# Limits and Continuity

## 1 The Definition of Limit

Suppose we are given a function $f(x)$ and a point $x_0$ in its domain. The basic idea of a limit is the following:

> $L$ is the **limit** of $f(x)$ as $x$ approaches $x_0$ if whenever $x$ is close to (but not equal to) $x_0$, $f(x)$ is close to $L$.

The formal definition is as follows.

**Definition 1.1** *Suppose $f(x)$ is a function and let $x_0$ be a number in its domain. Fix a real number $\epsilon > 0$ (think of $\epsilon$ as a very small positive number like 0.00001). If there exists a number $L$ and a number $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \quad \textit{implies that} \quad |f(x) - L| < \epsilon$$

*then we say that $L$ is the **limit** of $f$ as $x$ approaches $x_0$. In this case, we write*

$$\lim_{x \to x_0} f(x) = L.$$

What this says is that $L$ is the limit of $f(x)$ as $x$ approaches $x_0$ if whenever we fix a nonzero radius $\epsilon$ around $L$, there is a radius $\delta$ around $x_0$ such that any $x$ no more that a distance $\delta$ from $x_0$ determines a value $f(x)$ which is within the radius $\epsilon$ of $L$.

### Examples

1. If $f(x) = x^2$ and $x_0 = 1$, then the graph of $f(x)$ is a parabola with vertex at the origin and passing through the point $(1, 1)$. Graphically, it appears that $f(x)$ approaches $f(1) = 1$ as as $x$ approaches 1. A table of values such as the the one in Table 1 also supports the idea.

| $x$ | $f(x)$ |
|---|---|
| 1.1 | 1.21 |
| 0.9 | 0.81 |
| 1.01 | 1.0201 |
| 0.99 | 0.9801 |
| 1.001 | 1.002001 |
| 0.999 | 0.998001 |
| 1.0001 | 1.00020001 |
| 0.9999 | 0.99990001 |

2. The limit of a function $f$ as $x$ approaches $x_0$ may not exist. For example, if

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

then $\lim_{x\to 0} f(x)$ does not exist. For $x > 0$, $f(x) = 1$ and so you might think the limit is 1. But if $x < 0$, then $f(x) = -1$. Consequently, if we take $\epsilon = 1/2$, then no matter how small we choose the radius $\delta$ around $x_0 = 0$ it will always contain a positive number $a$ and its negative $-a$ and

$$|f(a) - f(-a)| = |1 - (-1)| = |1 + 1| = 2 > \frac{1}{2} = \epsilon$$

3. The limit $L$ of $f$ as $x$ approaches $x_0$ does not have to equal $f(x_0)$. For instance, if $f : \mathbb{R} \to \mathbb{R}$ is the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then $f(0) = 0$ but $\lim_{x\to 0} f(x) = 1$.

4. Let $f(x) = x$ and fix any real number $x_0$. Then for any $\epsilon > 0$, take $\delta = \epsilon$ and suppose $x$ is such that $|x - x_0| < \delta = \epsilon$. Then $|f(x) - f(x_0)| = |x - x_0| < \epsilon$. This shows that $\lim_{x\to x_0} x = x_0$.

# 2 Basic Laws of Limits

Using the definition to compute limits is often difficult. Our first theorem gives some "limit laws" which can be used to calculate the limits of many functions. In order to prove these laws, one must apply the definition. We will merely state the laws without proof here.

**Theorem 2.1 (Limit Laws)** *Let $f(x)$ and $g(x)$ be functions and fix a real number $x_0$. Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$.*

*(i) Constant Law: If $c$ is a constant, then*

$$\lim_{x\to x_0} c = c.$$

*(ii) Limit of the Identity:*

$$\lim_{x\to x_0} x = x_0.$$

*(iii) Sum and Difference Laws:*

$$\lim_{x\to x_0} (f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = L \pm M.$$

*(iv) Product Law:*

$$\lim_{x\to x_0} (f(x)g(x)) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right) = LM.$$

*(v) Quotient Law: If $M \neq 0$, then*

$$\lim_{x\to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L}{M}.$$

(vi) *Root Law: If $n$ is a positive integer and $x_0 > 0$ whenever $n$ is even, then*

$$\lim_{x \to x_0} \sqrt[n]{x} = \sqrt[n]{x_0}.$$

(vii) *General Root Law: If $n$ is a positive integer and $L > 0$ whenever $n$ is even, then*

$$\lim_{x \to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to x_0} f(x)} = \sqrt[n]{L}.$$

**Examples**

1. Let's use the limit laws to compute

$$\begin{aligned}\lim_{x \to 3}(2x^2 - 5x + 10) &= \lim_{x \to 3} 2x^2 - \lim_{x \to 3} 5x + \lim_{x \to 3} 10 \\ &= (\lim_{x \to 3} 2)(\lim_{x \to 3} x)^2 - (\lim_{x \to 3} 5)(\lim_{x \to 3} x) + 10 \\ &= (2)(3)^2 - (5)(3) + 10 = 13\end{aligned}$$

# 3 One-Sided Limits

Consider the function $f(x) = \sqrt{x}$. This function is undefined for all $x < 0$, and so the function *cannot* approach a limit as $x$ approaches 0 because the function simply does not exist to the left of 0. (Note that even though the limit doesn't exist, $f(0) = \sqrt{0} = 0$). However, there is a reasonable way that we can talk about the limit of this function as $f$ approaches 0 through values to the *right* of 0 (i.e., positive values of $x$).

The idea of a one-sided limit of a function is as follows:

> $L$ is the **right-hand limit** of a function $f$ as $x$ approaches $x_0$ if whenever $x > x_0$ and $x$ is close to (but not equal to) $x_0$, $f(x)$ is close to $L$ and in this case we write
>
> $$\lim_{x \to x_0^+} f(x) = L.$$
>
> $L$ is the **left-hand limit** of a function $f$ as $x$ approaches $x_0$ if whenever $x < x_0$ and $x$ is close to (but not equal to) $x_0$, $f(x)$ is close to $L$ and in this case we write
>
> $$\lim_{x \to x_0^-} f(x) = L.$$

The following are useful:

$$\lim_{x \to 0^+} \sqrt{x} = 0 \quad \text{but} \quad \lim_{x \to 0^-} \sqrt{x} \text{ does not exist.}$$

The next theorem relates one-sided limits to the (two-sided) limit.

**Theorem 3.1** *If $f(x)$ is a function which is defined in an open interval around $x_0$ except possibly at $x_0$, then*

$$\lim_{x \to x_0} f(x) = L$$

*if and only if*

$$\lim_{x \to x_0^+} = L = \lim_{x \to x_0^-} f(x).$$

This theorem implies that the limit laws of Theorem **??** hold for one-sided limits.

# 4 Infinite Limits

Recall that the function $f(x) = \frac{1}{x}$ has an asymptote at $x = 0$; as $x$ gets closer to 0 from the right side, $f(x)$ is positive and gets larger and larger and as $x$ gets closer to 0 from the left side, $f(x)$ is negative and gets larger and larger in absolute value. This means that the limit of $f(x)$ as $x$ approaches 0 does not exist. However, we have a way of denoting this behavior of $f$ near 0 as follows:

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

In general, if a function increases without bound or decreases without bound as it gets closer to $x_0$, we write

$$\lim_{x \to x_0} f(x) = +\infty \quad \text{or} \quad \lim_{x \to x_0} f(x) = -\infty.$$

There is another limit associated with infinity. This limit corresponds to horizontal asymptotes. For instance, $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$. Unlike the infinite limit of the previous paragraph, this limit exists and we write

$$\lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x}.$$

More generally, if $f(x)$ has a horizontal asymptote at $y = L$, then we write

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

We will use one-sided and infinite limits when we sketch graphs of functions. They will also come in handy when we talk about functions whose domain is a closed interval.

# 5 Continuity

The idea of continuity of a function $f(x)$ at a point $x_0$ is that $f(x_0)$ is defined and as $x$ gets close to $x_0$, $f(x)$ gets close to $f(x_0)$. The formal definition is given in terms of limits:

**Definition 5.1** *Let $f(x)$ be a function and let $x_0$ be a point in the domain of $f(x)$ which is contained in an open interval $(a, b)$ which contained in the domain of $f(x)$. We say that $f(x)$ is **continuous at** $x_0$ if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

*We say that $f(x)$ is **continuous** if it is continuous at each point $x_0$ of its domain. If $f(x)$ is not continuous at some point $x_0$ in its domain, then we say that $f(x)$ is **discontinuous at** $x_0$.*

If $n$ is a non-negative integer, then a **polynomial function of degree** $n$ is a function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where each $a_i$ is a real number and $a_n \neq 0$.

**Theorem 5.1** *If $f(x)$ is a polynomial function, then $f(x)$ is continuous everywhere.*

*Proof.* This follows from the sum, constant, and identity limit laws as follows. Let $x_0$ be any real number. Then

$$\begin{aligned}\lim_{x\to x_0}(a_nx^n &+ a_{n-1}x^{n-1}+\cdots+a_1x+a_0)\\ &= \lim_{x\to x_0} a_nx^n + \lim_{x\to x_0} a_{n-1}x^{n-1}+\cdots+\lim_{x\to x_0} a_1x+\lim_{x\to x_0} a_0\\ &= (\lim_{x\to x_0} a_n)(\lim_{x\to x_0} x)^n + (\lim_{x\to x_0} a_{n-1})(\lim_{x\to x_0} x)^{n-1}+\cdots+(\lim_{x\to x_0} a_1)(\lim_{x\to x_0} x)+a_0\\ &= a_nx_0^n+a_{n-1}x_0^{n-1}+\cdots+a_1x_0+a_0=f(x_0)\end{aligned}$$

which is what we needed to show. □

The next theorem is useful for us to determine easily whether a function is continuous. Its proof follows directly from the limit laws of Theorem **??** and the definition of continuity.

**Theorem 5.2** *Let $f(x)$ and $g(x)$ be continuous functions with the same domain $D$ and let $c$ be a constant.*

*(i) (Identity Function) The identity function $id(x)=x$ is continuous.*

*(ii) (Constant Multiple of a Function) The function $cf$ given by $(cf)(x)=cf(x)$ is continuous.*

*(iii) (Sums and Differences of Functions) The functions $f\pm g$ given by $(f\pm g)(x)=f(x)\pm g(x)$ are continuous.*

*(iv) (Product of Functions) The function $fg$ given by $(fg)(x)=f(x)g(x)$ is continuous.*

*(v) (Quotients of Functions) The function $f/g$ given by $(f/g)(x)=\frac{f(x)}{g(x)}$ is continuous at every point in $D$ except where $g(x)=0$.*

*(vi) (Compositions of Functions) If $g(x)$ is in $D$ whenever $x$ is in $D$, then the function $f\circ g$ given by $(f\circ g)(x)=f(g(x))$ is continuous.*

*(vii) (Roots of Functions) If $n\geq 2$ is any integer, then the function $h(x)=\sqrt[n]{f(x)}$ is continuous on its domain except where $f(x)=0$ when $n$ is even.*

One consequence of this theorem is that quotients of polynomials

$$f(x)=\frac{p(x)}{q(x)}$$

(called **rational functions**) are continuous wherever $q(x)\neq 0$.

We will often be interested in computing the limit of a continuous function at a point in its domain. If we know $f(x)$ is continuous at $x_0$, then

$$\lim_{x\to x_0} f(x)=f(x_0)$$

**Examples**

1. We will prove that $f(x)=\sqrt{4-x^2}$ is continuous on $(-2,2)$. First, if $x_0$ is in the interval $(-2,2)$, then $4-x_0^2>0$. Hence, $f(x_0)=\sqrt{4-x_0^2}$. On the other hand,
$$\lim_{x\to x_0}(4-x^2)=\lim_{x\to x_0}4-(\lim_{x\to x_0}x)^2=4-x_0^2>0$$

since $x_0$ is in $(-2, 2)$. Consequently, we can apply the generalized root law and obtain

$$\lim_{x \to x_0} \sqrt{4 - x^2} = \sqrt{\lim_{x \to x_0} (4 - x^2)} = \sqrt{4 - x_0^2} = f(x_0)$$

which shows that $f(x)$ is continuous at any $x_0$ in $(-2, 2)$.

2. Let's compute the following limit:

$$\lim_{t \to -3} \frac{t^2 + 6t + 9}{t^2 - 9}$$

We want to compute the limit of a rational function and we know it is continuous except where $0 = t^2 - 9 = (t - 3)(t + 3)$. That is, it is continuous except where $t = \pm 3$. As we want to compute the limit of this function at $t = -3$, we cannot use continuity to do it. Note that $t^2 + 6t + 9 = (t + 3)^2$ and since we are only looking at values close to $t = -3$ and not $t = -3$ itself, we can cancel the factor $(t + 3)$ from the function and obtain

$$\lim_{t \to -3} \frac{(t + 3)^2}{(t + 3)(t - 3)} = \lim_{t \to -3} \frac{t + 3}{t - 3}$$

Now this new rational function is continuous except at $t = 3$ and so we can use continuity to compute its limit at $t = -3$:

$$\lim_{t \to -3} \frac{(t + 3)^2}{(t + 3)(t - 3)} = \lim_{t \to -3} \frac{t + 3}{t - 3} = \frac{(-3) + 3}{(-3) - 3} = \frac{0}{-6} = 0$$

3. Let's try

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x}$$

The previous theorem asserts that $\sqrt{x}$ is continuous on $(0, \infty)$ and therefore that the function $\frac{3 - \sqrt{x}}{9 - x}$ is continuous on $(0, 9) \cup (9, \infty)$. As it is not continuous at $x = 9$, we cannot use continuity to compute the limit at this point. Let's multiply top and bottom of the function by $3 + \sqrt{x}$ (this is the **conjugate** of $3 - \sqrt{x}$) to obtain

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}} = \lim_{x \to 9} \frac{9 - x}{(9 - x)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{1}{3 + \sqrt{x}}$$

This last function is continuous for all $x$ in $(0, \infty)$ and in particular at $x = 9$. Consequently,

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} = \lim_{x \to 9} \frac{1}{3 + \sqrt{x}} = \frac{1}{3 + \sqrt{9}} = \frac{1}{3 + 3} = \frac{1}{6}$$

# 6 One-Sided Continuity

When a function is defined on a closed interval (or a half-closed interval) we need to modify our definition of continuity using one-sided limits to discuss **continuity from the right** or **continuity from the left**. To check continuity from the right, we must verify that

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

and to check continuity from the left, we must verify that

$$\lim_{x \to x_0^-} f(x) = f(x_0).$$

For instance, $f(x) = \sqrt{x}$ has domain $[0, \infty)$. It is continuous on $(0, \infty)$ and right continuous at $x = 0$.

# 7 Intermediate Value Theorem

If you have two points, one above the $x$-axis and one below the $x$-axis, you cannot trace out a path from one point to the other without crossing the $x$-axis. This corresponds to an important property of continuous functions which is called the Intermediate Value Theorem.

**Theorem 7.1 (Intermediate Value Theorem)** *Let $a < b$ be real numbers. If $f(x)$ is continuous on $[a, b]$ and $y_0$ lies between $f(a)$ and $f(b)$, then there exists $c$ in $(a, b)$ such that $f(c) = y_0$.*

The Intermediate Value theorem is not true in general if $f$ is not continuous or the domain of $f$ is not a closed interval.

**Example.** We will use the Intermediate Value Theorem to prove that the equation $x^4 + 2x - 1 = 0$ has a solution in the interval $[0, 1]$. First, note that solutions of the equation correspond to roots of the function $f(x) = x^4 + 2x - 1$, which is continuous on $[0, 1]$ because it is a polynomial. Now, $f(0) = -1$ and $f(1) = 2$, and we have $f(0) < 0 < f(1)$. Therefore, there exists a number $c$ in the interval $(0, 1)$ such that $f(c) = 0$. That is, $c^4 + 2c - 1 = 0$ and so $c$ is a solution of the equation.

# 8 Trigonometric Limits

There are three important limits involving trigonometric functions. These three limits will be our basic tools in computing limits of any function which is built from the six basic trigonometric functions. We also will give the proofs of these limits so that you can see the process of proving a limit formally.

**Theorem 8.1 (Fundamental Trigonometric Limits)** *Let $\theta$ be an angle given in radians (so it is just a real number). Then*

*(i)*

$$\lim_{\theta \to 0} \sin \theta = 0;$$

*(ii)*

$$\lim_{\theta \to 0} \cos \theta = 1;$$

*(iii)*

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

*Proof.* We will use the definition of the limit. Figure 1 should help visualize what what is going on. Let $\epsilon > 0$ be given and take $\delta = \epsilon$. Suppose $\theta$ is such that $0 < |\theta - 0| = |\theta| < \delta = \epsilon$ (note that $\theta$ can be positive or negative). Consider the point $P = (\cos \theta, \sin \theta)$ on the unit circle. If $d$ is the distance from $P$ to $Q = (1, 0)$, then $d$ is the length of the hypotenuse of the triangle $\triangle PQR$ and $|\sin \theta|$ is the length of the edge $\overline{PR}$ of the same triangle and so $|\sin \theta| < d$. Recall that the length of the arc $\widehat{PQ}$ is $|s| = |r\theta| = |\theta|$ (where $r = 1$ is the radius of the circle). Clearly the length of the line segment $\overline{PQ}$ is less than the length of the arc $\widehat{PQ}$ when $|\theta| > 0$ and so we have

$$|\sin \theta| < d < |s| = |\theta| < \epsilon$$

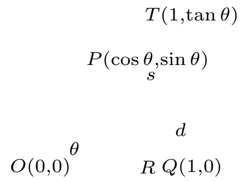


Figure 1: Proof of the Fundamental Trigonometric Limits

so by the the definition of the limit, $\lim_{\theta\to 0}\sin\theta = 0$ as claimed.

To prove the second limit, we will compute $d^2$ (where $d$ is the length from $P$ to $Q$ in the figure) using the distance formula:

$$\begin{aligned} d^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2 \\ &= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta \\ &= (\sin^2\theta + \cos^2\theta) + 1 - 2\cos\theta = 2(1-\cos\theta). \end{aligned}$$

This time, take $\delta = \sqrt{2\epsilon}$ in the definition of the limit. Then again using the fact that the length $d$ of the line segment $\overline{PQ}$ is less than the length $s$ of the arc $\widehat{PQ}$, we have

$$|\cos\theta - 1| = 1 - \cos\theta = \frac{1}{2}d^2 < \frac{1}{2}|s|^2 = \frac{1}{2}|\theta|^2 < \frac{1}{2}\delta^2 = \epsilon.$$

This shows that $\lim_{\theta\to 0}\cos\theta = 1$.

To prove the last limit, consider the similar triangles $\triangle OPR$ and $\triangle OTQ$. Using this similarity of triangles, we have

$$|\overline{QT}| = \frac{|\overline{QT}|}{1} = \frac{|\overline{RP}|}{|\overline{OR}|} = \left|\frac{\sin\theta}{\cos\theta}\right| = |\tan\theta|.$$

(Note that $\overline{QT}$ is tangent to the unit circle at the point $P = (1,0)$ and so the equation $|\overline{QT}| = |\tan\theta|$ may indicate to you why the name was chosen for the trigonometric function tan). Computing the area of the triangle $\triangle OPR$, the *sector* $OPQ$, and the triangle $\triangle OTQ$, we have respectively:

$$A_1 = \frac{1}{2}\sin\theta\cos\theta, \quad A_2 = \frac{1}{2}r^2\theta = \frac{1}{2}\theta, \quad A_3 = \frac{1}{2}\tan\theta.$$

Since $A_1 < A_2 < A_3$, we have

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{\sin\theta}{2\cos\theta} \quad\Rightarrow\quad \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

(note that we can divide by $\sin\theta$ since $\theta \neq 0$). Taking the reciprocal of these three expressions reverses the inequality and we have

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}.$$

Now we will take the limit of each of these functions as $\theta$ goes to 0, noting that we maintain the direction of the inequality but lose the strictness of it (since a function may approach a given limit but never actually attain the value):

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le \lim_{\theta\to 0}\frac{1}{\cos\theta} \quad\Rightarrow\quad 1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1.$$

The last limit follows. □

Note that these proofs required the formula for arc length $s = r\theta$ which requires that $\theta$ be given in radians, NOT IN DEGREES! Consequently, all results in calculus involving trigonometric functions require that the domain be given in radians.

We will now use the sum laws for the sine and cosine to prove that sin and cos are continuous functions on all of the real numbers.

**Theorem 8.2** *The functions* $\sin, \cos : \mathbb{R} \to [-1, 1]$ *are continuous.*

*Proof.* Let $x_0$ be any real number. If $h$ is a real number and $x = x_0 + h$, then $x$ approaches $x_0$ if and only if $h$ approaches 0. Thus, using the sum identity for sin as well as the limit laws, we have

$$\begin{aligned}\lim_{x\to x_0}\sin(x) = \lim_{h\to 0}\sin(x_0+h) &= \lim_{h\to 0}\big(\sin x_0\cos h + \sin h\cos x_0\big)\\ &= \big(\lim_{h\to 0}\sin x_0\big)\big(\lim_{h\to 0}\cos h\big) + \big(\lim_{h\to 0}\sin h\big)\big(\lim_{h\to 0}\cos x_0\big)\\ &= (\sin x_0)(1) + (0)(\cos x_0) = \sin x_0.\end{aligned}$$

This shows that $\sin x$ is continuous.

Similarly, if we use the sum law for cos along with the limit laws, we compute

$$\begin{aligned}\lim_{x\to x_0}\cos(x) = \lim_{h\to 0}\cos(x_0+h) &= \lim_{h\to 0}\big(\cos x_0\cos h - \sin h\sin x_0\big)\\ &= \big(\lim_{h\to 0}\cos x_0\big)\big(\lim_{h\to 0}\cos h\big) - \big(\lim_{h\to 0}\sin h\big)\big(\lim_{h\to 0}\sin x_0\big)\\ &= (\cos x_0)(1) - (0)(\sin x_0) = \cos x_0\end{aligned}$$

showing that $\cos x$ is continuous. □

**Examples.**

1. Let's evaluate the limit

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta} = \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{1}{\cos\theta}\right) = 1$$

2. Now let's try

$$\lim_{x\to 0}\frac{1-\cos 2x}{x}$$

If we multiply top and bottom by $1 + \cos 2x$, then we get

$$\lim_{x\to 0}\frac{1-\cos^2 2x}{x(1+\cos 2x)} = \lim_{x\to 0}\frac{\sin^2 2x}{x(1+\cos 2x)}$$

by the Pythagorean identity. Multiply top and bottom by 2, and we obtain

$$\lim_{x\to 0}\frac{2\sin^2 2x}{2x(1+\cos 2x)} = 2\left(\lim_{u\to 0}\frac{\sin u}{u}\right)\left(\lim_{u\to 0}\frac{\sin u}{(1+\cos u)}\right) = (2)(1)\left(\frac{0}{1+1}\right) = 0$$

where $u = 2x$. Note that $\lim_{x\to 0} 2x = 0 = \lim_{u\to 0} u$.