Implicit Differentiation

1 Implicit Differentiation

In this section, we will discuss a nice tool to differentiate any equation, even those equations which do not correspond to a function. As an example, consider the equation of a circle

$$x^2 + y^2 = 1.$$

This equation does not correspond to a function; however, if we solve for y we get two equations, one for the upper semicircle and one for the lower semicircle:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}.$$

These two equations, which we found explicitly from the one original equation, do in fact correspond to functions.

The power of implicit differentiation is that we do not need to express one variable explicitly in terms of the other variable(s). Instead, we assume one variable is defined implicitly in terms of the other(s) and view the equation as an identity; that is, the equation holds for every point in a certain set. For example, the equation $x^2 + y^2 = 1$ holds true for every point which is a distance of 1 unit from the origin (0,0) (i.e., all points on the unit circle). We then use the chain rule to differentiate both sides of the identity with respect to one of the variables. Since we are viewing the equation as an identity, both sides must still be equal to each other after we differentiate them.

Examples.

1. We will differentiate the equation $x^2 + y^2 = 1$ with respect to x. We assume this is an identity and that y is defined implicitly in terms of x. Differentiating both sides with respect to x yields

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Notice that if we differentiate the two equations representing functions that come from $x^2 + y^2 = 1$, we get

$$\frac{dy}{dx} = -\frac{x}{y} = \mp\frac{x}{\sqrt{1 - x^2}}.$$

2. Consider the following equation:

$$x^3 + y^3 = 3xy.$$

The graph of this equation is called the *folium of Descartes*. It is very challenging to solve this equation explicitly for y in terms of x. This curve was first proposed by René Descartes as a challenge to Pierre de Fermat in the 1600's to find its tangent line at any point where a tangent line is defined. We will use implicit differentiation to do just that. Differentiating both sides of this equation with respect to x, we obtain

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}.$$

Note that the point $(\frac{3}{2}, \frac{3}{2})$ is on the folium. To find the tangent line to the graph at that point, we evaluate the derivative at that point, this time using both the x and the y coordinates:

$$\frac{dy}{dx}|_{(\frac{3}{2},\frac{3}{2})} = \frac{\frac{3}{2} - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \frac{3}{2}} = -1.$$

One powerful use of implicit differentiation is that of **related rates**; i.e., rates that are related to one another in some way, given by an equation. The idea is as follows:

1. Determine what you are asked to find and express it mathematically using differential notation.
2. Decide what variables you need; a picture is often helpful. Each variable can be thought of as a function of time (implicitly defined) because the quantities the variables represent change as time passes.
3. Express what you are given as an equation involving the necessary variables.
4. Using implicit differentiation, take the derivative of each side of the equation with respect to time. You will then have a second equation relating the rates of change of the variables.
5. Find what you are asked for by evaluating and solving the two equation appropriately.

2 Linear Approximations

This section is not covered in my course, but is given here for any interested readers.

We defined the tangent line to a function $f(x)$ at a point x_0 to be the line which is indistinguishable from $f(x)$ for x-values near enough to x_0. Of course, such a line exists only if $f(x)$ is differentiable at x_0. For differentiable functions, we thus have the following:

> The linear function that most closely approximates $f(x)$ for x near x_0 is the linear function
>
> $$L(x) = L^f_{x_0}(x) = f'(x_0)(x - x_0) + f(x_0)$$
>
> which is called the **linear approximation to f near x_0**.

Note that the graph of $y = L^f_{x_0}(x)$ is the tangent line to the graph of $y = f(x)$ at x_0. We write $f(x) \approx L^f_{x_0}(x)$ for x near x_0, which we read "$f(x)$ is approximately $L^f_{x_0}(x)$ if x is close to x_0". Linear approximations are best used when x_0 is a point for which both $f(x_0)$ and $f'(x_0)$ are easy to compute.

Since linear functions are the easiest functions to deal with, the linear approximation to a function has historically been indispensable. In this age of computers, they are less important for applications but still vital in theoretical situations.

Examples.

1. Consider the function
$$f(x) = \frac{1}{\sqrt{1+x}}.$$
We will find $L_0^f(x)$. First we compute that $f(0) = 1$. Next, we find the derivative of f to be
$$f'(x) = D_x((1+x)^{-\frac{1}{2}}) = -\frac{1}{2(1+x)^{\frac{3}{2}}}.$$
Hence, $f'(0) = -\frac{1}{2}$. Therefore, the linear approximation to f at 0 is
$$L_0^f(x) = -\frac{1}{2}(x-0) + 1 = -\frac{1}{2}x + 1.$$
To demonstrate that this really is an approximation of f near 0, we evaluate
$$L_0^f(.1) = -\frac{1}{2}\frac{1}{10} + 1 = 1 - \frac{1}{20} = \frac{19}{20} = .95.$$
On the other hand, a calculator gives $f(.1) \approx .95346258924559$ so the approximation is good to two decimal places!

2. Lets compute $L_0^g(x)$ where $g(x) = \ln(1+x)$. First, $g(0) = \ln(1) = 0$. Also,
$$g'(x) = \frac{1}{1+x}$$
and so $g'(0) = 1$. Thus, $L_0^g(x) = x$. Note that $L_0^g(.01) = .01$ and a calculator gives $g(.01) = .009503308531681$.

3. Suppose we did not have a calculator and wanted to know approximately what the value for $\sqrt{80}$ is. One approach would be to use a linear approximation. In order to do this, we need a function and an appropriate point. We note that $\sqrt{80} = f(80)$, where $f(x) = \sqrt{x}$. Furthermore, $x_0 = 81$ is a perfect square which is close to 80. We have $f(81) = 9$ and we compute the derivative of f to be $f'(x) = \frac{1}{2\sqrt{x}}$. Hence, $f'(81) = \frac{1}{18}$ and therefore
$$L_{81}^f(x) = \frac{1}{18}(x-81) + 9.$$
Now this linear function allows us to easily compute
$$\sqrt{80} = f(80) \approx L_{81}^f(80) = -\frac{1}{18} + 9 = 9 - .0555... = 8.9444....$$
To see how close this approximation is, note that a calculator will give $\sqrt{80} \approx$ 8.9442719099992. Not too bad!

4. Let's use a linear approximation to approximate the number $\sin 32°$. Since $\sin 32° = \sin\frac{8\pi}{45} = f(8\pi/45)$ where $f(x) = \sin x$. Of course, $30° = \frac{\pi}{6}$ is close to $32°$ and is a special angle on the unit circle. Recall that $f(\pi/6) = \sin\frac{\pi}{6} = \frac{1}{2}$. Furthermore, $f'(\pi/6) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Hence we have
$$L_{\pi/6}^f(x) = \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) + \frac{1}{2}.$$
We compute therefore that
$$\sin 32° = f(8\pi/45) \approx L_{\pi/6}^f(8\pi/45) = \frac{\sqrt{3}}{2}(\frac{8\pi}{45} - \frac{pi}{6}) + \frac{1}{2} = \frac{\sqrt{3}}{2}\frac{\pi}{90} + \frac{1}{2} = \frac{\sqrt{3}\pi}{180} + \frac{1}{2}.$$
Of course, this is not something we are use to evaluating without a calculator, so in a sense we are no better off than we were before. I will leave it to you to use a calculator to see that this number is on fact close to $\sin 32°$.