BIORTHOGONAL WAVELET BASES WITH RATIONAL MASKS AND THEIR APPLICATION[1]

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Abstract. General principles of a construction of biorthogonal wavelet bases, associated with recursive filters, are considered. Fast algorithms of expansion of functions in these bases, including, algorithms, which are based on factorizations of polyphase matrices, are presented. Results of numerical simulation of image compression algorithms, based on expansion of images in the offered basis, are given. They allow to make a conclusion about high effectiveness of the offered algorithms from a point of view of computational complexity as well as from a point of view of value of the compression factors.

§1. Introduction.

For last 10 years wavelet bases became a powerful tool for many theoretical problems of analysis and for applied problems, relating to signal processing. The theory of wavelet bases, connected to the bases whose functions have compact support, had the greatest development. The algorithms of expansion of functions in such bases are realized as a collection of discrete convolutions with numerical sequences, which has only finite number of non-zero coefficients. In a radio engineering linear operators, acting on infinite sequences and commutative with a shift (i.e., the operators of a convolution), is called *(digital) filters.* A sequence of numbers, which forms the convolution kernel is called *impulse response* of a filter. Thus, the expansion in wavelet bases with compact support is realized with the aid of filters those have finite impulse response (FIR-filters). Fourier transform of impulse response of a filter is called a frequency characteristic. Hence, the frequency characteristic of a filter with finite impulse response is a trigonometric polynomial. In many divisions of mathematics, for example in the theory approximation, rational functions are more flexible and effective tool than polynomials. Here we consider wavelet bases, such that expansion in them is realized with filters with a rational frequency characteristic. Certainly, appearing new types of wavelet bases gives new possibilities for applied problems of signal and image processing.

Despite the fact, that this types of filters has infinite impulse response (IIR-filters), there is effective numerical realization in the form of a composition of well known in a radio engineering, so-called, recursive filters. Moreover, the computional complexity for realization of filters with a rational frequency characteristic are proportional to a sum of degrees of the numerator and the denominator, that is comparable to complexity of a

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realization of expansions in bases with compact supports, which are proportional to degrees of the appropriate polynomials.

For the first time the similar bases were investigated systematic in the paper of C.Herley and M.Vetterly [1], where the case when the corresponding bases are orthogonal and form the multiresolution analysis (MRA) of the space $\mathbb{L}^2(\mathbb{R})$ was considered. As it is well-known (see [4]), except for the Haar bases, there do not exist orthogonal bases of compactly supported wavelets, the functions of which are even or odd with an accuracy to a shift of argument. However, for bases, generated by recursive IIR-filters, it does not so. In [1] even and odd bases, associated with recursive filters, were constructed. Orthogonality of bases allows to receive results, which guarantees convergence and a stability of expansion algorithms with considerably smaller efforts, than in the cases , when the orthogonality is lacking. However, to use an orthogonality it is necessary, at least, to have a scalar product in an investigated space. But even if the scalar product is introduced, for an effective uze of orthogonal bases in applied problems it is required, that the Euclidean distance should be natural for the given problem. Unfortunately, not always it is so. In particular, in image processing biorthogonal pairs of bases are more effective than orthogonal wavelet bases. For example, for image compression, so-called, 9/7-wavelets with compact supports, constructed in [2], are most frequently used. As for the bases from [1], which combines an orthogonality and symmetry, in view of severe constraints their adaptation to a concrete problem is rather problematic. Outcomes of numerical simulation of image compression algorithms with use of such bases have shown, that their effectiveness is rather law at least for small degree of recursive filters. On the contrary, the tool of biorthogonal pairs of bases, generated by recursive filters, signicantly exceeds in flexibility the tool of biorthogonal pairs of bases with compact support.

About a possibility of a construction of biorthogonal pairs of filter banks with a rational frequency characteristic it is written, for example, in the book by G.Strang and T.Nguyen [7]. There a few examples of such bases were obtained as a collateral yield in a construction of "good" bases with compact support. Here we build just the biorthogonal theory of bases, associated with recursive filters. As we shall see, the use of such approach leads to essential economy of computing expenditures in some problems.

Primarily the theory of wavelet bases was developed independently of the theory of subband coding which arose a little bit earlier in a radio engineering . However, soon their almost complete identity has become clear. In any case, it may be stated for most practically important cases. In §2 we stop on their connection. Unfortunately, in a general case a question when a pair of filters, defining algorithm of subband coding, generates a pair of biorthogonal bases, remains open until now. At the moment this problem for the biorthogonal case, when the bases associated with subband coding belong to $\mathbb{L}^2(\mathbb{R})$ and have the compact support, was investigated the most completely in the works [2], [5], Therefore in §3 we temporarily disregard a problem on the correspondence with bases of wavelets and, remaining on a platform of subband coding, we shall construct biorthogonal pairs of filters with a rational frequency characteristic.

In §4 algorithm of subband coding and reconstruction of signals, based on, so-called, lifting scheme, is constructed for IIR filters. This scheme for FIR filters (see [3], [6]), gives asymptotically double gain in computational complexity as compared with a direct realization of convolution algorithms of subband coding.

In §5 the computational complexity of the proposed algorithms is calculated and comparison with the computational cost of a direct realization of algorithms of wavelet expansion is given.

In §6 we give results of numerical simulation of image compression algorithms, which allows to compare effectiveness of the new algorithms with the known ones in respect to compression factor as well as with respect to computational cost. Besides, we give arguments, which show that the filters, used in the simulation, associate with some pairs of biorthogonal wavelet bases of the space $\mathbb{L}^2(\mathbb{R})$.

§2. The multiresolution analysis of quasi-Banach spaces of distributions.

First we extend the definition of the Multiresolution analysis to quasi-Banach space differing from $\mathbb{L}^2(\mathbb{R})$. We recall, that quasi-Banach spaces are generated by a quasi-norm, i.e., functional, whose difference from a norm consists in the fact that an triangle inequality for it is valid only in the form $|f+g| \le C(|f|+|g|)$, where $C>1$. Such relaxation of norm properties allows to include in general consideration many useful spaces such as the Hardy spaces. The complete construction of the theory of Multiresolution for spaces of periodic distributions can be found in our paper [8] (see also [9], [11]), the case of periodic wavelets on a discrete grid is considered in [10]. The given below definitions of wavelets in quasi-Banach spaces of non-periodic distributions in no way are the complete theory. It is only a sketch of the possible scheme of construction of the wavelet theory, which is necessary here to us only to show a place of those special bases, construction of which we deal with.

We denote by S' and S respectively the space of tempered distributions and the space of trial functions

Definition 1. Collection of closed linear subspaces $\{V^j\}_{j\in\mathbb{Z}}$ of quasi-Banach spaces X, $S\subset X\subset S'$ is called *the Multiresolution analysis* (MRA) of the space X, if it satisfies the following conditions:

a) $\dots\subset V^{-2}\subset V^{-1}\subset V^0\subset V^1\subset\dots$;

b) $\overline{\cup V^j}=X$; c) $f(x)\in V^j \Leftrightarrow f(2x)\in V^{j+1}$;

d) there is distribution $\varphi\in V^0$, called *a scaling function*, such that the linear span of a system $\varphi(\bullet-k)$, $k\in\mathbb{Z}$, is dense in V^0.

According to item b) of Definition 1 function $\varphi(x)$ can be approximated by linear combinations of functions $\{\varphi(2x-k)\}$. We are interested in only a case, when MRA is formed by real spaces and *scaling equation*

$$\varphi(x)=\sqrt{2}\sum_{n\in\mathbb{Z}}h_n\varphi(2x-n), \tag{2.1}$$

where h_n are Fourier coefficients of a rational function

$$\sqrt{2}m_0(\omega)=\frac{\mathcal{P}(e^{i\omega})}{\mathcal{Q}(e^{i\omega})}=\sum_{n\in\mathbb{Z}}h_ne^{-in\omega}, \tag{2.2}$$

takes place. It is clear, that h_n is a real sequence. The functions $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ in (2.2) are Laurent polynomials. We note that we use a traditional factor $\sqrt{2}$, since if the functions $\{\varphi(x-n)\}$ form an orthonormal basis of the space $V^0 \in \mathbb{L}^2(\mathbb{R})$, the functions $\{\sqrt{2}\varphi(2x-n)\}$ constitute orthonormal basis of V^1. After application Fourier transform to (2.1) this relation can be rewritten in the form

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right). \tag{2.3}$$

If we assume, that $\hat{\varphi}$ is a continuous function and $\hat{\varphi}(0) \neq 0$, it is clear that $m_0(0) = 1$. Thus, applying (2.3) and assuming that $\hat{\varphi}(0) = (2\pi)^{-1/2}$, it can be proved, that

$$\hat{\varphi}(\omega) = (2\pi)^{-1/2}\prod_{k=1}^{\infty} m_0\left(\frac{\omega}{2^k}\right),$$

where the product in the right side converges uniformly on compact sets.

Let $\{V^j\}$ be MRA of a quasi-Banach space X. We denote by Z° the completion of the space S with respect to a quasi-norm of the space Z. In what follows we assume $X = X^\circ$. To introduce the concept of wavelet decomposition of a space X we need MRA $\{\tilde{V}^j\}$ of the dual space $Y = X^{*\circ}$. Under this assumption the spaces

$$W^j = \{f \in V^{j+1} | \langle f, g\rangle = 0, \text{ for any } g \in \tilde{V}^j\}$$

are called wavelet spaces. In analogous way the dual wavelet spaces

$$\tilde{W}^j = \{g \in \tilde{V}^{j+1} | \langle f, g\rangle = 0, \text{ for any } f \in V^j\}$$

are defined. Here we do not discuss any additional conditions for the correct definition of wavelet spaces. However, we mark that the MRA $\{\tilde{V}^j\}$ of the dual space should be chosen so that $V^j \cap W^j = 0$. Then we have

$$V^{j+1} = W^j \oplus V^j = \left(\bigoplus_{i=0}^{k} W^{j-i}\right) \bigoplus V^{j-k} = \left(\bigoplus_{i=0}^{+\infty} W^{j-i}\right) \bigoplus \left(\bigcap_{i\in\mathbb{Z}} V^i\right),$$

where the symbol $\oplus$ means a direct sum of spaces. The component of intersection of spaces V^i in the last formula could not appear in the case of MRA of $\mathbb{L}^2(\mathbb{R})$, because such intersection is always empty for $\mathbb{L}^2(\mathbb{R})$. As for a general case it is not. Easily to construct examples of MRA which have in intersection either polynomials of a fixed degree or Dirac's δ-function or its generalized derivatives.

Let us comment this construction. On the one hand, it is clear, that the classical definition MRA of the space $\mathbb{L}^2(\mathbb{R})$ requires generalization on other function space, in which scalar product is not defined. However, on the first sight it seems that Definition 1 envelop too wide class of spaces and for applications it is possible to be limited by classical function spaces, not passing to distributions. It would be possible to agree with it, if the spaces V^j were used only as approximations of the space X. However, in the definition of wavelet spaces two MRAs of mutually dual spaces participate. One of them serves for an approximation, and the second one does for the definition of a projection. And since, the narrower the initial space, the wider the dual space, then one may occur that it is not enough functionals, determined by usual functions, for the determination of a projection to a space of smooth functions. For example, at the definition of interpolational wavelets, the examples of which will be given below, the projectors are set by shifts of the Dirac δ-function which is the most simple example of a distribution. This example is a quite natural corollary that a dual space for the space of continuous functions is the space of finite Borel measures. We note, that the space of finite Borel measures is not separable. Therefore, if $X = \mathbb{C}(\mathbb{R})$ is the space of continuous functions, vanishing at infinity, the space of finite Borel measures $\mathbb{C}^*(\mathbb{R})$ does not coincide with $\mathbb{C}^{*\circ}(\mathbb{R})$. However, if we take little narrower space (for example, determined by any module of a continuity) as above X, we obtain that the space of Borel measures is a separable subspace of the space X^*.

We recall, that two function collections $\{f_n\} \subset X$ and $\{g_n\} \subset X^*$ is defined to be biorthogonal, if relations

$$\langle f_k, g_m \rangle = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

hold. Let us assume, there is a basis $\{\tilde{\varphi}(x-n)\}$ of the space $\tilde{V}^0$ biorthogonal to the basis $\{\varphi(x-n)\}$, which is generated by a rational function

$$\tilde{m}_0(\omega) = \frac{1}{\sqrt{2}} \frac{\tilde{\mathcal{P}}(e^{i\omega})}{\tilde{\mathcal{Q}}(e^{i\omega})} = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \tilde{h}_n e^{-in\omega}.$$

In view of a condition of a biorthogonality easily to obtain, that m_0 and $\tilde{m}_0$ satisfies the condition

$$m_0(\omega)\overline{\tilde{m}_0(\omega)} + m_0(\omega+\pi)\overline{\tilde{m}_0(\omega+\pi)} = 1.$$

We note since the convergence of infinite products, determining functions $\hat{\varphi}$ and $\hat{\tilde{\varphi}}$, implies equalities $m(0) = \tilde{m}(0) = 1$, we obtain, that from a biorthogonolity condition we have $m(\pi)\overline{\tilde{m}(\pi)} = 0$. It means, that at least one of the factors $m(\pi)$ and $\tilde{m}(\pi)$ is equal to zero.

From the definition W^j we obtain, that Fourier transform of function ψ, generating basis $\{\psi(x-n)\}$ of the space W^0, can be calculated by the formula

$$\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}),$$

where m_1 satisfies the equation

$$m_1(\omega)\overline{\tilde{m}_0(\omega)} + m_1(\omega+\pi)\overline{\tilde{m}_0(\omega+\pi)} = 0.$$

It follows from here

$$m_1(\omega) = \alpha(\omega)e^{-i\omega}\overline{\tilde{m}_0(\omega+\pi)},$$

where α is some π-periodic distribution. In what follows, following traditions in notation rather than a matter of fact, for a definitness we suppose $\alpha(\omega) \equiv 1$. In analogous way we introduce a function

$$\tilde{m}_1(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)},$$

which generates a basis in the dual wavelet space $\tilde{W}^0$. Thus, the matrix relation

$$\begin{pmatrix} m_0(\omega) & m_0(\omega+\pi) \\ m_1(\omega) & m_1(\omega+\pi) \end{pmatrix} \cdot \begin{pmatrix} \tilde{m}_0(\omega) & \tilde{m}_0(\omega+\pi) \\ \tilde{m}_1(\omega) & \tilde{m}_1(\omega+\pi) \end{pmatrix}^* = I, \tag{2.4}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, holds.

Thus, the equation (2.4) gives a necessary condition for the functions m_0 and $\tilde{m}_0$ to generate continuous scaling functions φ, $\tilde{\varphi}$, which determine a pair of dual MRA. As for sufficient conditions, at the moment this problem is far from a complete solution, though in most practically important cases it is possible to show, that the concrete solution of the equation (2.4) really generates a biorthogonal pair of bases of certain dual MRA.

In case of multiplying of the function $m_1(\omega)$ by π-periodic function $\alpha(\omega)$ the equality (2.4) is broken. For its restoration the function $\tilde{m}_1(\omega)$ should be multiplied by $1/\overline{\alpha(\omega)}$, by receiving, thus,

$$\begin{pmatrix} m_0(\omega) & m_0(\omega+\pi) \\ m_1(\omega)\alpha(\omega) & m_1(\omega+\pi)\alpha(\omega) \end{pmatrix} \cdot \begin{pmatrix} \tilde{m}_0(\omega) & \tilde{m}_0(\omega+\pi) \\ \tilde{m}_1(\omega)/\overline{\alpha(\omega)} & \tilde{m}_1(\omega+\pi)/\overline{\alpha(\omega)} \end{pmatrix}^* = I. \tag{2.4$'$}$$

Certainly, the choice of $\alpha(\omega)$ does not influence either a pair of dual MRA, which are determined by functions m_0, $\tilde{m}_0$, or on a choice of bases in the appropriate spaces V^j and $\tilde{V}^j$ or on a choice of wavelet spaces W^j and $\tilde{W}^j$. However, variating the function α gives an additional degree of freedom at a choice of wavelet bases to correspond in the greatest degree to a concrete applied problem. In what follows we shall see, that the basis, the most appropriate for image compression, uses this degree of freedom. The reason why this method was unclaimed in the current literature has an objective character. The fact is, that considering biorthogonal wavelet bases, associated with FIR-filters, we are forced to search polynomial for solutions of the equation (2.4). If m_0 and $\tilde{m}_0$ are polynomials, the traditional choice of functions m_1 and $\tilde{m}_1$ is also polynomial. If we shall multiply function m_1 by a π-periodic polynomial α, we are obliged to divide $\tilde{m}_1$ by $\overline{\alpha}$. Let us assume there is a π-periodic polynomial α such that the second matrix in (2.4$'$) is polynomial. In a degenerate case, when α is a monomial, i.e., accurate to multiplication by a constant we have $\alpha(\omega) = e^{i2k\omega}$, $k \in \mathbb{Z}$, such choice leads to a shift of the elements of bases in W^{-1} and $\tilde{W}^{-1}$ by 2, i.e., in fact only to a renumbering of the basis elements. If α is a nondegenerate polynomial, in order to the function $\tilde{m}_1/\overline{\alpha}$ is a polynomial it is required the coincidence of zeros of $\tilde{m}_1$ and a polynomial $\overline{\alpha}$. But then in view of π-periodicity of the polynomial α the polynomial $\tilde{m}_1(\omega+\pi)$ also has zero at those points, where $\alpha(\omega) = 0$. Thus, there is the point ω_0 at which the vector $(\tilde{m}_1(\omega_0), \tilde{m}_1(\omega_0+\pi))$ vanishes and the corresponding matrix in (2.4) becomes degenerate. It leads to violation of equality (2.4) not only at this point, but because of continuity of components in some its neighborhood. In the case, when rational solutions of the equation (2.4) are allowed, the zero of a determinant of one of the matrices in (2.4$'$) can put in the correspondence to poles of other determinant. Thus, though the traditional choice of functions m_1 and $\tilde{m}_1$ is quite justified, but it is unique possible only for polynomial solutions.

The dual MRA $\{V^j\}$ and $\{\tilde{V}^j\}$ in a certain sense are equivalent and can be considered as an imbedded sequence of spaces in the signal space as well as for the definition of projection operators, which in turn allow to determine spaces of wavelets. In what follows we stipulate to suppose, that the signals are projected to the spaces $\tilde{V}^j$ with the help of the spaces V^j. Such agreement is accepted in overwhelming majority publications.

We introduce the functions

$$h(z) = \sum_{n\in\mathbb{Z}} h_n e^{-in\omega} = \sqrt{2}m_0(\omega), \quad \tilde{h}(z) = \sum_{n\in\mathbb{Z}} \tilde{h}_n e^{-in\omega} = \sqrt{2}\tilde{m}_0(\omega), \tag{2.5}$$

where $z = e^{i\omega}$, and

$$g(z) = z^{-1}\tilde{h}(-z^{-1}), \quad \tilde{g}(z) = z^{-1}h(-z^{-1}). \tag{2.6}$$

Functions h and $\tilde{h}$ we call *a biorthogonal pair of filters.* Now equality (2.4) in case of real $\{h_n\}$ and $\{\tilde{h}_n\}$ can be rewritten in the form

$$M(z)\tilde{M}^t(z^{-1}) = 2I, \tag{2.7}$$

where the symbol t means a transposition of a matrix, and

$$M(z) = \begin{pmatrix} h(z) & h(-z) \\ g(z) & g(-z) \end{pmatrix}, \quad \tilde{M}(z) = \begin{pmatrix} \tilde{h}(z) & \tilde{h}(-z) \\ \tilde{g}(z) & \tilde{g}(-z) \end{pmatrix}$$

are, so-called, *modulation matrices*, whose components in our case are rational functions. From (2.6) and (2.7) it is easily to obtain that $\det|M(z)| = \det|\tilde{M}(z)| = -2z^{-1}$.

Now we explain a sense and assignment of modulation matrixes M and $\tilde{M}$. Let $\tilde{v}^0(x) = \sum_n c_n^0\tilde{\varphi}(x-n) \in \tilde{V}^0$ and it is required to find projections $\tilde{v}^{-1}(x) = \sum_n c_n^{-1}\tilde{\varphi}(x/2-n)$ and $\tilde{w}^{-1}(x) = \sum_n d_n^{-1}\tilde{\psi}(x/2-n)$ of distribution $\tilde{v}^0$ to the spaces $\tilde{V}^{-1}$ and $\tilde{W}^{-1}$. Using the refinement equation (2.1), we obtain

$$c_m^{-1} = \langle\tilde{v}^0(x), \varphi(x/2-m)\rangle = \sqrt{2}\langle\sum_{n\in\mathbb{Z}} c_n^0\tilde{\varphi}(x-n), \sum_{k\in\mathbb{Z}} h_k\varphi(x-k-2m)\rangle = \sqrt{2}\sum_{n\in\mathbb{Z}} c_n^0 h_{n-2m}.$$

In terms of z-transform of sequences the last equality can be written in the form $c^{-1}(z^{-2}) = \frac{1}{\sqrt{2}}(h(z)c^0(z^{-1}) + h(-z)c^0(-z^{-1}))$, where $c^0(z) = \sum_n c_n^0 z^{-n}$, $c^{-1}(z) = \sum_n c_n^{-1}z^{-n}$. We note, that here and in what follows the convergence of series is guaranteed by exponential decrease of sequences $\{h_n\}$ and $\{\tilde{h}_n\}$. The equality $d_m^{-1} = \sqrt{2}\sum_{n\in\mathbb{Z}} c_n^0 g_{n-2m}$ is obtained in analogous way. Hence, $d^{-1}(z^{-2}) = \frac{1}{\sqrt{2}}(g(z)c^0(z^{-1}) + g(-z)c^0(-z^{-1}))$

Thus, after passage to z-transform the matrix M becomes a matrix of passage from the space $\tilde{V}^0$ to a direct sum of the spaces $\tilde{V}^{-1}$ and $\tilde{W}^{-1}$, which is determined by the formula

$$\begin{pmatrix} c^{-1}(z^{-2}) \\ d^{-1}(z^{-2}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} h(z) & h(-z) \\ g(z) & g(-z) \end{pmatrix}\begin{pmatrix} c^0(z^{-1}) \\ c^0(-z^{-1}) \end{pmatrix}. \tag{2.8}$$

For reconstruction of coefficients of expansion of function in basis of the space $\tilde{V}^0$ in terms of its projections on the spaces $\tilde{V}^{-1}$ and $\tilde{W}^{-1}$ the formula

$$\begin{pmatrix} c^0(z^{-1}) \\ c^0(-z^{-1}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{h}(z^{-1}) & \tilde{g}(z^{-1}) \\ \tilde{h}(-z^{-1}) & \tilde{g}(-z^{-1}) \end{pmatrix}\begin{pmatrix} c^{-1}(z^{-2}) \\ d^{-1}(z^{-2}) \end{pmatrix}$$

holds. Certainly, the last formulae are convenient for analytical representation of operation of expansion in wavelets and reconstruction by projections. However, they contain many repeated operations and consequently they cannot be recommended for an immediate evaluation of coefficients.

In some sense an alternative method of representation of operation of expansion of the space $\tilde{V}^0$ in a direct sum of spaces $\tilde{V}^{-1}$ and $\tilde{W}^{-1}$ is its realization with the help of, so-called, polyphase matrix. For an arbitrary formal Laurent series $f(z) = \sum_n f_n z^{-n}$ we introduce the notation $f_e(z) = \sum_n f_{2n} z^{-n}$ and $f_o(z) = \sum_n f_{2n+1} z^{-n}$. Then the representation $f(z) = f_e(z^2) + z^{-1} f_o(z^2)$ is valid. Using this representation, by replacing them all components of the formula (2.8), after simple transformations of a right side we obtain

$$\begin{pmatrix} c^{-1}(z^{-1}) \\ d^{-1}(z^{-1}) \end{pmatrix} = \begin{pmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{pmatrix} \begin{pmatrix} c_e^0(z^{-1}) \\ c_o^0(z^{-1}) \end{pmatrix}. \tag{2.9}$$

Just this formula is an initial point for a construction of fast algorithms of decomposition and reconstruction of signals, joined by a common title *the lifting scheme*. A matrix in expression (2.9) is called *polyphase*. We denote it by $P(z)$. Polyphase matrix of a dual MRA is denoted by $\tilde{P}(z)$. It is easily to verify that

$$P(z)\tilde{P}^t(z^{-1}) = I. \tag{2.10}$$

Besides, since for polyphase matrices P and $\tilde{P}$ representations

$$P(z^2) = \frac{1}{2} M(z) \begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix}, \quad \tilde{P}(z^2) = \frac{1}{2} \tilde{M}(z) \begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix}$$

are valid, from the equalities $\det M = \det \tilde{M} = -2z^{-1}$ we have $\det P = \det \tilde{P} = 1$.

§3. Examples.

Equation (2.7) has many rational solutions. Though from further reasoning easily to see, in what way all possible solutions of such type can be obtained, here we are interested in only some of them, which have a minimal order of a numerator and denominator and correspond to wavelet bases, whose scaling functions are symmetric with respect to whole or half-integer points. Thus, we show, that after an appropriate shift of argument the corresponding wavelet bases are odd functions in the first case, and even ones in the second case. Therefore for a short we speak, that an odd or even case takes place respectively.

First we consider the even case. We note, that the multiplying the function $h(z)$ by z^k means passage from a scaling function $\varphi(z)$ to the scaling function $\varphi(z+k)$, i.e. passage to one of functions of a basis system of shifts. Easily to see, that, multiplying h, if it is necessary, by an appropriate integer degree of a variable z, we obtain, that in (2.5) $h_n = h_{-n}$ for all $n = 1, 2, \ldots$. The function $h(z)$ together with each pole at $z = z_0$ has a pole at $z = z_0^{-1}$. Thus, rational function h is representable as a ratio of two polynomials $S(z) = \sum_{n=-l}^{l} s_n z^{-n}$ and $Q(z) = \sum_{n=-m}^{m} q_n z^{-n}$ with symmetric sequences of coefficients $s_{-n} = s_n$ and $q_{-n} = q_n$, hence, invariant with respect to a change of variable $z \to z^{-1}$.

In the odd case, by multiplying h, if it is necessary, by an appropriate integer degree of a variable z, we obtain $h_n = h_{1-n}$ for all $n = 1, 2, \dots$. It is clear, that the function $f(z) = (1+z)h(z)$ has a symmetric sequence of Laurent coefficients. The sequence of Laurent coefficients of function $f(z)/((1+z)(1+z^{-1}))$ is symmetric, and this function is rational and representable in the form $S(z)/Q(z)$, where S and Q are Laurent polynomials with symmetric sequences of factors, invariant with respect to a change $z \mapsto z^{-1}$. Thus,

$$h(z) = \frac{(1+z^{-1})S(z)}{Q(z)}.$$

We show, that the filters h and $\tilde{h}$ can have distinguish evenness, i.e., when one function corresponds to the even case, and other does to the odd one, only for a pair of filters $h(z) = (1+z^{-1})/\sqrt{2}$ and $\tilde{h}(z) = \sqrt{2}$. In spite of the fact that the given pair of filters leads to two MRA, generated accordingly the Haar basis and basis, consisting of shifts δ-function, in view of a discontinuity the Haar functions these bases is not form a biorthogonal pair.

From (2.7) for an arbitrary pair h and $\tilde{h}$ we obtain

$$h(z)\tilde{h}(z^{-1}) + h(-z)\tilde{h}(-z^{-1}) = 2$$

or, substituting a representation $h(z)$ and $\tilde{h}(z)$ in the form of rational functions and reducing if necessary the common factors of a numerator and denominator, we obtain expression of the form

$$\frac{A(z)}{B(z)} + \frac{A(-z)}{B(-z)} = 2, \tag{3.1}$$

where A, B are Laurent polynomials of z. The addends in (3.1) should have common poles with consideration for a multiplicity. In particular, it follows from this that the polynomials $B(z)$ and $B(-z)$ have the same zeros. Thus, we obtain that in a polynomial B, at least after a multiplying, if it is necessary, by an appropriate factor z^k, only even degrees are present, i.e., $B(z) = \mathcal{B}(z^2)$, where $\mathcal{B}(z)$ is some Laurent polynomial. Thus, equation (3.1) can be rewritten in the form

$$A(z) + A(-z) = 2\mathcal{B}(z^2). \tag{3.2}$$

The last equation is a simple sourse of all possible MRA. Indeed, it is easily to see from relation (3.2), that the Laurent polynomial A can be chosen arbitrary within certain limits. As above $A_e(z) = \mathcal{B}(z)$ an arbitrary Laurent polynomial, that has not the roots on a unit circle $|z| = 1$, can be taken. Then, since $A(-1) = m_0(\pi)\overline{\tilde{m}_0(\pi)} = 0$, the polynomial A_o has to satisfy the equality $A_o(1) = A_e(1)$, i.e., sums of coefficients of polynomials A_o and A_e has to coincide. All possible biorthogonal filters h and $\tilde{h}$ come out a factorization $h(z)\tilde{h}(z^{-1}) = A(z)/B(z)$, where $h(1) = \tilde{h}(1) = \sqrt{2}$. Certainly, not all factorizations lead to biorthogonal pairs of wavelet bases, however, all such pairs of bases are generated by these factorizations.

Now we assume, that one of the functions h and $\tilde{h}$ generates an even MRA, and other does an odd one. Then the representation $A(z) = (1+z^{-1})\mathcal{A}(z)$, where $\mathcal{A}(z)$ is

Laurent polynomial with a symmetric sequence of coefficients, is valid. Substituting this representation in (3.2), we have

$$(1+z^{-1})\mathcal{A}(z)+(1-z^{-1})\mathcal{A}(-z)=2\mathcal{B}(z^2), \tag{3.3}$$

where without loss of generality it is possible to suppose that the polynomials $\mathcal{A}$ and $\mathcal{B}$ have not common zeros. The right member of equality (3.3) is invariant with respect to a change of variable $z\to z^{-1}$, since it is a product of polynomials, possessing this property. Hence, his left part is invariant with respect to such replacement. Consequently,

$$(1+z^{-1})\mathcal{A}(z)+(1-z^{-1})\mathcal{A}(-z)=(1+z)\mathcal{A}(z^{-1})+(1-z)\mathcal{A}(-z^{-1}).$$

And because of $\mathcal{A}(z)=\mathcal{A}(z^{-1})$, the last equality is transformed to the form

$$(z^{-1}-z)(\mathcal{A}(z)-\mathcal{A}(-z))=0.$$

It means a lack of odd degrees of the polynomial $\mathcal{A}$. But then, returning to (3.3), we see that $\mathcal{A}(z)=\mathcal{B}(z^2)$. So, in view of a lack of common divisors of these polynomials we have $\mathcal{A}\equiv 1$. A biorthogonal pair of filters $h(z)=(1+z^{-1})/\sqrt{2}$ and $\tilde{h}(z)=\sqrt{2}$ corresponds to the given case. Thus, h and $\tilde{h}$ can have distinguish evenness only for trivial case mentioned above, which does not generate biorthogonal wavelet bases.

So, we consider even and odd biorthogonal pairs with small orders of a numerator and denominator of functions h and $\tilde{h}$. We start from the odd case. The simplest right term in (3.2) can be taken as $\mathcal{B}\equiv 1$. At the first sight, this case can seem degenerate and corresponding to biorthogonal compactly supported bases. Nevertheless, one could appear, that the zero of one of the functions $h(z)$ and $\tilde{h}(z^{-1})$ has appeared a pole of other function. Then the corresponding factors in a product $h(z)\tilde{h}(z^{-1})$ are reduced and consequently are not taken in consideration in (3.1). The specific features of MRA forces us to consider not all solutions of equation (3.2). Principal of restrictions consists of the fact that since for the odd case $h(-1)=\tilde{h}(-1)=0$, then $A(z)=(1+z)(1+z^{-1})\mathcal{A}_1(z)$. Thus, the polynomial $\mathcal{A}_1(z)$ of a minimal possible degree is equal to a constant, $\mathcal{A}_1(z)\equiv 1/2$. And taking into account, that the rational functions $h(z)/(1+z^{-1})$ and $\tilde{h}(z)/(1+z^{-1})$ have symmetric sequences of Laurent factors, we obtain that a biorthogonal rational pair of filters of the minimal possible degree looks like

$$h(z)=\alpha\frac{1+z^{-1}}{\sqrt{2}(z^{-1}+a+z)},\qquad \tilde{h}(z)=\frac{1}{\sqrt{2}\alpha}(1+z^{-1})(z^{-1}+a+z),$$

where $\alpha=2+a$ is a factor which ensures a normalization $h(1)=\tilde{h}(1)=\sqrt{2}$.

For the wavelet theory only the case $|a|>2$ is a subject of interest. Indeed, the polynomial $1+az+z^2$ has the roots $z_{1,2}=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-1}$. For $|a|\le 2$ the both roots are equal to 1 in modulus, i.e., $z_{1,2}=e^{\pm i\omega_0}$. Hence, in this case the function $m_0(\omega)$ has singularities at points $\omega=\pm\omega_0$ and not only is nonsummable, but also is not even a distribution.

The considered example shows, in what way MRA (in our case MRA, generated by the Haar bases) may be varied by means of addition a factor $z^{-1}+a+z$ in a denominator of one of the filters h, $\tilde{h}$ and the same factor in a numerator of the other. From the qualitative point of view this method leads to "swapping of a smoothness" between one basis of a biorthogonal pair and its dual basis with preservation of a summarized smoothness of bases. Such method can be used for a variation of any biorthogonal pair and it should be taken into account, though in what follows we shall not mention about it.

Now we consider a less trivial case, when a polynomial $\mathcal{B}$ is nondegenerate. In the simplest case $\mathcal{B}(z) = z^{-1} + a + z$. Here for the above mentioned reasons we also have $|a| > 2$. Hence, we obtain, that the polynomial $\mathcal{A}_1$ of a minimal possible degree, satisfying (3.2), looks like $\mathcal{A}_1(z) = z^{-1} + (\frac{a}{2} - 1) + z$. Thus, there are two biorthogonal pairs:

$$h(z) = \sqrt{2}\frac{(1+z^{-1})(z^{-1}+(\frac{a}{2}-1)+z)}{z^{-2}+a+z^{2}}, \qquad \tilde{h}(z) = \frac{1+z^{-1}}{\sqrt{2}}$$

and

$$h(z) = \alpha\frac{(1+z^{-1})}{z^{-2}+a+z^{2}}, \qquad \tilde{h}(z) = \frac{(1+z^{-1})(z^{-1}+(\frac{a}{2}-1)+z)}{\alpha}, \text{where } \alpha = \frac{a+2}{\sqrt{2}},$$

corresponding to the given choice of the polynomial $\mathcal{B}$.

We consider the simplest solutions in the even case. Since in (3.1) $A(-1) = 0$, one of polynomials h or $\tilde{h}$ has zero at the point $z = -1$. And because of symmetry of the coefficient sequence the root is of even multiplicity. Hence, the polynomial $A(z)$ contains the factor $z^{-1} + 2 + z$. Thus, in the simplest case $A(z) = \frac{1}{2}(z^{-1} + 2 + z)$, $\mathcal{B}(z) \equiv 1$,

$$h(z) = \sqrt{2}, \qquad \tilde{h}(z) = \frac{z^{-1}+2+z}{2\sqrt{2}}.$$

The given pair of FIR-filters leads to a pair of a biorthogonal MRA, first of which is generated by shifts of δ-function, and other is generated by shifts of linear B-spline. At the first glance, one can seem that this example is degenerate and has not prospect to be used in applications. Indeed, "imperfections" of this pair consist in the fact that one of spaces of a dual pair consists of distributions, besides the functions of bases of piecewise linear wavelets have not zero mean value. The second property seems especially unpleasant from the generally accepted point of view, because it is considered, that good for approximation bases (in particular for compression of an information) should have many zero moments or, what is the same, the polynomial $h(z)$ should have a high multiplicity of zero at the point $z = -1$. Nevertheless, it turns out, that in most cases use of this pair of bases for image compression leads to bigger to compression factors, than use quite satisfactory Haar bases. Taking into account high computing effectiveness of these bases, it is possible to assume, that they quite can be used in applications.

Now we consider the case, when $\mathcal{B}(z) = z^{-1} + a + z$. For the polynomial $\mathcal{A}_1$, chosen before, four variants of an even factrization $h(z)\tilde{h}(z^{-1}) = (z^{-1} + 2 + z)\mathcal{A}_1(z)$ are possible:

$$h(z) = \frac{(z^{-1}+2+z)(z^{-1}+(\frac{a}{2}-1)+z)}{\sqrt{2}(z^{-2}+a+z^{2})}, \qquad \tilde{h}(z) = \sqrt{2};$$

$$h(z)=\frac{(a+2)(z^{-1}+2+z)}{2\sqrt{2}(z^{-2}+a+z^{2})},\qquad \tilde{h}(z)=2\sqrt{2}\frac{z^{-1}+(\frac{a}{2}-1)+z}{a+2};$$

$$h(z)=\frac{z^{-1}+2+z}{2\sqrt{2}},\qquad \tilde{h}(z)=2\sqrt{2}\frac{z^{-1}+(\frac{a}{2}-1)+z}{z^{-2}+a+z^{2}};$$

$$h(z)=\frac{(z^{-1}+(\frac{a}{2}-1)+z)(z^{-1}+2+z)}{\sqrt{2}(a+2)},\qquad \tilde{h}(z)=\frac{\sqrt{2}(a+2)}{z^{-2}+a+z^{2}}.$$

We note, that for all four cases $\tilde{h}(-1)\neq 0$.

It is clear, that for realization of the relation $h(-1)=\tilde{h}(-1)=0$ in the even case it is necessary to require that a polynomial $A(z)$ in (3.2) has zero of at least of the fourth order at the point -1. Thus, representation $A(z)=(1+z)^2(1+z^{-1})^2\mathcal{A}_2(z)$ is valid. Let again $\mathcal{B}(z)=z^{-1}+a+z$. Then, solving equation (3.2) for $\mathcal{A}_2(z)$ of the form $bz^{-1}+c+bz$, we obtain

$$\mathcal{A}_2(z)=\frac{6-a}{16}z^{-1}+\frac{a-2}{4}+\frac{6-a}{16}z,$$

where, as well as earlier, $|a|>2$. Thus, assuming $b=\frac{6-a}{16}$, $c=\frac{a-2}{4}$, we have two different biorthogonal pairs

$$h(z)=\frac{2(z^{-1}+2+z)(bz^{-1}+c+bz)}{\sqrt{2}(z^{-2}+a+z^{2})},\qquad \tilde{h}(z)=\frac{z^{-1}+2+z}{2\sqrt{2}}\tag{3.4}$$

and

$$h(z)=\frac{4(z^{-1}+2+z)(bz^{-1}+c+bz)}{\sqrt{2}(a+2)},\quad \tilde{h}(z)=\frac{\sqrt{2}(a+2)(z^{-1}+2+z)}{4(z^{-2}+a+z^{2})}.\tag{3.5}$$

Formulae (3.4) and (3.5) have two important special cases, when the algorithms of wavelet decomposition are much simplified. The first of these cases $a=6$ leads to the simplified biorthogonal pair

$$h(z)=\frac{z^{-1}+2+z}{2\sqrt{2}}.\qquad \tilde{h}(z)=\frac{2\sqrt{2}(z^{-1}+2+z)}{z^{-2}+6+z^{2}},$$

The second case under the limit passage $a\to\infty$ leads to the known (see [2] and [4, p. 277]) biorthogonal pair of compactly supported bases that is determined by the functions

$$h(z)=\frac{(z^{-1}+2+z)(-z^{-1}+4-z)}{4\sqrt{2}},\qquad \tilde{h}(z)=\frac{z^{-1}+2+z}{2\sqrt{2}}.$$

At last, two more important pairs of filters are obtained at $a=10/3$. For this value of the parameter a at the point -1 filters which have a maximal (equal to 6) summarized multiplicity of zero are obtained. Filters (3.4) is transformed thus to filters

$$h(z)=\frac{\sqrt{2}(z^{-1}+2+z)^2}{3z^{-2}+10+3z^{2}},\qquad \tilde{h}(z)=\frac{z^{-1}+2+z}{2\sqrt{2}}$$

and filters (3.5) is transformed to filters

$$h(z)=\frac{(z^{-1}+2+z)^2}{8\sqrt{2}},\qquad \tilde{h}(z)=\frac{4\sqrt{2}(z^{-1}+2+z)}{3z^{-2}+10a+3z^{2}}.$$

We note, that except for two pairs (3.4) and (3.5) in this case there are four more different factorizations, for which one of the filters h or $\tilde{h}$ has not zero at the point -1.

§4. Method of a lifting.

The greatest progress in a problem of acceleration of algorithms of function expansion in wavelet bases and their reconstruction by expansion coefficients is connected to, so-called, lifting scheme, accepting a completed form in [3]. Close ideas were realized in [6] and [15]. Just upon the approach, considered in [6], we lean here.

We introduce necessary notation. Let $\Pi(z)$ be Laurent polynomial $\Pi(z)=\sum_{n=k_1}^{k_2}p_nz^n$, where $k_1,k_2\in\mathbb{Z}$, then $l(\Pi)\overset{\text{def}}{=}k_1$, $u(\Pi)\overset{\text{def}}{=}k_2$, $L(\Pi)\overset{\text{def}}{=}p_{l(\Pi)}$, $U(\Pi)\overset{\text{def}}{=}p_{u(\Pi)}$. The value $d(\Pi)\overset{\text{def}}{=}u(\Pi)-l(\Pi)$ we call an order of a polynomial Π.

At first we consider a case, when each of rational functions h and $\tilde{h}$ has not roots, being poles of another function. In this case, as it was shown in the previous section the denominators of these functions are Laurent polynomials of z^2, that allows to write components of a polyphase matrix P in the form

$$h_e(z)=\frac{E_1(z)}{Q_1(z)},\quad h_o(z)=\frac{O_1(z)}{Q_1(z)},\quad g_e(z)=\frac{E_2(z)}{Q_2(z)},\quad g_o(z)=\frac{O_2(z)}{Q_2(z)}. \tag{4.1}$$

The fact that a determinant of a polyphase matrix is equal to the identity imples the relation

$$\det\begin{pmatrix}E_1(z) & E_2(z)\\ O_1(z) & O_2(z)\end{pmatrix}=Q_1(z)Q_2(z). \tag{4.2}$$

holds. It is clear from here that two inequalities

$$\mathcal{L}(P)\overset{\text{def}}{=}l(Q_1Q_2)-\min\{l(E_1O_2),l(O_1E_2)\}\geq 0;$$

$$\mathcal{U}(P)\overset{\text{def}}{=}\max\{u(E_1O_2),u(O_1E_2)\}-u(Q_1Q_2)\geq 0$$

are valid. The value $\mathcal{D}(P)\overset{\text{def}}{=}\mathcal{L}(P)+\mathcal{U}(P)$ is said to be *the defect* of matrix P.

Matrices of the form

$$T^{+}(k,a)=\begin{pmatrix}1 & az^k\\ 0 & 1\end{pmatrix},\qquad T^{-}(m,b)=\begin{pmatrix}1 & 0\\ bz^m & 1\end{pmatrix},$$

where $a,b\in\mathbb{R}$; $k,m\in\mathbb{Z}$, are said to be *elementary liftings.*

Theorem 1. *Any polyphase matrix P with rational components of the form (4.1) can be represented in the form*

$$P(z)=P_0(z)T^{\sigma(1)}(k_1,a_1)T^{\sigma(2)}(k_2,a_2)\dots T^{\sigma(\mathcal{D}(P))}(k_{\mathcal{D}(P)},a_{\mathcal{D}(P)}), \tag{4.3}$$

where $P_0(z)$ is a matrix with a zero defect, $k_i\in\mathbb{Z}$, $a_i\in\mathbb{R}$, $\sigma(i)$ is an appropriate sign; $i=1,2,...,\mathcal{D}(P)$.

Remark 1. In a statement of Theorem 1 it does not affirm that all elementary liftings in (4.3) are nondegenerated. Some of them can be equal to the identity matrix.

Remark 2. Factrization (4.3) is determined by not a unique way.

Proof of Theorem 1. To prove the theorem it suffices to show that any polyphase matrix P of a non-zero defect can be represented in the form $P(z)=P'(z)T^{\sigma}(k,a)$, where $\mathcal{D}(P^1)<\mathcal{D}(P)$.

First we prove the following auxiliary statement.

Lemma 1. *If for a polyphase matrix $P(z)$ with rational components of the form (4.1) the inequality $\mathcal{L}(P)>0$ (or $\mathcal{U}(P)>0$) is valid, then $l(E_1)+l(O_2)=l(E_2)+l(O_1)$ (or $u(E_1)+u(O_2)=u(E_2)+u(O_1)$) and the determinant of the matrix*

$$A=\begin{pmatrix} L(E_1)z^{l(E_1)} & L(E_2)z^{l(E_2)} \\ L(O_1)z^{l(O_1)} & L(O_2)z^{l(O_2)} \end{pmatrix} \quad \left(\text{or } B=\begin{pmatrix} U(E_1)z^{u(E_1)} & U(E_2)z^{u(E_2)} \\ U(O_1)z^{u(O_1)} & U(O_2)z^{u(O_2)} \end{pmatrix}\right)$$

is equal to zero.

Proof of Lemma 1. In the case when the equality $l(E_1)+l(O_2)=l(E_2)+l(O_1)$ is violated we come to contradiction to equality (4.2), since the minimal degree in the left side equals $\min\{l(E_1O_2),l(E_2,O_1)\}$. By assumption of Lemma 1 it is less than minimal power of the right term of equality (4.2).

For the same reason the determinant of the matrix A is equal to zero, since in the opposite case the minimal power of a polynomial in the left part (4.2) would be less than minimal power of the right member.

The statement, concerning the matrix B, is proved in similar way.

Let us return to proving Theorem 1.

Let for a definiteness $\mathcal{L}(P)>0$. Then it follows from Lemma 1 that

$$l(E_1)+l(O_2)=l(E_2)+l(O_1). \tag{4.4}$$

For a definiteness we also assume, that

$$d(E_1)\le d(O_1). \tag{4.5}$$

We consider two cases: 1) $\mathcal{U}(P)>0$; 2) $\mathcal{U}(P)=0$.

In the first case, according to Lemma 1, we have $u(E_1)+u(O_2)=u(E_2)+u(O_1)$. Subtracting (4.4) from here, we obtain $d(E_1)+d(O_2)=d(E_2)+d(O_1)$, that join with (4.5) gives $d(E_2)\le d(O_2)$. Thus, by carrying out transformation $E_1'(z)=E_1(z)$, $E_2'(z)=E_2(z)$, $O_1'(z)=O_1(z)-az^kE_1(z)$, $O_2'(z)=O_2(z)-az^kE_2(z)$, where $a=L(O_1)/L(E_1)$, $k=l(O_1)-l(E_1)$, taking into account that the determinant of the matrix A is equal to zero, we obtain $l(O_1')=l(O_1)+1$, $l(O_2')=l(O_2)+1$. At the same time, the inequalities $u(O_1')\le u(O_1)$, $u(O_2')\le u(O_2)$ are valid. Hence, the defect of the matrix

$$P'(z)\overset{\text{def}}{=}P(z)\begin{pmatrix} 1 & -az^k \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

is strictly less than the defect of the matrix P. As was to be proved.

Let now $\mathcal{U}(P)=0$. We show that transformation (4.6) reduces the defect of the matrix P at least by 1. If $d(E_2)\le d(O_2)$, the mentioned above reasoning remain in a force. We consider the case $d(E_2)>d(O_2)$. Because after transformation (4.6) we have $\mathcal{L}(P")\le\mathcal{L}(P)-1$, it remains to prove an invariance of the value $\mathcal{U}(P)$. Easily to see, that $u(O_2')=u(E_2)+k$ and $u(O_1')\le u(O_1)$. Thus, $u(E_2O_1')\le u(E_2O_1)$ and $u(E_1O_2')=u(E_1)+u(E_2)+k\le u(O_1)+u(E_2)=u(E_2O_1)$. Hence,

$$\mathcal{U}(P")=\max\{u(E_1O_2'),u(O_1'E_2)\}-u(Q_1Q_2)\le \max\{u(E_1O_2),u(O_1E_2)\}-u(Q_1Q_2)=\mathcal{U}(P)=0.$$

Thus, $\mathcal{D}(P') \le \mathcal{D}(P) - 1$.

In conclusion of the proof of Theorem 1 we note that in the case, when $d(E_1) > d(O_1)$, it should be applied an elementary lifting, being the upper triangular matrix.

As it will be shown in §5, realization of algorithms leaning upon the factorization, guaranteed by Theorem 1, has a double asymptotic gain in computational complexity for fixed degrees of denominators of $h(z)$ and $\tilde{h}(z)$ and degree of at least one of numerators, goes to infinity, provided that both functions $h(z)$ and $\tilde{h}(z)$ have a general form, i.e., there are no zeros or respectively little amount of them among coefficients of their numerators between a maximum and minimum degree.

However, in practice most often it is necessary to deal with wavelets which with accurate to a shift of argument are even or odd functions. Just such examples were considered in §2.

Unfortunately, for sequences h_n and $\tilde{h}_n$ with half-integer symmetry we were not succeeded to find a lifting realization, taking into account a symmetry. The importance of this problem is obvious, since such sequences determine bases of odd wavelets (elementary example of such functions are Haar's wavelets).

We consider the case of even wavelets. About a possibility of taking account of the symmetry at a realization of the lifting scheme, when $h(z)$ and $\tilde{h}(z)$ are Laurent polynomials, it was mentioned (without a proof) in [3]. In those paper for all examples, corresponding to the even case, the lifting realization is given with the taking account of evenness.

The problem here is as follows. When $h(z) = S(z)/Q(z)$, where S, Q are Laurent polynomials, for which $S(1/z) = S(z)$, $Q(1/z) = Q(z)$, it is possible to economize at a realization of formulae (2.8) or (2.9), accounting a symmetry of a numerator, using the formula

$$c_k^{-1} = h_0 c_{2k}^0 + \sum_{m=1}^{M} h_m (c_{2k+m}^0 + c_{2k-m}^0),$$

where $2M = d(S)$. Asymptotically at large m at the account of a symmetry the amount of operations is equal $3/4$ of the amount of operations which is necessary in a general case. Naturally, it would be necessary at a lifting realization additionally to the double gain to obtain the same additional gain in computational complexity.

So, we suppose that the both rational functions $h(z)$ and $\tilde{h}(z)$ have a sequences of Laurent coefficients which are symmetric with respect to the coefficient with the zeroth subscript. Then without loss of generality it is possible to state this about polynomials E_1, E_2, O_1, O_2 in notation of (4.1). Then the following statement takes place.

Theorem 2. *Let a polyphase matrix $P(z)$ has rational components of the form (4.1), generated by sequences $h_n = h_{-n}$, $\tilde{h}_n = \tilde{h}_{-n}$. Then the matrix $P(z)$ has an even defect and can be representted in the form*

$$P(z) = P_0(z) \left(\prod_{i=1}^{\mathcal{D}(P)/2} T^{\sigma(i)}(\sigma(i)k_i, a_i) T^{\sigma(i)}((-1)\cdot\sigma(i)(k_i+1), a_i) \right),$$

where $P_0(z)$ is a matrix with a zero defect, $k_i \in \mathbb{N} \cup 0$, $a_i \in \mathbb{R}$, $\sigma(i)$ is a sign chosen in appropriate way.

Proof of Theorem 2. First we show that the defect of the matrix $P(z)$ is an even number. Indeed, from a symmetry of factors it is easily to receive that $l(E_1) = -u(E_1)$, $l(O_1) = 1 - u(O_1)$, $l(O_2) = -u(O_2)$, $l(E_2) = -1 - u(E_2)$. Hence, $l(E_1O_2) = -u(E_1O_2)$ and $l(E_2O_1) = -u(E_2O_1)$, moreover, $l(Q_1Q_2) = -u(Q_1Q_2)$. Therefore $\mathcal{L}(P) = \mathcal{U}(P)$, that implies $\mathcal{D}(P) = 2\mathcal{L}(P) = 2\mathcal{U}(P)$.

Thus, it remains to show that any matrix $P(z)$ with a non-zero defect, satisfying conditions of Theorem 2, can be represented in the form

$$P(z) = P_1(z)T^{\sigma}(\sigma k, a)T^{\sigma}((-1)\cdot\sigma(k+1), a),$$

where P_1 is a matrix, also satisfying conditions of Theorem 2.

Since $\mathcal{L}(P) = \mathcal{U}(P) > 0$, by Lemma 1 we have $l(E_1) + l(O_2) = l(E_2) + l(O_1)$ and $u(E_1) + u(O_2) = u(E_2) + u(O_1)$. Moreover, the determinants of matrices

$$A = \begin{pmatrix} L(E_1) & L(E_2) \\ L(O_1) & L(O_2) \end{pmatrix}, \quad B = \begin{pmatrix} U(E_1) & U(E_2) \\ U(O_1) & U(O_2) \end{pmatrix}$$

are equal to zero. In view of a symmetry of the sequences h_n and $\tilde{h}_n$ we have $A = B$. In view of the fact that $d(E_1)$ and $d(O_2)$ are even numbers and $d(O_1)$ and $d(E_2)$ are odd ones, either fulfilment inequalities a) $d(E_1) > d(O_1)$, $d(E_2) > d(O_2)$ or b) $d(E_1) < d(O_1)$, $d(E_2) < d(O_2)$ takes place.

We assume that inequalities a) are valid. Then, choosing $a = L(E_1)/L(O_1)$ and $m = u(E_1) - u(O_1)$, we obtain that for polynomials

$$E_1'(z) = E_1(z) - a(z^m + z^{-m-1})O_1(z),$$

$$E_2'(z) = E_2(z) - a(z^m + z^{-m-1})O_2(z)$$

relations $E_1'(z) = E_1'(z^{-1})$, $E_2'(z) = z^{-1}E_2'(z^{-1})$, $d(E_1') = d(E_1) - 2$, $d(E_2') = d(E_2) - 2$ hold. Thus, taking as above $P_1(z)$ a matrix, obtained of $P(z)$ by the change $E_1 \to E_1'$, $E_2 \to E_2'$, we have

$$P_1(z) = P(z)\begin{pmatrix} 1 & 0 \\ -a(z^m + z^{-m-1}) & 1 \end{pmatrix} = P(z)T^{-}(m, -a)T^{-}(-(m+1), -a) \tag{4.7}$$

or $P(z) = P_1(z)T^{-}(-(m+1), a)T^{-}(m, a) = P_1(z)T^{-}(m, a)T^{-}(-(m+1), a)$.

The case b) is considered in similar way. Operating in the same way as for the case a), we obtain

$$P(z) = P_1(z)T^{+}(-m, a)T^{+}(m+1, a),$$

where $a = L(O_1)/L(E_1)$, $m = l(O_1) - l(E_1)$, and the matrix $P_1(z)$ differs from $P(z)$ by the change

$$O_1(z) \to O_1'(z) = O_1(z) - a(z^{-m} + z^{m+1})E_1(z),$$

$$O_2(z) \to O_2'(z) = O_2(z) - a(z^{-m} + z^{m+1})E_2(z).$$

Thus, Theorem 2 is proved.

§5. Numerical realization of algorithms of subband coding.

Now we consider realization of decomposition and reconstruction algorithms in a basis of which the lifting scheme lays. First we explain what advantage is given by a factorization of a polyphase matrix, guaranteed by Theorem 1. Let us calculate how many arithmetical operations (additions and multiplications) is required for a direct realization of the formula. For this purpose in notation of §4 the action of a polyphase matrix $P(z)$ to a signal it is conveniently to represent in the form

$$P(z)\begin{pmatrix} c_e^0(z^{-1}) \\ c_o^0(z^{-1}) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{Q_1(z)} & 0 \\ 0 & \dfrac{1}{Q_2(z)} \end{pmatrix} \begin{pmatrix} E_1(z) & O_1(z) \\ E_2(z) & O_2(z) \end{pmatrix} \begin{pmatrix} c_e^0(z^{-1}) \\ c_o^0(z^{-1}) \end{pmatrix}. \tag{5.1}$$

Easily to see that the first matrix multiplication in (5.1) by a signal vector can be realized, using on average for $d(E_1)+d(O_1)+d(E_2)+d(O_2)+3$ arithmetical operations for a signal sample. Indeed, for deriving one factor in Laurent expansion of the first component it is required $2d(E_1)+2d(O_1)+3$ operations, and for deriving one Laurent factor of the second component it is required $2d(E_2)+2d(O_2)+3$ operations.

We stop in more details on a realization of the second factor with the help of a method, which in a radio engineering refers to as a digital recursive filter. Since for $|a|<1$ we have

$$\sum_{n\in\mathbb{Z}} \beta'_n z^n \overset{\text{def}}{=} \frac{1}{1-az}\sum_{n\in\mathbb{Z}} \beta_n z^n = \sum_{n\in\mathbb{Z}} \beta_n z^n \sum_{n=0}^{+\infty} a^n z^n = \sum_{n\in\mathbb{Z}} \left(\sum_{k=0}^{+\infty} \beta_{n-k} a^k\right) z^n, \tag{5.2}$$

then

$$\beta'_n = \sum_{k=0}^{+\infty} \beta_{n-k}a^k = \sum_{k=0}^{+\infty} \beta_{n-1-k}a^{k+1} + \beta_n = (\sum_{k=0}^{+\infty} \beta_{n-1-k}a^k)a + \beta_n = \beta'_{n-1}a + \beta_n.$$

The obtained recurrence formula is called *an elementary recursive filter.*

If there exists N such that $\beta_n = 0$ for $n < N$, therefore $\beta'_n = 0$, and the recursive filter allows an exact realization of formula (5.2). Certainly, in this case the interior sum in (5.2) is finite and consequently it allows an exact realization. However, the computational complexity for its realization are much higher than complexity of the recursive filter.

In the case when there are non-zero b_n with an arbitrary large negative numbers, the recursive filter can be realized, beginning with some number N. Therefore there is a necessary error in its realization, which decreases with each pitch as a geometrical progression with a ratio a.

Similarly an operator is determined by the formula

$$\sum_{n\in\mathbb{Z}} \beta'_n z^n \overset{\text{def}}{=} \frac{1}{1-az^{-1}}\sum_{n\in\mathbb{Z}} \beta_n z^n,$$

can be realized as an elementary recursive filter $\beta'_n = \beta'_{n+1}a + \beta_n$.

Laurent polynomials Q_1 and Q_2 in (5.1) can be represented in the form

$$Q_1(z) = Az^{l_1} \prod_{i=1}^{d(Q_1)} \left(1 - \left(\frac{a_i}{z}\right)^{\operatorname{sign}(1-|a_i|)}\right), \quad Q_2(z) = Bz^{l_2} \prod_{i=1}^{d(Q_2)} \left(1 - \left(\frac{b_i}{z}\right)^{\operatorname{sign}(1-|b_i|)}\right),$$

where $\{a_i\}$, $\{b_i\}$ are roots of the polynomials Q_1 and Q_2, and without loss of generality the factors Az^{l_1} and Bz^{l_2} can be considered equal to the identity. We note that the roots of the polynomials Q_1 and Q_2 cannot be equal to the identity by modulus, because in this case, on the one hand, we have unstable filters, and on the other hand, such functions h and $\tilde{h}$ obviously does not correspond to any biorthogonal pair of MRA. Hence, multiplication by a diagonal matrix in the formula (5.1) can be realized with the help of $d(Q_1)+d(Q_2)$ elementary recursive filters. Therefore the average computational complexity of all recursive filters are equal to $d(Q_1) + d(Q_2)$. Thus, total computational complexity of a direct realization of formula (5.1) constitutes $d(E_1) + d(O_1) + d(E_2) + d(O_2) + d(Q_1) + d(Q_2) + 3$ operations per a signal sample.

The computational complexity for lifting realization of formula (5.1) are represented in the following statement.

Theorem 3. *The computational complexity of lifting realization of the formula (5.1) does not exceed $\mathcal{D}(P) + 3d(Q_1) + 3d(Q_2) + 3$ operations per a signal sample.*

Remark 1. If $h(z)$ and $\tilde{h}(z)$ are Laurent polynomials, then as a special case of the Theorem 3 we obtain, that the complexity is equal to $\mathcal{D}(P) + 3$, that is result, obtained in [3] and [6].

Proof of Theorem 3. According to Theorem 1, in formula (2.9) action of a polyphase matrix to the signal vector can be represented as a product of $\mathcal{D}(P)$ elementary liftings and matrix P_0 of a zero defect. The action of each lifting $T^\sigma(k, a)$ to a signal consists in multiplication of a half of signal samples by number a, a shift of coordinates of the same samples by k positions and adding the result to the second half of samples. Thus, the cost of one lifting is equal to one operation per a signal sample (as it usually is a shift of coordinates is not included in computational expenditures). Thus, we obtain it is necessary $\mathcal{D}(P)$ arithmetical operations for realization $\mathcal{D}(P)$ elementary liftings.

Denoting by E_1', E_2', O_1', O_2' the numerators of the appropriate elements of the matrix P_0, we obtain that for a realization of multiplication by a matrix $P_0(z)$ it is required at most $d(E_1') + d(E_2') + d(O_1') + d(O_2') + d(Q_1) + d(Q_2) + 3$ operations per a sample. But since the matrix P_0 has a zero defect, $d(E_1') + d(O_2') \le d(Q_1) + d(Q_2)$ and $d(E_2') + d(O_1') \le d(Q_1) + d(Q_2)$, therefore the total expenditures are estimated from above by the value $3d(Q_1) + 3d(Q_2) + 3$. As was to be proved.

It is clear, that for lifting scheme a possibility of additional reduction of amount of arithmetical operations by one forth in the case of symmetric sequences $h(z)$ and $\tilde{h}(z)$ follows from Theorem 2. Such possibility appears because of the fact, that for a realization of each elementary lifting one operation per a sample is required and it is visible from (4.7) composition of two elementary liftings $T^+(m, a)$ and $T^+(-(m+1), a)$ can be realized for 1.5 operations per a sample (3 operations with a half of samples).

§6. Applications to image compression

We begin from motivation of application of considered types of wavelets to signal and image processing. We consider an elementary digital device, for a realization of which wavelet bases can be applied. This is an equalizer. It is intended for frequency correction of a signal. The principle of an operation consists in the following. An initial signal $f(t)$ with a finite spectrum (Fourier transform) is represented as a sum of signals f_0, f_1,f_2, … ,f_n, which have disjoint spectra, located respectively on intervals $[0, \omega_1]$, $[\omega_1, \omega_2[$, $[\omega_2, \omega_3[$, … $[\omega_n, \omega_{n+1}]$, the union of which covers the whole spectrum of the signal f. Then the obtained signals are summed with required weights α_k, that gives a signal $\tilde{f} = \sum \alpha_k f_k$ at the output of a device . This problem can be solved by means of Fast Fourier Transform. Computational complexity of such procedure is equal to $O(N \log N)$, where N is number of signal samples. Clearly that the computational complexity of algorithm can be reduced due to a partition of a signal to short segments. However, in this case the problem, connected to a glueing together of separate pieces after processing, can occur.

It is usually presupposed that the points ω_k form a logarithmic scale, i. e., $\omega_{k+1}/\omega_k = c$, $k = 1, 2, \dots, n$. In the case when $c = 2$, it is conveniently to use representation of a signal f in the form of a sum of its projections to some collection of spaces $\tilde{V}^m$, $\tilde{W}^m$, $\tilde{W}^{m+1}$, … , $\tilde{W}^{m+n}$. Where the operators of projection to $\tilde{W}^{m+k}$ should well approximate a bandpass filter, passing a harmonics in an interval $[\omega_k, \omega_{k+1}[$ without distortions and damping all remaining harmonics. Thus, the problem is reduced to a choice of 2π-periodic function m_0, approximating the function

$$\chi(x) = \begin{cases} 1, & |x| < \pi/2, \\ 0, & \pi/2 \le |x| \le \pi \end{cases}$$

In [12] in some sense optimal solutions of this problem, when $m_0 = \tilde{m}_0$ is polynomial of fixed degree were found. The optimality of a choice consists in the fact that polynomial solution which has as the much as possible flat graph near to zero was chosen. It was provided by the greatest possible multiplicity of zero of derivative of a polynomial m_0 at points 0 and $\pm\pi$. Such choice ensures a large absolute value of derivative (so-called, steepness of a filter) at points $\pm\pi/2$.

It is clear that the rational functions can approximate function χ with a much higher degree. As much as possible flat performance near to zero among rational functions of an equal degree of a numerator the functions $m_0(\omega) = \tilde{m}_0(\omega) = h(e^{i\omega})/\sqrt{2}$, constructed (see, for example, [1]) on the base of Butterworth filters

$$H(z)h(z^{-1}) = \frac{2(1+z^{-1})^N(1+z)^N}{(z^{-1}+2+z)^N + (-z^{-1}+2-z)^N},$$

give.

If to take for a comparison the Daubechies and Butterworth bases with functions m_0 with the same multiplicity of zero (equal to N) at points $\pm\pi$, the corresponding algorithms of decomposition and reconstruction of signals have almost identical computational complexity.

At the same time the ratio of derivative of a function m_0 for the Butterworth wavelets at points $\pm\pi$ for $N = 2$ to the derivative of the Daubechies wavelets is equal to 4/3, and at $N = 3$ is equal to 8/5. At growth N the distinction derivatives becomes even more significant. If the Butterworth filter at $N = 2$ ensures the same derivative (even a little bit large), as a the Daubechies filter at $N = 3$, to receive the same derivative, which gives a filter of a Butterworth at $N = 3$, it is required to take the Daubechies filter of order 7.

Unfortunately, a similar sort of the simple argumentation cannot be serious argument for the benefit of application of any bases to problems of image compression. At the present there are no justified reasons about quantitative characteristics of bases, which influence a degree of compression. One of such properties, which is presented in majority of the publications, is a multiplicity of zero of functions $m(\omega)$ and $\tilde{m}(\omega)$ at points $\pm\pi$. It is accepted to consider, that increasing a multiplicity of zero leads to increasing of possible compression factor for images. Our research, the results of which are presented below, refutes this statement in some respects.

We have conducted numerical simulation for comparison the bases described in §3 and best biorthogonal pairs of compact supported bases for their effectiveness for image compression.

All studies were carried out for 8-bit (256 grades of grey) images. An image as the function of two variables was periodized by means of a mirror prolongation beyond the edges and then was represented as coefficients of expansion in wavelet bases, which were a tensor product (see, for example, [4, Chapter 10]) of one-dimensional bases.

The obtained coefficients were coded by algorithm SPIHT [13] without additional arithmetic coding. For reconstruction of an image an initial segment of a code of required length was taken. The reconstruction was carried out by code segments of a length equal to from 1/10 up to 1/150 of a file length which is necessary to keep image in a standard format, requiring 1 byte per a pixel.

The measured in decibels peak signal-to-noise ratio (PSNR), determined by the formula

$$PSNR = 10\lg\frac{255^2}{\sum_{i=1}^{N}\sum_{j=1}^{M}|x_{ij}-\tilde{x}_{ij}|^2},$$

where x_{ij}, $\tilde{x}_{ij}$ are integers in bounds from 0 up to 255, defining value of intensity of a pixel with coordinates (i, j) accordingly for original and for a reconstructed image, was taken as a criterion for the quality of a reconstructed image.

Among the investigated bases, associated with recursive filters, the biorthogonal pair

$$h(z) = \frac{1+z}{\sqrt{2}}, \qquad g(z) = \frac{z^{-3} - 3z^{-2} + 3z^{-1} - 1}{4\sqrt{2}},$$

$$\tilde{h}(z) = \frac{(1+\alpha)^2}{4\sqrt{2}}\,\frac{z^{-1} + 3 + 3z + z^2}{(1+\alpha z^2)(1+\alpha z^{-2})},$$

$$\tilde{g}(z) = \frac{(1+\alpha)^2}{\sqrt{2}}\,\frac{-z^{-2} + z^{-1}}{(1+\alpha z^2)(1+\alpha z^{-2})},$$

where $\alpha = 3 - 2\sqrt{2}$, has given the best results for compression. Such pair allows to reach approximately the same compression factors as the best of known bases with finite masks (for example, 9/7-wavelets from [2]). The difference in PSNR for the given pair and 9/7-bases for overwhelming majority of images is in range ±0.3 dB. However, our algorithm has significant advantage in computational complexity.

The standard pyramidal algorithm of expansion of an one-dimensional signal in 9/7-bases has complexity 14 operations (see [3]) of addition and multiplication per a signal sample. As much of operations it is required for reconstruction of a signal by the coefficients. We show, that our algorithm can be realized for 7 operations per a signal sample for decomposition of an one-dimensional signal and for 11 operations per a sample for reconstruction.

The standard realization of image expansion in two-dimensional wavelet bases, consists in a sequential realization of one-dimensional algorithms at first for rows of an image and then for columns. Thus, the relationships of computational complexity for different bases are kept after passage to two-dimensional algorithms.

The polyphase matrices of algorithms of decomposition and reconstruction have the form

$$P(z)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & z\\ -\frac{3z^{-1}-1}{4} & \frac{z^{-1}+3}{4}\end{pmatrix},\tilde{P}(z)=\frac{(1+\alpha)^2}{\sqrt{2}(1+\alpha z^{-1})(1+\alpha z)}\begin{pmatrix}\frac{3+z}{4} & \frac{1+3z}{4}\\ -z^{-1} & 1\end{pmatrix}.$$

That does not allow to use outcomes §4 for realization of algorithms of expansion and restoration directly . We note, that the reason is not that matrices $P(z)$ and $\tilde{P}(z)$ in view of transfer of a polynomial from a denominator of the second row of the matrix P to a denominator of the second row of the matrix $\tilde{P}$ have a determinant different from the identity, but in view of the fact that before transfer of a polynomial the initial matrices have zero defects. That does senseless application of Theorem 1. Thus, probably, for realization of one-dimensional algorithm of decomposition it makes a sense to use the direct form of decomposition algorithm (without passage to a polyphase matrix), which in view of a symmetry of filters requires 7 operations per a signal sample.

As to algorithm of reconstruction, despite above mentioned argument, we were succeed in finding a lifting factorization

$$\tilde{P}(z)=\begin{pmatrix}1 & \frac{1+3z}{4}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & \frac{(1+\alpha)^2}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{(1+\alpha z^{-1})(1+\alpha z)}\end{pmatrix}\begin{pmatrix}1 & 0\\ -z^{-1} & 1\end{pmatrix}.$$

It can be seen from here, the computational complexity of such algorithm are equal 11 operation per a signal sample. We note, that the direct form of a realization of reconstruction algorithm requires 15 operations per a sample.

In summary we mark, that the filters $h(z)$ and $\tilde{h}(z)$, considered in this section, really generate a biorthogonal pair of bases of the space $\mathbb{L}^2(\mathbb{R})$. Unfortunately, we do not know any general result, from which the given fact would immediately follows. One of the first results about a sufficient condition for existence of orthogonal wavelet basis, corresponding to some filter $h(z)=\tilde{h}(z)$ and satisfying the equation (2.7), belongs to S.Mallat [14].

Though we deal with biorthogonal pairs h and $\tilde{h}$ and we can not use the mentioned sufficient condition, however, in our case in view of a positivity of functions m_0 and $\tilde{m}_0$ on an interval $(-\pi,\pi)$ a Mallat's proof for our pair of recursive filters passes without essential modifications.

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