choice of VCs linearly depended on the input. Because of the natural requirement that those components depend only on z^d, an arbitrary linear operator can be represented in the form

$$\Big(\mathbb{X}_d(z^d), \dots, \mathbb{X}_r(z^d)\Big)^T = \mathbf{F}(z)\vec{\mathbf{X}}(z) = \mathbb{F}(z^d)\vec{\mathbb{X}}(z^d),$$

where $\mathbf{F}(z)$ is a matrix of the size $(r-d+1)\times(r+1)$ whose elements are defined by the formula $\mathbf{F}_{k,j}(z) = \mathbf{F}_k(ze_d^j)$, $\mathbf{F}_k$ are arbitrary Laurent polynomials, $e_d^j := e^{2\pi i j/d}$. The more general case when $\mathbf{F}_k$ may belong to some subclass of Laurent series deserves the consideration as well. However, within the framework of our paper we restrict ourselves with the polynomial case providing an opportunity to work with convolutions of the data with finite sequences.

5 Analysis Operators with Maximum Number of Vanishing Moments

As was mentioned above, for bi-orthogonal wavelet bases, the approximation order n of the MRA $\{V^j\}$ immediately implies that all the wavelets generated in the dual MRA $\{\tilde{V}^j\}$ have n vanishing moments.

For wavelet frames, the situation changes dramatically. No MRA approximation order can guarantee the same number of vanishing moments. Actually, even one vanishing moment can not be guaranteed.

We give a method for "the correction" of a dual frame allowing to construct a dual frame (to be more precise, a dual filter bank) with the maximum number of vanishing moments defined by the MRA approximation order.

Theorem 1. *Let $\mathbf{M}(z)$ be a modulation matrix of the size $(r+1)\times d$ and of the rank d, $d < r+1$, generated by Laurent polynomials $H_i(z)$, $i = 0, 1, \dots r$, where $H_0(z) = (1 + z + \dots + z^{d-1})^n R(z)$, $R(1) = 1$. Then there exists a dual polynomial matrix $\tilde{\mathbf{M}}(\omega)$ defining an analysis operator with n vanishing moments.*

Remark 1. *The statement of Theorem 1 is quite elementary and probably may be proved with different methods. However, the VC method gives a convenient constructive method for computationally efficient implementation of analysis operators for the fixed $\mathbf{M}(\omega)$ in a parametric form.*

Proof. Since the matrix $\mathbf{M}(z)$ minors of the size $d\times d$ cannot turn into 0 simultaneously for any $z \neq 0$, there exist (cf. Section 4) polynomial matrices $\mathbf{M}_e(z)$, $\tilde{\mathbf{M}}_e(z)$ satisfying (13).

Generally speaking, the matrix (filter bank) $\tilde{\mathbf{M}}(z)$ does not provide the maximum number of vanishing moments. We now find an algorithm for the correction of the matrix $\tilde{\mathbf{M}}(z)$ increasing the number of the vanishing moments up to the optimum value n.

Suppose the polynomials $\left\{H_0\left(ze_d^k\right)\right\}_{k=0}^{d-1}$ do not have common roots. Then the equations

$$\sum_{k=0}^{d-1} F_j(ze_d^k)H_0(ze_d^k) = G_{0,j}(z^d) + (1-z^d)^n A_j(z^d), \quad j = 0, 1, r-d+1, \tag{15}$$

where $A_j(z)$ are arbitrary Laurent polynomials, have (not unique) solutions $F_j(z)$. Of course, solutions to (15) can be more easily found in the polyphase form

$$\sum_{k=0}^{d-1} F_{j,k}(z)H_{0,k}(z) = G_{0,j}(z) + (1-z)^n A_j(z), \quad j = 0, 1, r-d+1, \tag{16}$$

where $F_{j,k}$ and $H_{0,k}$ are polyphase components of F_j and H_0. In the case of the common roots, A_j cannot be chosen absolutely free. However, if the polynomials $G_{0,j}(z) + (1-z)^n A_j(z)$ has the same roots with the same multiplicity, then equations (16) still have solutions.

We assign for the "signal" extension components $\mathbf{X}_j(z)$, $j = d, \dots, r$, the formal Laurent series

$$\mathbb{X}_j(z^d) := \sum_{k=0}^{d-1} F_j(ze_d^k)\mathbf{X}(ze_d^k) \tag{17}$$

or

$$\mathbb{X}_j(z) := \sum_{k=0}^{d-1} F_{j,k}(z)\mathbb{X}_k(z). \tag{18}$$

Let $p(x)$ be an algebraic polynomial of the degree $n-1$. Then it belongs to any of the spaces $\mathbf{V}^j$ of the MRA generated by a refinable distribution associated with the symbol $H_0(z)$. In particular, in z-domain, we can represent it in the form of a formal Laurent series $\mathbf{X}(z) = \sum_{k\in\mathbb{Z}} P(k)z^k$, where $P(x)$ is an algebraic polynomial of the order $n-1$ uniquely defined by $p(x)$. In what follows, we suppose that $\mathbf{X}(z)$ is a polynomial input for an analysis operator.

Before proceeding to the proof we note that $L(z)\mathbf{X}(z) = 0$ for a Laurent polynomial $L(z)$ if and only if $L(z)$ has a root $z = 1$ of the multiplicity n. The polynomial $H_0(z)$ has a factor $1 + z + \cdots + z^{d-1} = (1-z^d)/(1-z)$ of the multiplicity n. So $H_0(z)$ has roots of the multiplicity n at the points $z = e_d^k$, $k = 1, \dots, d-1$. In particular, it means that

$$H_0(ze_d^k)\mathbf{X}(ze_d^l) = 0, \quad k \neq l \mod d. \tag{19}$$

We need to prove that only the first component of the vector $\tilde{\mathbf{M}}_e(1/z)\vec{\mathbf{X}}_e(z)$ is not equal to 0. Instead of verifying this property for the components, starting from the second one, we choose a different strategy. First we compose a vector $\vec{\mathbf{Y}}(z)$ whose first component coincide with the first component of the vector $\tilde{\mathbf{M}}_e(1/z)\vec{\mathbf{X}}_e(z)$ and remaining components are equal to 0. Then we show that $\mathbf{M}_e(z)\vec{\mathbf{Y}}(z) = \vec{\mathbf{X}}_e(z)$. Since $\mathbf{M}(z)$ is a non-degenerate matrix, this implies the equality $\vec{\mathbf{Y}}(z) = \tilde{\mathbf{M}}_e(1/z)\vec{\mathbf{X}}_e(z)$.

The first component of the vector $\mathbf{Y}(z)$ can be represented in the form

$$\begin{aligned}
\mathbb{Y}_1(z^d) &= \mathbf{Y}_1(z) = \sum_{k=0}^{d-1} \tilde{H}_0(z^{-1}e_d^{-k})\mathbf{X}(ze_d^k) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})\mathbb{X}_{d+j-1}(z^d) \\
&= \sum_{k=0}^{d-1} \tilde{H}_0(z^{-1}e_d^{-k})\mathbf{X}(ze_d^k) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d}) \sum_{k=0}^{d-1} F_j(ze_d^k)\mathbf{X}(ze_d^k) \\
&= \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})F_j(ze_d^k) \right) \mathbf{X}(ze_d^k). \qquad (20)
\end{aligned}$$

We need to check the validity of the equalities

$$\mathbf{X}(ze_d^l) = H_0(ze_d^l) \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})F_j(ze_d^k) \right) \mathbf{X}(ze_d^k),\ l = 0, \dots, d-1; \qquad (21)$$

$$\mathbb{X}_{d+q-1}(z^d) = G_{0,q}(z^d) \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})F_j(ze_d^k) \right) \mathbf{X}(ze_d^k),\ q = 1, \dots, r+1-d. \qquad (22)$$

Note that since the factor with $H_0(ze_d^l)$ in (21) depends only on z^d, we need to prove (21) only for $l = 0$.

Due to (19) and the elementary identities

$$\mathbf{X}(z)H_0(z)\tilde{H}_0(z^{-1}) = \mathbf{X}(z) \sum_{k=0}^{d-1} H_0(ze_d^k)\tilde{H}_0(z^{-1}e_d^{-k}),$$

and

$$\mathbf{X}(z)H_0(z)F_j(z) = \mathbf{X}(z) \sum_{k=0}^{d-1} H_0(ze_d^k)F_j(ze_d^k),$$

we have

$$\begin{aligned}
& H_0(z) \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})F_j(ze_d^k) \right) \mathbf{X}(ze_d^k) \\
= \ & H_0(z) \left(\tilde{H}_0(z^{-1}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})F_j(z) \right) \mathbf{X}(z) \\
= \ & \left(\sum_{k=0}^{d-1} H_{(}ze_d^k)\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d}) \sum_{k=0}^{d-1} H_0(ze_d^k)F_j(ze_d^k) \right) \mathbf{X}(z) \\
= \ & \left(\sum_{k=0}^{d-1} H_{(}ze_d^k)\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d}) \left(G_{0,j}(z^d) + (1-z^d)^n A_j(z^d) \right) \right) \mathbf{X}(z) \\
= \ & \mathbf{X}(z) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d})(1-z^d)^n A_j(z^d)\mathbf{X}(z) = \mathbf{X}(z).
\end{aligned}$$

To prove equalities (22) we reduce them to equivalent ones. We subtract linear combinations of (21) with the coefficients $F_q(ze_d^l)$ from the equality (22) with the number q. The equivalent form is

$$0 = -(1-z^d)^n A_q(z^d) \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d}) F_j(ze_d^k) \right) \mathbf{X}(ze_d^k),\ q = 1, \ldots, r+1-d. \tag{23}$$

So their validity follows just from the fact that $1 - z^d$ has the roots e_d^k, $k = 0, \ldots, d-1$ and $\mathbf{X}(z)$ is the z-transform of a polynomial of the degree $n-1$. □

In the next section we show that formulas (17) and (15) describe all possible analysis operators with n vanishing moments.

6 Related Results

Theorem 1 guarantees the existence of the analysis operator with the following properties:

1. *The approximation order of the analysis operator is equal to n*, i.e., polynomials up to the order $n-1$ can be recovered using the coefficients of $\mathbb{Y}_1(z)$ by the formula $\mathbf{X}(z) = H_0(z)\mathbb{Y}_1(z^d)$.
2. *The analysis operator has n vanishing moments*, i.e., polynomials up to the order $n-1$ have zero framelet coefficients.

For regular bi-orthogonal wavelet bases, these properties are equivalent. Moreover, the validity of those listed properties for polynomials implies (actually, is equivalent to) the same properties for any distribution from $\mathbf{V}^0$. For wavelet frames, we do not have neither the equivalency nor the transition from polynomials to the entire space $\mathbf{V}^0$. Recall that Property 2 is stronger than Property 1. The same properties extended from polynomials to the entire space $\mathbf{V}^0$ we will call Property 3 and Property 4 correspondingly. Now we give necessary and sufficient conditions within the VC method for each of the listed properties, applying the constructive approach used above in the proof of the theorem. In particular, we prove that our choice of the coefficients in (17) satisfying (15) is not only sufficient but also necessary to provide the maximum number of vanishing moments.

We introduce the vectors $\vec{\mathbb{G}}_0(z)$ and $\vec{\tilde{\mathbb{G}}}_0(z)$ which are the first columns of the matrices $\mathbb{G}(z)$ and $\tilde{\mathbb{G}}(z)$ and the vectors $\vec{\mathbf{H}}_0(z) := (H_0(z), \ldots, H_0(ze_d^{d-1})^T$ and $\vec{\tilde{\mathbf{H}}}_0(z) := (\tilde{H}_0(z), \ldots, \tilde{H}_0(ze_d^{d-1})^T$.

Let us suppose that the polynomials $F_j(z)$ generating the matrix $\mathbf{F}(z)$ are arbitrary and, generally speaking, do not satisfy (15). Then the right part of (17) still can be represented in the matrix form as $\mathbf{F}(z)\vec{\mathbf{X}}(z)$, where the (j,k)th component of the matrix $\mathbf{F}(z)$ is $F_j(ze_d^k)$. Then instead of (15) we have

$$\mathbf{F}(z)\vec{\mathbf{H}}_0(z) = \vec{\mathbb{G}}(z^d) + \vec{\mathbb{P}}(z^d),$$

for some vector $\vec{\mathbb{P}}(z)$ with polynomial entries.

Let a formal Laurent series $\mathbf{X}(z)$ be the z-transform of coefficients of the decomposition of some function (distribution) $f(z)$ in the translates of the scaling function in $\mathbf{V}^1$. Obviously, if this function belongs to $\mathbf{V}^0$, then the representation

$$\mathbf{X}(z) = H_0(z)\mathbb{Q}(z^d),$$

where $\mathbb{Q}(z)$ is a formal Laurent series, takes place. In particular, if $f(z) = \varphi(x)$, we have $\mathbb{Q}(z) \equiv 1$.

Representing (20) in the matrix form, we have

$$\begin{aligned}
\mathbb{Y}^1(z^d) &= \vec{\mathbf{X}}^T(z)\left(\vec{\tilde{\mathbf{H}}}_0(1/z) + \mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right) = \mathbb{Q}(z^d)\vec{\mathbf{H}}_0^T(z)\left(\vec{\tilde{\mathbf{H}}}_0(1/z) + \mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)\\
&= \mathbb{Q}(z^d)\left(\vec{\mathbf{H}}_0^T(z)\vec{\tilde{\mathbf{H}}}_0(1/z) + \vec{\mathbf{H}}_0^T(z)\mathbf{F}^T(z)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)\\
&= \mathbb{Q}(z^d)\left(\vec{\mathbf{H}}_0^T(z)\vec{\tilde{\mathbf{H}}}_0(1/z) + \left(\vec{\mathbb{G}}_0^T(z^d) + \vec{\mathbb{P}}^T(z^d)\right)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)\\
&= \mathbb{Q}(z^d)\left(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right).
\end{aligned}$$

Thus, (21) can be rewritten in the form

$$\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d) = \vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)\left(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)$$

or

$$\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)\vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d}) = 0. \tag{24}$$

Recall that the validity of the relation (24) means that a function (distribution) from $\mathbf{V}^0$ represented by $\mathbb{Q}(z)$ is mapped to itself by the the first component of the analysis operator. Let $\mathbb{Q}(z)$ be a function summable on the unit circle. Since $H_0(z)$ and $\vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})$ are Laurent polynomials, (24) takes place only if

$$\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z) = 0, \tag{25}$$

i.e., $\vec{\mathbb{P}}(z)$ has to be orthogonal to $\vec{\tilde{\mathbb{G}}}_0(z)$. However, such a choice of $\vec{\mathbb{P}}(z)$ guarantees the same projection property for any distribution $\mathbb{Q}(z)$, i.e., the first component of the analysis operator provides us with a projector from $\mathbf{V}^1$ to $\mathbf{V}^0$ (and from V^1 to V^0). Thus, (25) is necessary and sufficient for Property 3. Of course, the condition (25) is sufficient to provide the maximum approximation order for the analysis operator.

At the same time, (25) is not necessary for Property 1 since the analysis operator provides the approximation order n if the operator above is a projector only on algebraic polynomials up to the order $n-1$. For polynomials of the order $n-1$, the periodic distribution $\mathbb{Q}(e^{i\omega})$ is a linear combination of the δ-function and its derivatives up to the order $n-1$ with the support at $\omega = 0$. Since $\mathbf{H}_0(1) \neq 0$, (24) takes place if and only if the polynomial $\vec{\mathbb{P}}^T(e^{i\omega})\vec{\tilde{\mathbb{G}}}(e^{-i\omega})$ and its derivatives up to the order $n-1$ are equal to 0 at the point $\omega = 0$. Thus,

$$\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z) = (1-z)^n P(z), \tag{26}$$

where $P(z)$ is an arbitrary Laurent polynomial. Condition (26) is necessary and sufficient to provide the approximation order n. If the associated framelets form a Bessel system, (26) means that the approximation order with truncated expansions in the obtained bi-frame is equal to n. For bi-orthogonal Riesz bases, this property implies that the ℓ^2-norm of the discarded coefficients has the order $O(d^{-jn})$, where j defines the space approximating the function. In particular, this accords with the fact that the MRA approximation order and a number of vanishing moments of dual wavelets give essentially equivalent descriptions of bi-orthogonal bases. The requirement of n vanishing moments is stronger than the same wavelet frame approximation order (cf. [7]).

In Section 5, we found a sufficient condition for n vanishing moments of dual framelets provided that the approximation order of the MRA is not less than n. Now we prove the necessity of that condition.

The condition (26) is necessary and sufficient to satisfy the group of equalities (21). In other words, it guaranties the recovery of a polynomial using only $\mathbb{Y}_1(z^d)$. Conditions (22) has no influence on the recovery properties. However, they prohibit the leakage of coefficients to framelet spaces. The essence of this effect is as follows. If for elements of V^0 ($\mathbf{V}^0$) condition (21) is valid but (22) is not, then, applying the analysis operator to V^0 ($\mathbf{V}^0$), we may have non-zero framelet coefficients. At the same time, using those non-zero coefficients as an input for the synthesis operator, we have the zero output. This effect is undesirable when a compact representation of a function is a priority.

If $\vec{\mathbf{X}}(z) = \vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d)$ the condition (22) can be rewritten in the form

$$\mathbf{F}(z)\vec{\mathbf{H}}_0(z)\mathbb{Q}(z^d) = \vec{\mathbb{G}}_0(z^d)\mathbb{Q}(z^d)\left(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)$$

or

$$(\vec{\mathbb{G}}_0(z^d) + \vec{\mathbb{P}}(z^d))\mathbb{Q}(z^d) = \vec{\mathbb{G}}_0(z^d)\mathbb{Q}(z^d)\left(1 + \vec{\mathbb{P}}^T(z^d)\vec{\tilde{\mathbb{G}}}_0(z^{-d})\right)$$

Hence,

$$\vec{\mathbb{P}}(z)\mathbb{Q}(z) = \vec{\mathbb{G}}_0(z)\mathbb{Q}(z)\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(1/z). \tag{27}$$

Let $\mathbb{Q}(z) \not\equiv 0$ represent a function summable on the unit circle with. Due to (25), the right part of (27) is equal to 0. Hence, $\vec{\mathbb{P}}(z) \equiv \vec{0}$ is a necessary (and sufficient) condition for Property 4 under which the analysis operator is the identity operator on the spaces V^j.

If $\mathbb{Q}(z)$ is a distribution representing a polynomial of the order n, then, by (26), the right part of (27) is equal to zero again. Therefore, the components of the vector $\vec{\mathbb{P}}(z)$ have to have zeros of the multiplicity n at the point $z = 1$. Thus, the method we used in the proof of Theorem 1 gives all possible analysis operators providing n vanishing moments.

It means that for at least one $j = J$ we have

$$\sum_{k=0}^{d-1} F_J(ze_d^k)H_0(ze_d^k) = G_{0,J}(z^d) + (1 - z^d)^N B(z^d),$$

where $N < n$, $B(1) \neq 0$. Thus, (23) turns into

$$0 = -(1-z^d)^N B(z^d) \sum_{k=0}^{d-1} \left(\tilde{H}_0(z^{-1}e_d^{-k}) + \sum_{j=1}^{r+1-d} \tilde{G}_{0,j}(z^{-d}) F_j(ze_d^k) \right) \mathbf{X}(ze_d^k).$$

Summarizing the reasonings above, we can formulate the following theorem.

Theorem 2. *Let $\mathbb{M}_e(z)$, $\det \mathbb{M}_e(z) = z^k$, be an extension of the polyphase matrix of a wavelet frame with an MRA approximation order n,*

$$\tilde{\mathbb{M}}_e := \begin{bmatrix} \tilde{\mathbb{M}}(z) \\ \tilde{\mathbb{G}}(z) \end{bmatrix} := \mathbb{M}_e^{-1}(z^{-1}).$$

Then all possible polynomial analysis operators $\tilde{\mathbb{M}}^{\#}(z)$ linearly depending on the input can be represented in the form

$$\tilde{\mathbb{M}}^{\#}(z) = \tilde{\mathbb{M}}(z) + \mathbb{F}^T(z^{-1})\tilde{\mathbb{G}}(z) \tag{28}$$

with an arbitrary polynomial matrix $\mathbb{F}(z)$.

Let $\vec{\mathbb{P}}(z) = \mathbb{F}(z)\vec{\mathbb{H}}_0(z) - \vec{\mathbb{G}}_0(z)$. The analysis operator $\tilde{\mathbb{M}}^{\#}(z)$ satisfies:

- **Property 1** *if and only if $\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(z) = (1-z)^n P(z)$, where $P(z)$ is an arbitrary Laurent polynomial;*
- **Property 2** *if and only if $\vec{\mathbb{P}}(z) = (1-z)^n \vec{P}(z)$, where $\vec{P}(z)$ is an arbitrary Laurent polynomial vector;*
- **Property 3** *if and only if $\vec{\mathbb{P}}^T(z)\vec{\tilde{\mathbb{G}}}_0(z) = 0$;*
- **Property 4** *if and only if $\vec{\mathbb{P}}(z) \equiv \vec{0}$.*

Note that, while the matrix $\tilde{\mathbb{M}}^{\#}(z)$ may have a greater degree than the original matrix $\tilde{\mathbb{M}}(z)$, in many practically important cases, the matrices $\mathbb{F}(z)$ and $\tilde{\mathbb{G}}(z)$ have low dimensions. For this reason, sometimes formula (28) allows to reduce the computational costs significantly if to compute $\vec{\mathbb{Y}}(z)$ as $\tilde{\mathbb{M}}^T(1/z)\vec{\mathbb{X}}(z) + \tilde{\mathbb{G}}^T(1/z)\mathbb{F}(z)\vec{\mathbb{X}}(z)$ instead of the direct computation $\tilde{\mathbb{M}}^{\#T}(1/z)\vec{\mathbb{X}}(z)$.

7 Examples

Example 1. We consider the tight frame generated by the piecewise liner B-spline ("the hat-function") with the symbol

$m_0(\omega) = \dfrac{e^{-i\omega} + 2 + e^{i\omega}}{4}$, $H_0(z) = H_0(e^{i\omega}) = m_0(\omega)$. In the polyphase form

$$H_0(z) = \frac{1}{\sqrt{2}}(H_{0,1}(z^2) + z^{-1}H_{0,2}(z^2)),$$

where $H_{0,1}(z) = \dfrac{1}{\sqrt{2}}$, $H_{0,2}(z) = \dfrac{1+z}{2\sqrt{2}}$. The standard choice of framelets gives a polyphase matrix

$$\mathbb{M}(z) = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1+z}{2\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1+z}{2\sqrt{2}} \\ 0 & \dfrac{1-z}{2} \end{pmatrix}.$$

of a tight frame. The graphs of the scaling function and framelets are given on Figure 1.

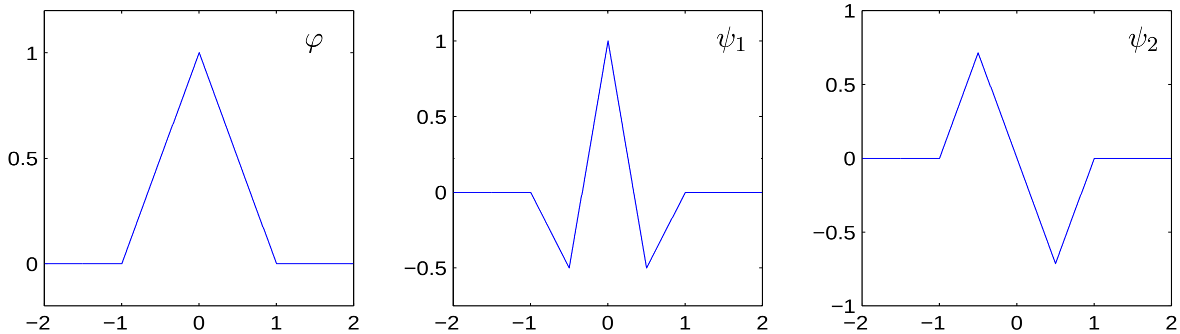

Fig. 1. Piecewise Linear Tight Frame

Let us extend the matrix $\mathbb{M}(z)$ up to the matrix

$$\mathbb{M}_e(z) := \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1+z}{2\sqrt{2}} & -\dfrac{1-z}{2\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1+z}{2\sqrt{2}} & \dfrac{1-z}{2\sqrt{2}} \\ 0 & \dfrac{1-z}{2} & -\dfrac{1+z}{2} \end{pmatrix}.$$

The matrix $\mathbb{M}_e(z)$ is paraunitary. Therefore, $\tilde{\mathbb{M}}(z) = \mathbb{M}(z)$, $\tilde{\mathbb{G}}_0(z) = \mathbb{G}_0(z) := -\frac{1-z}{2\sqrt{2}}$, where we use the notation from Section 4, turning down the vector symbols for $\mathbb{G}_0$ and $\tilde{\mathbb{G}}_0$.

We are going to describe polyphase matrices of minimal degrees defining dual filter banks providing two vanishing moments. We mention that even for $\tilde{\mathbb{M}}(z) = \mathbb{M}(z)$ the frame approximation order is equal to 2 (cf. [7]).

We need to solve the equation (16) for a various choice of $A_0(z)$. (Anti)symmetric solutions of the minimal degree is a special subject of our interest. For this reason, we consider a special choice $A_0(z) = A(1 - z^{-1})/2\sqrt{2}$, where A is an arbitrary constant. The norming factor $1/2\sqrt{2}$ is chosen for our convenience. Thus, we have the equation

$$2F_{0,0}(z) + F_{0,1}(z)(1 + z) = z - 1 + A \cdot (1 - z)^2(1 - z^{-1}). \tag{29}$$

First let us assume $A = 0$. Then we have $\mathbb{P}(z) \equiv 0$, i.e., Property 4 takes place. The simplest solution to (29) is $F_{0,0}(z) \equiv -1$, $F_{0,1}(z) \equiv 1$. This solution gives the

VCs component $\mathbb{X}_3(z) = \mathbb{X}_2(z) - \mathbb{X}_1(z)$. It is clear that the degree of the corrected dual modulation matrix is greater by one than the degree of the initial matrix $\tilde{\mathbf{M}}(z)$. However, the components of the corrected matrix are not symmetric any more. The simplest (anti)symmetric filter banks are generated by the solution $F_{0,0}(z) = -\frac{1}{2}(1-z)$, $F_{0,1}(z) \equiv 0$. Then the corrected filters can be represented by formulas

$$\begin{aligned}
\tilde{H}_0^{\#}(z) &= \tilde{H}_0(z) + \frac{2 - z^{-2} - z^2}{8} = \frac{3}{4} + \frac{z^{-1}+z}{4} - \frac{z^{-2}+z^2}{8}, \\
\tilde{H}_1^{\#}(z) &= \tilde{H}_1(z) - \frac{2 - z^{-2} - z^2}{8} = \frac{1}{4} - \frac{z^{-1}+z}{4} + \frac{z^{-2}+z^2}{8} = \frac{1}{8}(1-z)^2(z^{-2}+1), \\
\tilde{H}_2^{\#}(z) &= \tilde{H}_2(z) + \frac{-z^{-2}+z^2}{4\sqrt{2}} = \frac{z^{-1}-z}{2\sqrt{2}} + \frac{-z^{-2}+z^2}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}(1-z)^2(-z^{-2}+1),
\end{aligned}$$

The scaling function and the framelets are shown on Figure 2.

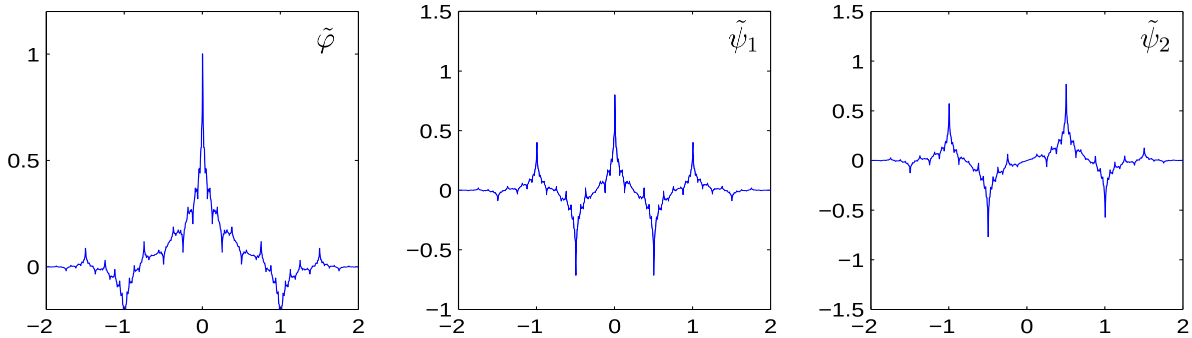

Fig. 2. Dual Almost Bi-orthogonal Frame

Obviously, the obtained pair of the scaling functions coincides with the famous pair generating 5/3 bi-orthogonal wavelet basis (cf. [2] [5]). This case is degenerate. The minor of $\tilde{\mathbb{M}}^{\#}(z)$ corresponding to framelets is equal to zero. We see that both $\tilde{H}_1^{\#}(z)$ and $\tilde{H}_2^{\#}(z)$ have the common factor $(1-z)^2$. At the same time, complementary factors depend on z^2. It means that wavelet spaces $\tilde{W}^1$ and $\tilde{W}^2$ coincide and, moreover, they coincide with the space W^* generated by a wavelet with a symbol $(1-z)^2$, i.e., the classical wavelet from the 5/3 pair. Note that the original piecewise linear tight frame does not possess that property.

If $A \neq 0$, the preserving symmetry solution to (29) of minimal degree has the form $F_{0,0} = -\frac{1}{2}(1-z) + 2(1-z)A$, $F_{0,1} = -(1-z)(1+z^{-1})A$. Thus, the corrected symbols of the dual scaling function and the framelets can be represented in the form

$$\begin{aligned}
\tilde{H}_0^A(z) &= \tilde{H}_0^{\#}(z) - A\frac{2-z^{-2}-z^2}{2} + A\frac{(2-z^{-2}-z^2)(z^{-1}+z)}{4}\\
&= \frac{3-4A}{4} + (1+A)\frac{z^{-1}+z}{4} - (1-4A)\frac{z^{-2}+z^2}{8} - A\frac{z^{-3}+z^3}{4},\\
\tilde{H}_1^A(z) &= \tilde{H}_1^{\#}(z) + A\frac{2-z^{-2}-z^2}{2} - A\frac{(2-z^{-2}-z^2)(z^{-1}+z)}{4}\\
&= \frac{1+4A}{4} - (1+A)\frac{z^{-1}+z}{4} + (1-4A)\frac{z^{-2}+z^2}{8} + A\frac{z^{-3}+z^3}{4},
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_2^A(z) &= \tilde{H}_2^{\#}(z) + A\frac{z^{-2}-z^2}{\sqrt{2}} + A\frac{(z-z^{-1})(2+z^{-2}+z^2)}{2\sqrt{2}}\\
&= (1-A)\frac{z^{-1}-z}{2\sqrt{2}} + (1-4A)\frac{-z^{-2}+z^2}{4\sqrt{2}} - A\frac{z^{-3}+z^3}{2\sqrt{2}},
\end{aligned}$$

Examples of dual wavelet frames for $A = 0.16$, 0.2 are given on Figures 3 and 4.

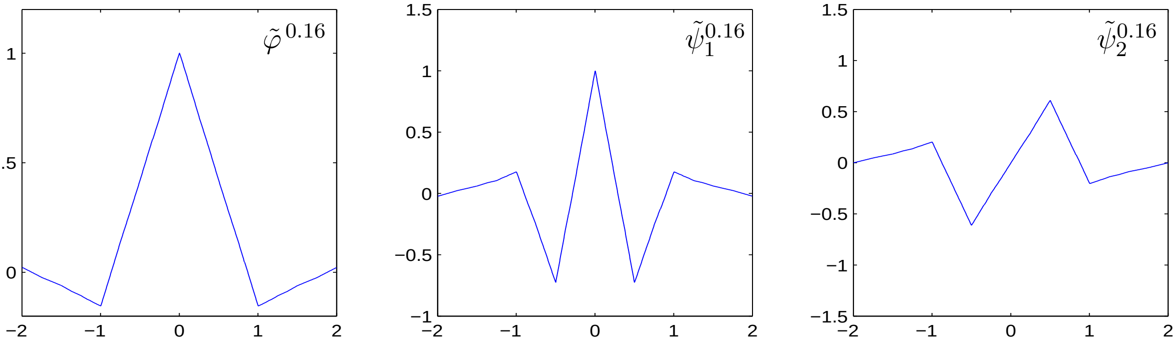

Fig. 3. Dual Frame with 2 Vanishing Moments, $A = 0.16$

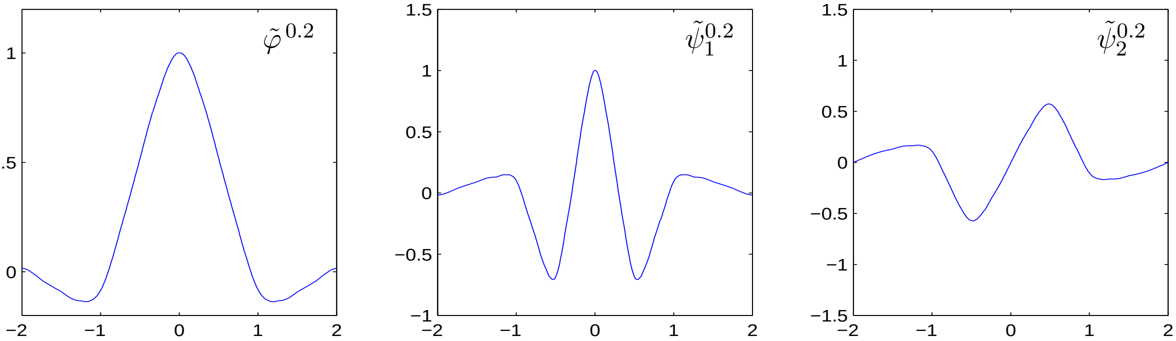

Fig. 4. Dual Frame with 2 Vanishing Moments, $A = 0.2$

Example 2.

Now we consider the tight frame generated by piecewise parabolic splines ([3]), [16]). The corresponding symbols generate the extended polyphase matrix

$$\mathbb{M}_e(z) = \sqrt{2}\begin{pmatrix} \frac{3+z}{8} & \frac{1+3z}{8} & \sqrt{6}\frac{-1+z}{8} \\ \frac{-3+z}{8} & \frac{-1+3z}{8} & \sqrt{6}\frac{1+z}{8} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}.$$

The graphs of framelets are given on Figure 5.

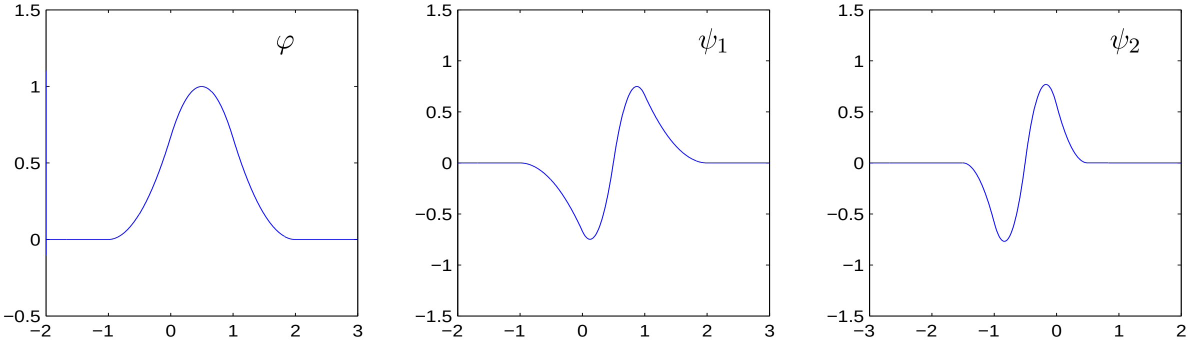

Fig. 5. Piecewise Parabolic Tight Frame.

The correcting coefficients can be computed from the equation

$$F_{0,0}(z)(3+z) + F_{0,1}(z)(1+3z) = \sqrt{6}(-1+z) + A\cdot(1-z)^3. \tag{30}$$

If $A = 0$, the simplest solution of (30) is $F_{0,0}(z) = -\frac{\sqrt{6}}{2}$, $F_{0,1}(z) = \frac{\sqrt{6}}{2}$. This correction leads to a symmetric scaling function (distribution) and a framelet with the symbols

$$\tilde{H}_0^{\#}(z) = \frac{3}{4}(1+z) - \frac{1}{4}(z^{-1}+z^2),$$
$$\tilde{H}_1^{\#}(z) = \frac{3}{4}(-1+z) + \frac{1}{4}(z^{-1}-z^2),$$

Unfortunately, the symbol $\tilde{H}_0^{\#}(z)$ above generates a scaling function with the non-integrable square (cf. [5], Section 8.3.4). Note that since the symbols $H_1(z)$ and $\tilde{H}_0^{\#}(z)$ are orthogonal, we have $\tilde{H}_2^{\#}(z) = 0$. Thus, $H_0(z)$, $H_1(z)$, $\tilde{H}_0^{\#}(z)$, $\tilde{H}_1^{\#}(z)$ form a quadruple of biorthogonal symbols.

To construct a genuine symmetric dual frame we need to allow the larger support of framelets. All such dual frames with the mask support of the length 8 can be represented in the form

$$\begin{aligned}
\tilde{H}_0^{\#}(z) &= \left(\frac{3}{4} - A(2-B)\right)(1+z) + \left(A(3+3B) - \frac{1}{4}\right)(z^{-1}+z^2) \\
&- A(1+3B)(z^{-2}+z^3) + AB(z^{-3}+z^4), \\
\tilde{H}_1^{\#}(z) &= \left(\frac{3}{4} - A(4+7B)\right)(-1+z) + \left(\frac{1}{4} - A(3+5B)\right)(z^{-1}-z^2) \\
&+ A(1+3B)(z^{-2}-z^3) + AB(z^{-3}-z^4), \\
\tilde{H}_2^{\#}(z) &= \frac{2A}{\sqrt{3}}\left((3+4B)(-z^{-1}+1) + (1+3B)(z^{-2}-z) - B(z^{-3}-z^2)\right),
\end{aligned}$$

where A and B are arbitrary parameters. Of course, as we mentioned above, not all pairs of the parameters lead to Bessel systems. We present a few relatively smooth examples.

We note that for $B = 0$ we have shorter supports. The choice $A = 0.08$ gives wavelets close to piecewise constant functions (Fig. 6). For $A = 0.09$ we have much smoother framelets (Fig. 7).

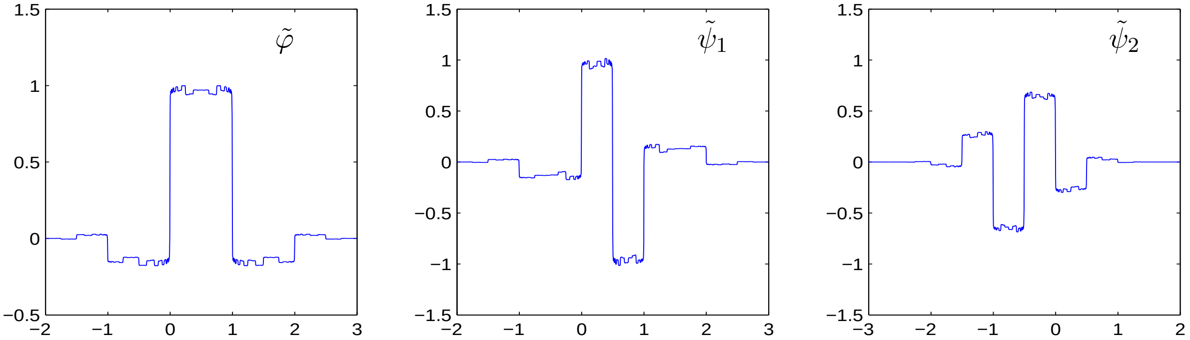

Fig. 6. Dual Frame with 3 Vanishing Moments, $A = 0.08$, $B = 0$.

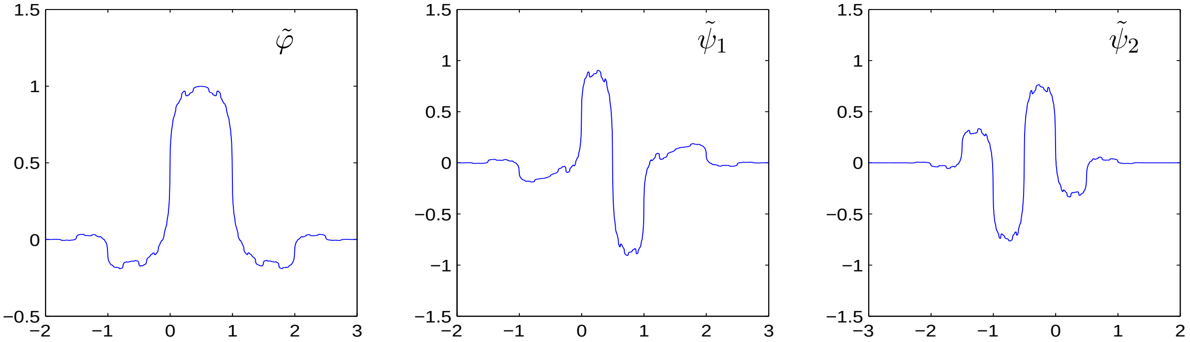

Fig. 7. Dual Frame with 3 Vanishing Moments, $A = 0.09$, $B = 0$.

In the case of masks of the length 8 (i.e., $B \neq 0$), we have more flexibility. In particular, for $A = 0.0896$, $B = -0.075$ we have framelets closer to piecewise constant functions (Figure 8) and for $A = 0.2$, $B = -0.275$ we have very smooth framelets (Figure 9) with 3 vanishing moments.

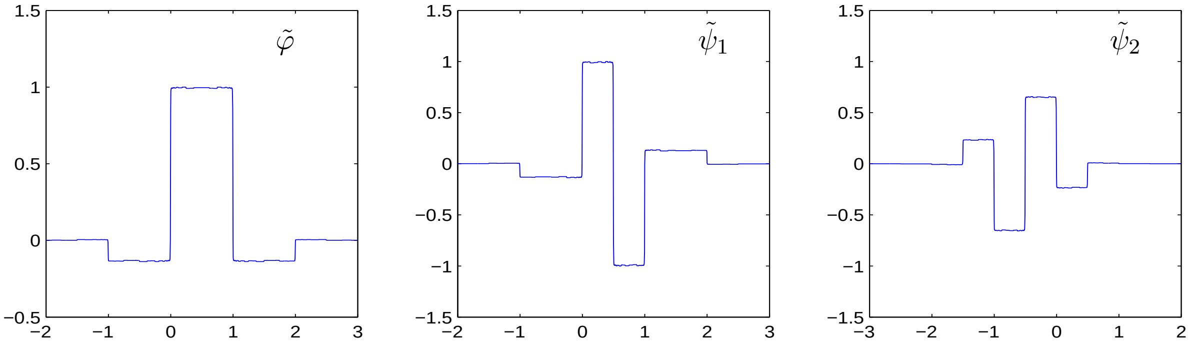

Fig. 8. Dual Frame with 3 Vanishing Moments, $A = 0.0896$, $B = -0.075$.

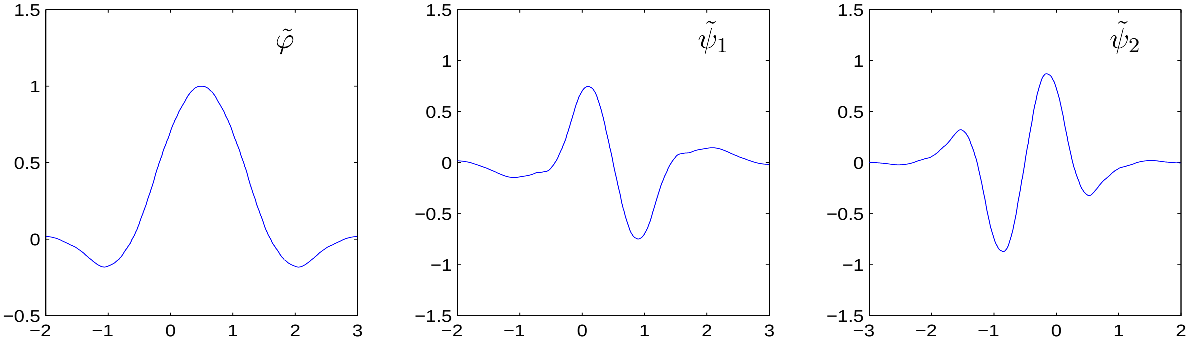

Fig. 9. Dual Frame with 3 Vanishing Moments, $A = 0.2$, $B = -0.275$.

References

[1] C. de Boor, R. A. DeVore, and A. Ron, On the construction of multivariate (pre)wavelets, Constr. Approx., **9** (1993), 123–166.

[2] A. Cohen, I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math., **45** (1992), 485–500

[3] C. K. Chui and W. He, Compactly supported tight frames associated with refinable functions, Appl. and Comp. Harm. Anal, **8** (2000), 293–319.

[4] C. K.Chui, W. He, and J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. and Comput. Harm. Anal., **13** (2002), 224–262.

[5] I. Daubechies, Ten Lectures on Wavelets, SIAM, 1992.

[6] I. Daubechies and B. Han, Pairs of dual wavelet frames from any two refinable functions, Constr. Approx., **20** (2004), 325–352.

[7] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. and Comput. Harm. Anal., **14** (2003), 1–40.

[8] R. J. Duffin and A. C. Shaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72** (1952), 147–158.

[9] Q. T. Jiang, Parameterizations of masks for tight affine frames with two symmetric/antisymmetric generators, Adv.Comput.Math., **18** (2003), 247–268

[10] M. J. Lai and J. Stöckler, Construction of compactly supported wavelet frames, preprint 2003.

[11] W. Lawton, S. L. Lee, and Z. Shen, An algorithm for matrix extension and wavelet construction, Math. of Computation, **65** (1996), 723–737.

[12] S. Mallat, Multiresolution approximation and wavelet orthonormal bases of $\mathbb{L}^2(\mathbb{R})$, Trans. Amer. Math. Soc., **315** (1989), 69 – 87.

[13] A. Petukhov, Explicit construction of framelets, Appl. and Comput. Harm. Anal., **11** (2001), 313 – 327.

[14] A. Petukhov, Explicit construction of framelets, Preprint, IMI of the University of South Carolina, #3, 2000 http://www.math.sc.edu/˜imip/00papers/0003.ps.

[15] A. Petukhov, Framelets with many vanishing moments, Approximation Theory X: Wavelets, Splines, and Applications, C. Chui, L. Shumaker, and J.Stöckler (eds.), Vanderbilt Univ. Press, 2002, 425 – 432.

[16] A. Petukhov, Symmetric framelets, Constr. Approx., **19** (2003), 309–328.

[17] A. Ron and Z. Shen, Affine systems in $\mathbb{L}_2(R^d)$: the analysis of the analysis operator, J. Functional Anal. **148** (1997), 408 – 447.

[18] A. Ron and Z. Shen, Affine systems in $\mathbb{L}_2(R^d)$ II: dual system, J. Fourier Anal. and Appl., **3** (1997), 617–637.

[19] I. Selesnik, Smooth wavelet tight frames with zero moments, Appl. and Comput. Harmon. Anal., **10** (2001), 163–181.

[20] I. W. Selesnick and A. F Abdelnour. Symmetric wavelet tight frames with two generators, Appl. and Comput. Harm. Anal., **17**(2004), 211–225.

[21] G. Strang and G. Fix, A Fourier analysis of the finite element variational methods, C.I.M.E. II, Ciclo 1971, Constructive Aspects of Functional Analysis (G. Geymonat, ed.), 1973, 793–840.