Trigonometric rational wavelet bases

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We propose a construction of periodic rational bases of wavelets. First we explain why this problem is not trivial.

Construction of wavelet basis is not possible neither for the case of algebraic polynomials nor for the case of rational algebraic functions. Of course algebraic polynomials do not belong to $\mathbb{L}^2(\mathbb{R})$. Nevertheless, they can belong to the closure of $\mathbb{L}^2(\mathbb{R})$ in topology of the generalized convergence. However, there is not polynomial bases, because any shift of polynomial is a polynomial of the same degree. So dimension of linear span of a set of polimomial shifts has a finite dimension. As for rational bases the reason of non-existance is different. It is clear that the rational function φ, whose shifts $\{\varphi(x-n)\}$ constitutes a basis of the space V^0, cannot have poles in the real line. Let d is a maximal distance from poles of $\varphi(x)$ to the real line. The function $\varphi(x/2)$ generates a basis in $V^{-1} \subset V^0$. A maximal distance from poles of $\varphi(x/2)$ to the real line is equal to $2d$. It contradicts to the fact that the function $\varphi(x/2)$ in some sence can be approximated by linear combinations of $\{\varphi(x-n)\}$.

Recall classic construction of periodic polynomial wavelets. It is based on periodization of non-periodic Multiresolution analysis (MRA), consisting of entire functions. As above MRA Meyer's wavelets [1] or any their modifications can be taken. These examples of triginometric polynomial MRA allowed to resolve many non-trivial problems of Analysis relating to constructing orthogonal polynomial bases with special properties.

We intend to propose MRA that possesses the following three properties:
1) *the MRA consists of rational trigonometric functions*;
2) *the uniform limit function of the sequence of the periodic interpolating scaling functions* $\varphi^n(2^{-n}x)$ *is the Shannon scaling function*;

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3) *the corresponding limit function for Fourier transforms* $\hat{\varphi}^n(2^n\omega)$ *is a Riemann-like function which is discontinuous at binary-rational points and continuous at the remaining points.*

We note that due to property 2) our construction of rational periodic bases cannot be reduced to the periodization of non-periodic MRA. It follows from the fact that the periodization of the Shannon scaling fuction leads to polynomial wavelets (they were studied in [2]).

Unexpected property 3) requires additional explaination. Fourier transform of periodic function is only defined at a discrete set of points. So the limit function of the $\hat{\varphi}^n$ is defined only at binary-rational points. However, it can be extended to entire real line by continuity.

We use technics developed in our paper [3] (see also [4]) which gives possibility of direct construction. We denote by $\overset{\circ}{X}$ the completion of a set of infinitely differentiable functions in the norm of a Banach space X.

Definition 1 *We say that the sequence* $\{V^j\}_{j=0}^{\infty}$ *of linear subspaces of Banach space* X *forms a multiresolution analysis (MRA) of the space* X*, if it satisfies the conditions:*

1) a) $V^0 \subset V^1 \subset \ldots \subset V^j \subset \ldots \subset X$*;*

b) $\dim V^j = 2^j$, $j = 0, 1, 2, \ldots$*;*

2) $\bigcup_{j\geq 0} V^j$ *is dense in* $\overset{\circ}{X}$*;*

3) a) V^0 *consists of constants;*

b) if $f(x) \in V^j$*, then* $f(2x) \in V^{j+1}$, $j = 0, 1, 2, \ldots$*;*

c) if $f(x) \in V^{j+1}$*, then* $f(x/2) + f(x/2 + \pi) \in V^j$ $j = 0, 1, 2, \ldots$*;*

d) every function $f \in V^{j+1}$ *can be represented in the form* $f(x) = f_1(x) + f_2(x + 2\pi \cdot 2^{-j-1}) + f_3(2x)$*, where* $f_1, f_2, f_3 \in V^j$, $j = 1, 2, \ldots$

We denote by $\hat{f}(n)$, $n \in \mathbb{Z}$, Fourier coefficients of 2π-periodic distribution f:

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

In what follows we use the notation $J = 2^j$, $N = 2^n$. The following Lemma was proved in [3].

Lemma 1 *Let a sequence* $\{V^j\}_{j=0}^{\infty}$ *of spaces forms MRA of the space* X*. Then for every* $j \geq 0$ *there exists a basis* $v_n^j(x)$*,* $n = 0, 1, \ldots, J-1$*, satisfying for* $m \in \mathbb{Z}$ *the conditions:*

1) $v_0^j \equiv 1,\ j \geq 0$;
2) for any $k \neq n,\ 0 \leq k, n < J,\ m \in \mathbb{Z}$, we have $\hat{v}_n^j(k + Jm) = 0$;
3) the recurrent relations

a) $\hat{v}_n^{j+1}(n + 2Jm) = \hat{v}_n^j(n + 2Jm)$ for odd n (here and in what follows we suppose $v_l^j \equiv v_{l+J}^j$);

b) $\hat{v}_n^{j+1}(n + 2Jm) = \hat{v}_{n/2}^j(n/2 + Jm)$ for even n hold.

In particular, from Lemma we have

Corollary 1 *Any function $v_n^j,\ j > 0,\ 0 < n < J$ can be represented in the form of the series*

$$v_n^j(x) = \sum_{m \in \mathbb{Z}} c_{n+Jm}^j \exp(i(n + Jm)x),$$

with nonzero coefficients.

Last two statements give very simple tool for constructing all possible rational MRA. Indeed, the recursive relations in Lemma allow reconstruct whole MRA by the function v_1^1. For the wide class $\mathfrak{H}$ (see [3]) of spaces which contains almost all classical spaces such as Lebesgue L^p, $1 \leq p \leq \infty$, Besov — Lizorkin — Triebel spaces it was proved the following statement.

Theorem 1 *A function $v_1^1 \in \overset{\circ}{X} \in \mathfrak{H}$ generates MRA of X if and only if*

$$v_1^1(x) = \sum_{n=-\infty}^{+\infty} a_n \exp(i(2n + 1)x), \tag{1}$$

where all a_n are distinct from 0.

Thus, there exists one-to-one correspondence between rational functions (v_1^1) of the form (1) and all possible rational MRA of any space $X \in \mathfrak{H}$.

We consider MRA generated by the simplest real-valued rational function

$$v_1^1(x) \overset{\text{def}}{=} \sum_{k=0}^{+\infty} q^k (e^{i(2k+1)x} + e^{-i(2k+1)x}) = \frac{e^{ix}}{1 - qe^{2ix}} + \frac{e^{-ix}}{1 - qe^{-2ix}} = \frac{2(1-q)\cos x}{1 + q^2 - 2q\cos 2x},$$

were $q \in \mathbb{R}$, $|q| < 1$.

Easily to see that the functions v_n^j, obtained by the recurrent formulae from Lemma, are rational. Hence, this MRA consists of rational trigonometric functions.

Now we explain how an interpolating scaling function can be constructed. An arbirary function φ^j whose shifts $\{\varphi^j(x-2\pi k2^{-j})\}_{k=0}^{J-1}$ constitute bases of V^j can be represented in the form (see [3])

$$\varphi^j(x)=\sum_{k=0}^{2^j-1}\alpha_k v_k^j(x),$$

where all $\alpha_k\neq 0$. It can be easily proved, if we take $\alpha_k=(2^j v_k^j(0))^{-1}$, the functions φ^j satisfies interpolation conditions

$$\varphi^j(2\pi k2^{-j})=\begin{cases}1, & k=2^j m, \quad m\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Besides, the sequence

$$\tilde{\varphi}^j(x)\ \stackrel{\text{def}}{=}\ \begin{cases}\varphi^j(2^{-j+1}x), & |x|<\pi,\\ 0, & |x|>\pi;\end{cases}$$

uniformly on $\mathbb{R}$ converges to the Shannon scaling function $\phi(x)=\frac{\sin x}{x}$. Despite the fact that series $\sum_{k\in\mathbb{Z}}\frac{\sin(2^j x+2\pi k)}{(2^j x+2\pi k)}$ are not absolutely convergent, for any $j=0,1,2,\ldots$ it converges in the sense of principal value to a trigonometric polynomial $\phi^j(x)$ of the order 2^{j-1}. It is clear $\phi^j\neq\varphi^j$ for $j>0$.

Every positive integer number which is not divisible by J can be represented in the form $k=mN+rJ$, where $0\leq n<j$, $0<m<2^{j-n}$, $m=2l+1$, $r=0,1,\ldots$

Using direct calculations, it can be proved that we have

$$\hat{\varphi}^j(k)=\frac{1-q^{J/(2N)}}{q^l+q^{J/(2N)-l-1}}q^{l+rJ/(2N)},\qquad \hat{\varphi}^j(sJ)=\begin{cases}1, & s=0,\\ 0, & s\in\mathbb{Z}\setminus 0.\end{cases}$$

Let us introduce the sequence of functions $\Phi^j(\omega)=\hat{\varphi}^j(2^j\omega)$ which are determined at points ω such that their representation in the binary form requires at most j digits after the decimal point. This sequence determines a limit function Φ at binary-rational points. The function Φ is discontinuous everywhere but it can be extended by continuity to the entire real line.

References

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