Applications of the Derivative

1 Maxima and Minima of Functions on Closed Intervals

Often, we want to get the most from our resources. For instance, we may want to maximize our profits, or minimize the friction, or use the least amount of a material to build a box, subject to some fixed constraints. This means that we would like to find the maximum and minimum values of a function which describes the quantity in question.

Informally speaking, a maximum value of a function f is the largest value that the function assumes. Graphically, this corresponds to the biggest y-value that the equation $y = f(x)$ attains. Similarly, the minimum of f is the smallest value that the function assumes. The formal definition is as follows.

Definition 1.1 *Let $f(x)$ be a function with domain D. If there exists a point c in D such that $f(c) \leq f(x)$ for all x in D, then we call $f(c)$ the* ***minimum value*** *of f on D. If there exists a point d in D such that $f(d) \geq f(x)$ for all x in D, then we call $f(d)$ the* ***maximum value*** *of f on D.*

Here are some examples to illustrate the idea. Keep these examples in mind as we present some important theorems about maximums and minimums later.

1. If $C \in \mathbb{R}$ is a constant, then the constant function $f(x) = C$ has both a maximum and a minimum value of C which occurs at every x value.
2. The parabola $f(x) = x^2$ has a minimum value of 0 at $x = 0$ but it has no maximum value on $\mathbb{R}$. On the other hand, the function $g(x) = -x^2$ has a maximum value of 0 at $x = 0$ but it has no minimum value on $\mathbb{R}$.
3. The linear function $f(x) = x$ has neither a maximum nor a minimum on $\mathbb{R}$. However, it does have both a maximum and a minimum value on the closed interval $[0, 1]$; namely, $f(0) = 0$ is the minimum and $f(1) = 1$ is the maximum.
4. The cubic polynomial $f(x) = x^3 - 3x$ has neither a maximum nor a minimum on $\mathbb{R}$. However, there are two special points on this function that we should consider. First $f(-1) = 2$ is the biggest value that f attains for x values which are sufficiently close to $x = -1$. Second, $f(1) = -2$ is the smallest value that f attains for x values which are close enough to $x = 1$.
5. The absolute value function $f(x) = |x|$ has a minimum value 0 at $x = 0$ but no maximum value on $\mathbb{R}$.
6. The function $f(x) = \sin x$ has a maximum value 1 and a minimum value -1 on $\mathbb{R}$. Note that there are infinitely many real numbers x which yield the maximum and infinitely many that yield the minimum.

There is an important situation in which we can be guaranteed that a maximum value and a minimum value exist for a function. This situation is when the function is continuous on a closed interval, as the next theorem asserts.

Theorem 1.1 *If $f(x)$ is continuous on the closed interval $[a, b]$, then there are points c and d in $[a, b]$ for which $f(c) \leq f(x) \leq f(d)$ for all x in $[a, b]$.*

We will not prove this theorem here, but you should convince yourself that it is true. Try it out on the examples at the beginning of this section by taking a closed interval where the function is continuous and see if each of the functions attains a maximum and a minimum value in that interval.

It is important that the function be continuous on a closed interval for the function to be guaranteed to have a maximum and a minimum. Consider the following examples of functions for which the conclusion of the theorem fails.

1. The function $f(x) = \frac{1}{x}$ is continuous on the *open* interval $(0, 1)$ but it does not have a maximum value nor a minimum value on $(0, 1)$.
2. The function
$$g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
is defined on the closed interval $[-1, 1]$ but is not continuous there and it does not have a minimum value. It does have a maximum value 1 however.

Now that we know that a continuous function attains a maximum and a minimum value on a closed interval, we will try to develop a way of finding these points. Go back to the examples at the beginning of the section. What can we say about the tangent line of these functions at the maximum and minimum points (when we take the domain to be $\mathbb{R}$)? Notice that two things can happen: either the tangent line is horizontal (for instance $f(x) = x^2$ at $x = 0$) or the tangent line does not exist (like in $f(x) = |x|$ at $x = 0$). This is true in general as we will show. Before we do, we want to say something about points like those in $f(x) = x^3 - 3x$ at $x = -1$ and $x = 1$.

Definition 1.2 *Let $f(x)$ be a function with domain D. If there is a point c in D such that $f(c) \leq f(x)$ for all x in a sufficiently small open interval around c, then $f(c)$ is said to be a* ***local minimum value*** *of f. If there is a point d in D such that $f(d) \geq f(x)$ for all x in a sufficiently small open interval around d, then we say that $f(d)$ is a* ***local maximum value*** *of f.*

Theorem 1.2 *Let $f(x)$ be differentiable on its domain D. If c is a point of D for which $f(c)$ is a local maximum value or a local minimum value of $f(x)$, then $f'(c) = 0$.*

We will not prove this theorem here, but a consequence of the theorem is a way to find the maximum and minimum values of a continuous function on a closed interval. We state it (without proof) as the following theorem.

Theorem 1.3 *Suppose $f(x)$ is continuous on $[a, b]$. If c is a point of $[a, b]$ for which $f(c)$ is a maximum or minimum value of $f(x)$ on $[a, b]$, then one of the following must be true:*

(i) $f'(c) = 0$;

(ii) $f'(c)$ *does not exist;*

(iii) $c = a$ *or* $c = b$.

What this theorem means to us is that if we wish to find the maximum and minimum values of a continuous function $f(x)$ on a closed interval $[a, b]$, we need only do the following:

1. Make a list consisting of the endpoints a and b of the interval, the points in $[a, b]$ where $f'(x) = 0$, and the points in $[a, b]$ where $f'(x)$ does not exist.
2. Evaluate the function $f(x)$ at each of the points in the list.
3. Choose the smallest of these function values and the largest of these function values; these values will be respectively the minimum and the maximum value of $f(x)$ on $[a, b]$.

1.1 Applications to Optimization Problems

One important application for the theorems in this section is optimization problems. Often we are interested in getting the most from our resources. For example, perhaps you want to design a can with a certain fixed volume but using the least amount of material to do it, or maybe build a box with a fixed surface area having the largest possible volume. It turns out that we can do it many times by turning it into a problem of maximizing or minimizing a certain function on a closed interval domain.

Here is some suggestions when doing these types of problems.

1. Decide what quantity you need to maximize or minimize. This should guide you through the rest of the problem.
2. Construct a function which expresses the quantity you wish to maximize or minimize with respect to certain variables.
   (a) Draw a picture if you can.
   (b) Consider the separate pieces of the function that are connected in the function by '+' and '-' signs.
   (c) Use dimensional analysis to decide what you need to multiply by or divide by to get the correct quantity in the function.
3. Find the largest domain for which the function makes sense for the problem. The theorems of this section apply only if this is a closed interval.
4. Decide whether the function is continuous on the domain. Again, the theorems of this section apply only if this function is continuous on a closed interval domain.
5. Use the theorems of this section to find all the points in the closed interval where a maximum or a minimum can occur.
6. Evaluate the function at all of these points.
7. By considering the maximum or minimum values, answer the question. This may require some more computations depending on what you are trying to find.

Example. Suppose you want to make a cylindrical can with a bottom but no top having a fixed surface area of 300π in^2. What is the maximum possible volume of such a can?

To answer, this question, you first need to decide what quantity you are to optimize; in this case, it is the volume of a cylindrical can. Draw a picture of a such a can; this is just a cylinder with a bottom but no top, with radius r and height h. The volume function of such a can is $V(r,h) = \pi r^2 h$. The problem is that this is a function of two variables. We will use the constraint on the surface area to get rid of one of the variables in the function. The surface area formula for a cylinder and a circle gives the equation $2\pi rh + \pi r^2 = 300\pi$. Solving for h, we obtain

$$h = \frac{300 - r^2}{2r}$$

Substituting this into $V(r,h)$, we obtain

$$V(r) = \frac{\pi r^2(300 - r^2)}{2r} = 150\pi r - \frac{\pi}{2}r^3$$

We have simplified $V(r)$ to a polynomial, which is continuous everywhere, from a rational function which is not defined at $r = 0$. There is no harm in assuming that $V(r)$ is defined at $r = 0$.

We need a closed interval that describes the set of all r values that make sense to the problem. First, r must be positive, so let's start the interval at $r = 0$. But r cannot be too big, because then the surface area of the can would exceed 300π in^2. How big should r be? Well, as r gets larger, h must get smaller to compensate. The smallest that h can be is 0, and so setting $h = 0$ in the equation $2\pi rh + \pi r^2 = 300\pi$ and solving for r gives $r = \pm\sqrt{300} = \pm 10\sqrt{3}$. Consequently, our closed interval is $[0, 10\sqrt{3}]$.

As noted previously, the polynomial $V(r)$ is continuous everywhere so it is continuous on this closed interval. Consequently, the values for r where the maximum could occur must be $r = 0$, $r = 10\sqrt{3}$, or where $V'(r) = 0$ (since polynomials are differentiable everywhere). We compute $V'(r) = 150\pi - \frac{3\pi}{2}r^2$ and so $V'(r) = 0$ whenever $r = \pm 10$. Only $r = 10$ lies in the interval $[0, 10\sqrt{3}]$. Evaluating $V(r)$ at each of these three points gives $V(0) = 0 = V(10\sqrt{3})$ and $V(10) = 1000\pi$. Thus, the maximum possible volume of such a can is 1000π in^3.