# Derivatives of Transcendental Functions

## 1 Derivatives of Trigonometric Functions

The goal now is to find the derivatives are of the six trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$. First we will find the derivatives of the functions $\sin x$ and $\cos x$. Then, since the other trigonometric functions are expressed in terms of these two, we will apply our differentiation rules that we already have to find the derivatives of the other four functions.

In the proof of the fundamental trigonometric limit

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \tag{1}$$

we used the fact that $\theta$ is measured in *radians*! As we will use this limit in computing the derivative of $\sin x$ and $\cos x$, this means that the formulas only hold if the domain of the trigonometric functions are given in radians. Recall from trigonometry that radians are actually just real numbers, without units. That is, the $x$-values we put into the functions $\sin x$ and $\cos x$ are NOT MEASURED IN DEGREES but are real numbers without units! With this in mind, it is useful to have the following conversion relationship between degrees and radians:

$$\pi = 180^\circ.$$

Using the basic trigonometric limit (1), we can compute another useful trigonometric limit:

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0. \tag{2}$$

With these two limits, we are ready to show the following differentiation rules for $\sin x$ and $\cos x$ using the definition of the derivative.

**Theorem 1.1** *The functions* $\sin(x), \cos(x)$ *are differentiable everywhere and their derivatives are given by*

$$D_x(\sin x) = \cos x \quad \textit{and} \quad D_x(\cos x) = -\sin x.$$

*Proof.* Let $x$ be a real number. Using the definition of the derivative, the sum identity for $\sin x$, and the trigonometric limits (1) and (2), we compute

$$\begin{aligned}
D_x(\sin x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \sin h - \sin x(1 - \cos h)}{h} \\
&= (\cos x)\left(\lim_{h \to 0} \frac{\sin h}{h}\right) - (\sin x)\left(\lim_{h \to 0} \frac{(1 - \cos h)}{h}\right) \\
&= (\cos x)(1) - (\sin x)(0) = \cos x.
\end{aligned}$$

Similarly,

$$\begin{aligned} D_x(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\ &= \lim_{h\to 0} \frac{(\cos x\cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{-\sin x \sin h - \cos x(1-\cos h)}{h} \\ &= -(\sin x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) - (\cos x)\left(\lim_{h\to 0}\frac{(1-\cos h)}{h}\right) \\ &= -(\sin x)(1) - (\cos x)(0) = -\sin x. \end{aligned}$$

Since $\sin x$ and $\cos x$ exist for every real number $x$, these limits exist for each real number $x$ and so the functions $\sin x$ and $\cos x$ are differentiable on $\mathbb{R}$. □

The product rule, quotient rule, power rule, and chain rule can by used in conjunction with the derivatives of $\sin x$ and $\cos x$. In fact, we will use these rules to find the derivatives of the other four trigonometric functions now.

**Theorem 1.2** *The functions* $\tan(x), \sec(x), \cot(x), \csc(x)$ *are differentiable on their domains and*

$$\begin{array}{ll} D_x(\tan x) = \sec^2 x & D_x(\cot x) = -\csc^2 x \\ D_x(\sec x) = \sec x \tan x & D_x(\csc x) = -\csc x \cot x. \end{array}$$

*Proof.* By the quotient rule, $\tan x$ is differentiable on its domain and

$$\begin{aligned} D_x(\tan x) &= D_x\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{D_x(\sin x)\cos x - \sin x D_x(\cos x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x. \end{aligned}$$

The derivative of $\cot x = \frac{\cos x}{\sin x}$ is similar.

By the chain rule and power rule, $\sec x$ is differentiable on its domain and we compute

$$\begin{aligned} D_x(\sec x) &= D_x\left((\cos x)^{-1}\right) \\ &= (-1)(\cos x)^{-2}(D_x(\cos x)) \\ &= (-\cos x)^{-2}(-\sin x) \\ &= \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x. \end{aligned}$$

The derivative of $\csc x = (\sin x)^{-1}$ is similar. □

# 2 Derivatives of Exponential and Logarithmic Functions

In this section, we will find the derivatives of exponential and logarithmic functions. First, remember what an exponential function is.

**Definition 2.1** *If $a > 0$, then the function*

$$f(x) = a^x$$

*is called an **exponential function** and the number $a$ is called its **base**.*

**Examples.**

1. The function $f(x) = 2^x$ is the exponential function with base 2. It is useful when you talk about doubling a quantity. For instance, what if you want to know how thick a folded paper is if you fold it onto itself 100 times. Well, after the first fold, it is twice as thick. After the second fold it is four times as thick as it originally was. After three folds it is eight times as thick. We conclude that
$$\text{thickness} = 2^{100}(\text{thickness of the paper}).$$
Suppose you start with a paper which is 0.004 inches thick. The resulting thickness after 100 folds would then be
$$(0.004)2^{100} = (0.004)(1.26765 \times 10^{30}) = 5.07060 \times 10^{27}$$
inches thick; i.e., it is about $8.00284 \times 10^{22}$ miles. To give you an idea of how thick that is, the distance to the sun in about $9.3 \times 10^7$ miles. This means that this thickness is more than *860 trillion times the distance from the earth to the sun*! This example illustrates how quickly an exponential functions grows.

2. The most important exponential function, as we will soon see, is the exponential function $f(x) = e^x$ whose base is Euler's number defined by the following limit:
$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$
Euler's number $e$ is known to be an irrational number and it is approximately
$$e \approx 2.718281828459045.$$

Let $a > 0$, $b > 0$, and let $r$ and $s$ be real numbers. The following are the laws of exponents:

| | | |
|---|---|---|
| 1. $a^{r+s} = a^r a^s$ | 2. $(a^r)^s = a^{rs}$ | 3. $(ab)^r = a^r b^r$ |
| 4. $a^{-r} = \frac{1}{a^r}$ | 5. $a^0 = 1$. | |

Notice that we have stated these laws for *real valued exponents*. We know what is meant by a natural number exponent, and a negative integer exponent, and an exponent of 0. We also know what we mean when the exponent is rational (in terms of powers and roots). It is not clear what we mean by an irrational number exponent. We will not go into that in detail here. We will only say that they are important for exponential functions and they are basically defined by extending an exponential function defined on the rational numbers to a continuous function defined on the real numbers (using the fact that for each irrational number, there is a rational number arbitrarily close to it).

The graph of an exponential function $f(x) = a^x$ for $a > 0$ has the following properties: it is an increasing function on $\mathbb{R}$, it passes through the points $(0, 1)$ and $(1, a)$, grows very quickly to the right of $x = 1$ and decays to 0 as $x$ goes to $-\infty$.

Before we find the derivative of the exponential, we state one important theorem.

**Theorem 2.1** *The function $f(x) = e^x$ is differentiable at the point (0,1) and*

$$\lim_{x \to 0} \frac{e^x - 1}{x} = f'(0) = 1.$$

This is a difficult theorem to prove, but it will allow us to prove the following if we accept it as true (feel free to analyze the limit using MAPLE!).

**Theorem 2.2** *The function $f(x) = e^x$ is differentiable on $\mathbb{R}$ and*

$$D_x(e^x) = e^x.$$

*Proof.* We just use the definition of the derivative and the previous theorem as follows:

$$D_x(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot f'(0) = e^x$$

since $f'(0) = 1$. □

The function $f(x) = e^x$ is the only function with the property that $f(x) = f'(x)$. This turns out to be a very important property, particularly in differential equations (which we will discuss later on).

Now to discuss the derivative of a logarithmic function. Recall that the logarithm is the inverse of the exponential function in the sense of the following definition.

**Definition 2.2** *Let $a > 0$. The **logarithm with base a** is the function $f(x) = \log_a x$ defined for $x > 0$ with the property that*

$$y = \log_a x \quad \textit{if and only if} \quad a^y = x.$$

We have the following important properties of the logarithm with base $a > 0$.

| | | | |
|---|---|---|---|
| 1. | $\log_a(xy) = \log_a x + \log_a y$ | 2. | $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ |
| 3. | $\log_a(x^r) = r \log_a x$ | 4. | $\log_a 1 = 0$ |
| 5. | $\log_a a^x = x$ | 6. | $a^{\log_a x} = x.$ |

The **natural logarithm** is the logarithm of base $e$ and is denoted $\ln x := \log_e x$. As with exponential functions, it is the most important of the logarithmic functions. To compute its derivative, we use the following theorem.

**Theorem 2.3 (Inverse Function Theorem)** *Suppose $f(x)$ is a function with domain $D$ and range $R$. If $f(x)$ is differentiable on $D$ and $f'(x) \neq 0$ for all $x$ in $D$, then $f(x)$ has an inverse function $f^{-1}(x)$ with domain $R$ and range $D$, and $f^{-1}(x)$ is differentiable on $R$ with*

$$D_y(f^{-1})(y) = \frac{1}{f'(f^{-1}(y))}$$

*for all $y$ in $R$.*

*Proof.* Since $f(f^{-1}(y)) = y$ and $f(x)$ is differentiable, we can differentiate both sides of the equation and we get

$$f'(f^{-1}(y))(f^{-1})'(y) = 1 \quad \Rightarrow \quad (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

by the chain rule and the fact that $f'(f^{-1}(y)) \neq 0$ for all $x$ in $D$. (Note that $f^{-1}(y)$ is an element of $D$). □

Now we are ready to find the derivative of the natural logarithm.

**Theorem 2.4** *The natural logarithm function* $\ln x$ *is differentiable on* $(0, \infty)$ *and its derivative is*

$$D_x(\ln x) = \frac{1}{x}.$$

*Proof.* Since $e^x$ has domain $\mathbb{R}$ and range $(0, \infty$ and $D_x(e^x) = e^x \neq 0$ for all $x$, the Inverse Function Theorem applies and we get

$$D_x(\ln x) = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

as claimed. □

We finish our discussion with the next theorem which gives us formulas for the derivative of a general exponential function and a general logarithm function.

**Theorem 2.5** *If* $a > 0$*, then* $f(x) = a^x$ *and* $g(x) = \log_a x$ *are differentiable on their domains and*

$$D_x(a^x) = (\ln a)a^x \quad \textit{and} \quad D_x(\log_a x) = \frac{1}{x \ln a}.$$

*Proof.* First, note that

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

and so we use the chain rule to get

$$D_x(a^x) = e^{x \ln a} \ln a = a^x(\ln a).$$

For the logarithm, we use the change of base formula for logarithms:

$$\log_a x = \frac{\ln x}{\ln a}$$

and since the denominator is a constant, we have

$$D_x(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$

as claimed. □