CAPACITY THEORY AND ARITHMETIC INTERSECTION THEORY

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Abstract

We show that the sectional capacity of an adelic subset of a projective variety over a number field is a quasi-canonical limit of arithmetic top self-intersection numbers, and we establish the functorial properties of extremal plurisubharmonic Green's functions. We also present a conjecture that the sectional capacity should be a top self-intersection number in an appropriate adelic arithmetic intersection theory.

Contents

0. Introduction

In this paper we establish a formula for the sectional capacity of an adelic set in terms of arithmetic intersection theory. An adelic set is a subset of the adelic points of a projective variety over a number field. The sectional capacity of such a set is an arithmetic measure of its size relative to an ample divisor. The sectional capacity generalizes the classical logarithmic capacity to an adelic setting on varieties of arbitrary dimension.

Our main theorem asserts that the sectional capacity is a limit of top self-intersection numbers of metrized fractional line bundles in Gillet-Soulé's theory. These line bundles are supported on a canonical sequence of models determined by the nonarchimedean components of the set, with metrics given by smoothings of extremal

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plurisubharmonic Green's functions attached to the archimedean components of the set. The main theorem gives the first explicit relation between capacity theory and arithmetic intersection theory in dimensions $d \geq 2$; the connection is made via extremal plurisubharmonic Green's functions and H. Gillet and C. Soulé's arithmetic amplitude theorem.

The result above is the strongest one we can formulate within Gillet-Soulé's theory. However, we expect that the sectional capacity should be given exactly by a top self-intersection number of an adelic metrized line bundle with singular metrics, in an appropriate adelic intersection theory. The conjecture and the nature of the desired intersection theory are discussed in the final section of the paper.

We now introduce the three objects involved in the main theorem.

The sectional capacity

Let K be a number field, and let X/K be a smooth, connected, projective variety. Let D be an ample, effective, K-rational Cartier divisor on X, and for each place v of K, let $E_v \subset X(\mathbb{C}_v)$ be a nonempty set, disjoint from the support of D and stable under the group $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$ of continuous automorphisms of $\mathbb{C}_v/K_v$. ($\mathbb{C}_v$ denotes the completion of the algebraic closure of K_v.) Dehomogenizing at D, identify sections of $\mathscr{O}_X(nD)$ with functions in $K(X)$:

$$\Gamma(nD) = H^0\big(X, \mathscr{O}_X(nD)\big) = \big\{ f \in K(X) : \mathrm{div}(f) + nD \geq 0 \big\}.$$

For each v, $K_v \otimes_K \Gamma(nD) \subseteq K_v(X)$. For $f \in K_v(X)$, define

$$\| f \|_{E_v} = \sup_{z \in E_v} \big(|f(z)|_v \big).$$

In this generality it is possible that $\| f \|_{E_v} = \infty$. However, we are dealing only with sets and functions for which $\| f \|_{E_v}$ is always finite.

For each $n \geq 0$, put $\mathscr{B}(E_v, nD) = \{ f \in K_v \otimes_K \Gamma(nD) : \| f \|_{E_v} \leq 1 \}$. Let $\mathbb{A}_K$ be the adèle ring of K. Write $\mathbb{E} = \prod_v E_v$, and put

$$\mathscr{B}(\mathbb{E}, nD) = \big(\mathbb{A}_K \otimes_K \Gamma(nD) \big) \cap \prod_v \mathscr{B}(E_v, nD),$$

the "ball of adelic sections with sup norm at most 1 at each place." Let $\mathrm{vol}_{\mathbb{A}}$ be any Haar measure on $\mathbb{A}_K \otimes_K \Gamma(nD)$. Viewing $\Gamma(nD)$ as a lattice in $\mathbb{A}_K \otimes_K \Gamma(nD)$, write $\mathrm{covol}_{\mathbb{A}}(nD)$ for the volume of a fundamental domain for $(\mathbb{A}_K \otimes_K \Gamma(nD)) / \Gamma(nD)$. Write $\log(x)$ for $\ln(x)$. The ratio $\mathrm{vol}_{\mathbb{A}}(\mathscr{B}(\mathbb{E}, nD)) / \mathrm{covol}_{\mathbb{A}}(nD)$ is independent of the choice of Haar measure. T. Chinburg [10] defined the sectional capacity $S_\gamma(\mathbb{E}, D)$ by

$$-\log\big(S_\gamma(\mathbb{E}, D)\big) = \lim_{n \to \infty} \frac{(d+1)!}{n^{d+1}} \log\big(\mathrm{vol}_{\mathbb{A}}(\mathscr{B}(\mathbb{E}, nD)) / \mathrm{covol}_{\mathbb{A}}(nD)\big), \qquad (0.1)$$

provided the limit exists. Theorem C of [33, p. 8] asserts that under the following conditions, which we call the *standard hypotheses*, $S_\gamma(\mathbb{E}, D)$ exists and is finite (possibly zero):

(1) each E_v is nonempty and stable under $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$;

(2) each E_v is bounded away from $\mathrm{supp}(D)(\mathbb{C}_v)$ under the v-adic metric induced by the given projective embedding of X; and for all but finitely many v, E_v and $\mathrm{supp}(D)(\mathbb{C}_v)$ reduce to disjoint sets (mod v).

The extremal Green's function

For each archimedean place v of K, put $X_v = K_v \otimes_K X$, and view $X_v(\mathbb{C})$ as a complex manifold. Let $E_v \subseteq X_v(\mathbb{C})$ be a compact set, disjoint from $\mathrm{supp}(D)(\mathbb{C})$. Identifying sections $f \in K_v \otimes_K H^0(X, \mathscr{O}_X(nD))$ with functions $f(z) \in K_v(X_v)$, we define the extremal Green's function of E_v relative to D by

$$G(z; E_v, D) = \sup_{n \geq 1} \sup_{\substack{f \in K_v \otimes_K \Gamma(nD) \\ \|f\|_{E_v} \leq 1}} \left(\frac{1}{n} \log\big(|f(z)|\big) \right) \tag{0.2}$$

for $z \in Y_v(\mathbb{C}) = X_v(\mathbb{C}) \setminus \mathrm{supp}(D)(\mathbb{C})$. In the definition, the outer "sup" can be replaced by "lim sup" or by "lim" since $f \in K_v \otimes_K \Gamma(nD)$ implies $f^N \in K_v \otimes_K \Gamma(NnD)$.

It is not always true that $G(z; E_v, D)$ is finite for $z \notin E_v$. Recall that a compact set $E \subset X_v(\mathbb{C})$ is called *pluripolar* if for each $x \in E$ there is a plurisubharmonic function $\varphi(z)$ defined in a neighborhood U of x, such that $U \cap E \subset \{z : \varphi(z) = -\infty\}$. We prove that $G(z; E_v, D)$ is finite on $Y_v(\mathbb{C})$ if and only if E_v is not pluripolar by showing that it coincides with $\Phi(z; E_v, D)$, the "Siciak extremal function" or "extremal plurisubharmonic function of minimal growth along D," which has been extensively studied in the theory of plurisubharmonic functions (see, e.g., [18], [34], [35], [40]). Moreover, we show that $G(z; E_v, D)$ is a Weil function for D, so it induces a metric on $\mathscr{O}_{X_v}(D)$.

Arithmetic intersection numbers

By a *generically smooth arithmetic variety*, we mean an equidimensional projective scheme $\mathfrak{X}$, flat and proper over $\mathrm{Spec}(\mathbb{Z})$, with smooth generic fibre $X = \mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q}$. Write $d = \dim(X)$; so $d + 1 = \dim(\mathfrak{X})$. Let $\mathrm{Div}(\mathfrak{X})$ denote the group of Cartier divisors on X.

Definition 0.1

A *fractional Cartier divisor* on $\mathfrak{X}$ is an element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Div}(\mathfrak{X})$, that is, a symbol $\mathscr{D} = (1/n)\mathscr{D}'$, where $0 < n \in \mathbb{Z}$ and $\mathscr{D}'$ is a Cartier divisor on $\mathfrak{X}$. The associated fractional line bundle $\mathscr{L} = \mathscr{O}_{\mathfrak{X}}(\mathscr{D})$ gives well-defined line bundles $\mathscr{L}^{\otimes nm} = \mathscr{O}_{\mathfrak{X}}(m\mathscr{D}')$

on $\mathfrak{X}$ for all integers m. We call $\mathscr{D}$ ample if $\mathscr{D}'$ is ample. A metric $\{\!\|\ \|\!\}$ on $\mathscr{L}$ is specified by giving a metric $\{\!\|\ \|\!\}^{\otimes n}$ on the associated complex line bundle $\mathscr{L}^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{C} = \mathscr{O}_{\mathfrak{X}(\mathbb{C})}(\mathscr{D}'_{\mathbb{C}})$, where $\mathfrak{X}(\mathbb{C})$ is the complex analytic space associated to $\mathfrak{X}$, and $\mathscr{D}'_{\mathbb{C}}$ is the divisor $\mathscr{D}' \otimes_{\mathbb{Z}} \mathbb{C}$ on $\mathfrak{X}(\mathbb{C})$. We say that $\{\!\|\ \|\!\}$ is smooth and Hermitian if $\{\!\|\ \|\!\}^{\otimes n}$ is, and we define the first Chern form of $\{\!\|\ \|\!\}$ to be $1/n$ times that of $\{\!\|\ \|\!\}^{\otimes n}$. Given a line bundle L on X, we say that $\mathscr{L}$ induces L if $L^{\otimes n}$ extends to $\mathscr{O}_{\mathfrak{X}}(\mathscr{D}')$ on $\mathfrak{X}$.

Definition 0.2
A *metrized fractional line bundle* is a triple $(\mathfrak{X}, \mathscr{L}, \{\!\|\ \|\!\})$ consisting of a generically smooth arithmetic variety $\mathfrak{X}$, a fractional line bundle $\mathscr{L}$ on $\mathfrak{X}$, and a metric $\{\!\|\ \|\!\}$ on $\mathscr{L}$ invariant under complex conjugation.

We need Gillet and Soulé's arithmetic amplitude theorem (see [14]), strengthened using their arithmetic Riemann-Roch theorem for generically smooth arithmetic varieties (see [15]).

Let $(\mathfrak{X}, \mathscr{L}, \{\!\|\ \|\!\})$ be an ample metrized fractional line bundle such that $\{\!\|\ \|\!\}$ is a smooth Hermitian metric with positive first Chern form, invariant under complex conjugation. Write $X = \mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L = \mathscr{L} \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose that $\mathscr{L}^{\otimes n}$ is an ample metrized line bundle on $\mathfrak{X}$. For all $m > 0$, define the sup norm $\|\ \|_{\sup}$ for $s \in H^0(X, L^{\otimes nm})$ (and for $s \in \mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm})$) by

$$\|s\|_{\sup} = \sup_{x \in X(\mathbb{C})} \left(\{\!\|s(x)\|\!\}^{\otimes nm}\right).$$

Let $B_{\sup}(n, m)$ be the unit ball in $\mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm})$ relative to $\|\ \|_{\sup}$. (Note that $\mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm}) \cong \bigoplus_{v|\infty} K_v \otimes_K H^0(X, L^{\otimes nm})$, and $B_{\sup}(n, m) = \bigoplus_{v|\infty} B_v(n, m)$, where $B_v(n, m)$ is the unit ball in $K_v \otimes_K H^0(X, L^{\otimes nm}) = H^0(X_v, L_v^{\otimes nm})$ relative to the restriction $\|\ \|_{v,\sup}$ of $\|\ \|_{\sup}$ to $X_v = K_v \otimes_K X$.) Let vol_∞ be any Haar measure on the real vector space $\mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm})$. Then $H^0(\mathfrak{X}, \mathscr{L}^{\otimes nm})$ forms a $\mathbb{Z}$-lattice in $\mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm})$. Let

$$\mathrm{covol}_\infty \left(H^0(\mathfrak{X}, \mathscr{L}^{\otimes nm})\right) := \mathrm{vol}_\infty \left(\mathbb{R} \otimes_{\mathbb{Q}} H^0(X, L^{\otimes nm})/H^0(\mathfrak{X}, \mathscr{L}^{\otimes nm})\right)$$

be the volume of a fundamental domain for this lattice.

Given a metrized fractional line bundle $\overline{L} = (\mathfrak{X}, \mathscr{L}, \{\!\|\ \|\!\})$, the intersection number $(\mathfrak{X}, \mathscr{L}, \{\!\|\ \|\!\})^{d+1}$ is defined as follows. Let $\hat{c}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(\mathfrak{X})$ be its first arithmetic Chern class, and let $f : \mathfrak{X} \to \mathrm{Spec}(\mathbb{Z})$ be the structure morphism. Then

$$(\mathfrak{X}, \mathscr{L}, \{\!\|\ \|\!\})^{d+1} = f_*\left(\hat{c}_1(\overline{L})^{d+1}\right) \in \widehat{\mathrm{CH}}^1\left(\mathrm{Spec}(\mathbb{Z})\right) \cong \mathbb{R}, \tag{0.3}$$

where the isomorphism takes $((\mathfrak{a}), \lambda) \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z}))$ to $\lambda/2 - \log(\deg(\mathfrak{a}))$.

THEOREM 0.1 (Gillet, Soulé; M. Gromov)
Let $\mathfrak{X}$ be a generically smooth arithmetic variety. With the assumptions and notation above,

$$(\mathfrak{X}, \mathscr{L}, \|\ \|)^{d+1} = \lim_{m\to\infty} \frac{(d+1)!}{(nm)^{d+1}} \log\left(\mathrm{vol}_\infty(B_{\sup}(n,m))/\,\mathrm{covol}_\infty(H^0(\mathfrak{X}, \mathscr{L}^{\otimes nm}))\right). \quad (0.4)$$

This form of the arithmetic amplitude theorem (Th. 0.1) can be extracted from the discussion in [14, p. 888] using [15, Th. 8].

Outline

The paper is divided into four sections. The first contains the main theorem expressing the sectional capacity as a limit of top self-intersection numbers of metrized fractional line bundles. The second establishes properties of Green's function $G(z; E, D)$ needed for the first part. The third derives functoriality properties of $G(z; E, D)$, and the fourth gives the conjectural exact formula and discusses evidence for it.

1. The main theorem

Our main theorem is as follows.

THEOREM 1.1
Let K be a number field, and let X/K be a smooth, connected, projective variety of dimension d. Let an adelic set $\mathbb{E} = \prod_v E_v$ and an effective, ample, K-rational divisor D on X satisfying the standard hypotheses be given. Assume that $S_\gamma(\mathbb{E}, D) > 0$, and assume that each archimedean E_v is compact and not pluripolar. Put $L = \mathscr{O}_X(D)$. Then there is a canonical sequence of models $\{\mathfrak{X}_n\}_{1\le n<\infty}$ determined by the nonarchimedean part of $\mathbb{E}$, and a natural sequence of ample metrized fractional line bundles $(\mathfrak{X}_n, \mathscr{L}_n, \|\ \|_n)$ inducing L on X, where the $\|\ \|_n$ are smooth positive metrics given by smoothings of Green's functions $G(z; E_v, D)$ of the archimedean components of $\mathbb{E}$, such that

$$-\log\left(S_\gamma(\mathbb{E}, D)\right) = \lim_{n\to\infty} (\mathfrak{X}_n, \mathscr{L}_n, \|\ \|_n)^{d+1}. \quad (1.1)$$

More precisely, for each $n \ge 1$, let

$$S_n = \left\{ f \in \Gamma(nD) : \sup_{x\in E_v} (|f(x)|_v) \le 1 \text{ for each nonarchimedean } v \right\},$$

and let $O_K[S_n]$ denote the graded O_K-algebra generated in degree 1 by S_n. Then $\mathfrak{X}_n = \mathrm{Proj}(O_K[S_n])$, and $\mathscr{L}_n$ is a fractional line bundle induced by L on $\mathfrak{X}_n$ for which $L^{\otimes n}$ extends to $\mathscr{O}_{\mathfrak{X}_n}(1)$. The metric $\|\ \|_n$ is obtained by smoothing the Green's function *of an appropriate enlargement of the archimedean component of* $\mathbb{E}$*, constructed in the theorem. The enlargements have the property that their Green's functions are continuous and their capacities approach the capacity of the original set. Such a sequence of enlargements and smoothings is not unique, but any such sequence given by the construction in the theorem yields metrics* $\{\!\|\ \|\!\}_n$ *for which (1.1) holds.*

In the statement of the main theorem, we expect that the hypothesis E_v nonpluripolar is redundant, and it should follow from the fact that $S_\gamma(\mathbb{E}, D) > 0$ (see the remark following Lemma 2.3).

The proof of the main theorem uses results about extremal Green's functions proved in Section 2, together with Gillet-Soulé's arithmetic amplitude theorem and approximation theorems for the sectional capacity established in [33]. Note that although the limits (0.1) and (0.4) are similar in form, they have different origins. In the capacity case, the metric underlying the sup norm (the fibral metric $\{\!\| \cdot \|\!\}_{E_v}(x)$ on $\mathscr{O}_X(D)$ for which the canonical section $\mathbf{1}$ of $\mathscr{O}_X(D)$ satisfies $\{\!\|\mathbf{1}\|\!\}_{E_v}(x) = 1$ if $x \in E_v$, and zero otherwise) is not even continuous from fibre to fibre, while in the intersection theory case it is $\mathscr{C}^\infty$. Furthermore, the capacity case concerns only the generic fibre $X/\mathbb{Q}$, whereas the intersection theory case involves a model $\mathfrak{X}/\operatorname{Spec}(\mathbb{Z})$.

The proof depends on two key observations. The first is that for each archimedean v, the unit ball $\mathscr{B}(E_v, nD) \subset K_v \otimes_K \Gamma(nD)$ for

$$\|f\|_{E_v} = \sup_{z \in E_v} \big(|f(z)|\big)$$

coincides with the unit ball $B_{G_v}(n)$ for the sup norm $\|f\|_{G_v}$ with respect to the singular but smoothable metric on $\mathscr{O}_X(nD)$ associated to Green's function $G(z; E_v, D)$:

$$\|f\|_{G_v} = \sup_{z \in X(\mathbb{C}_v)} \left(\frac{|f(z)|}{\exp(nG(z; E_v, D))}\right).$$

The second is that for each n, if $(S_n)^m$ denotes the degree m component of $O_K[S_n]$, then for all sufficiently large m the set $(S_n)^m$ can be recovered as $H^0(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}(m))$ for the model $\mathfrak{X}_n = \operatorname{Proj}(O_K[S_n])$.

We need the following lemma. In the lemma, and in many other places in the paper, vol_v denotes an arbitrary Haar measure on $K_v \otimes_K \Gamma(nD)$. Given measurable subsets $\mathscr{B}_1, \mathscr{B}_2 \subset K_v \otimes_K \Gamma(nD)$ with finite, nonzero measure, the ratio $\mathrm{vol}_v(\mathscr{B}_1)/\mathrm{vol}_v(\mathscr{B}_2)$ is a generalized index, independent of the choice of Haar measure.

LEMMA 1.2

Let $\mathbb{E}$ *and* D *satisfy the hypotheses of Theorem 1.1, and fix an archimedean place* v *of* K*. Then for each* $\varepsilon > 0$*, there is a set* $E_v(\varepsilon)$ *containing* E_v *which is compact, stable under* $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$*, bounded away from* $\mathrm{supp}(D)$*, and such that*

(1) *for all sufficiently large n,*

$$\limsup_{m\to\infty} \frac{1}{(nm)^{d+1}} \log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\,\mathrm{vol}_v(\mathscr{B}(E_v(\varepsilon), nmD))\right) < \varepsilon;$$

(2) $G(z; E_v(\varepsilon), D)$ *is continuous.*

Proof

Since $S_\gamma(\mathbb{E}, D) > 0$, $\mathscr{B}(E_v, nD)$ is a bounded, convex subset of $K_v \otimes_K \Gamma(nD)$ for each $n > 0$. Let C be a constant such that $\dim_K(\Gamma(nD)) \le Cn^d$, and let $\mathscr{B}(E_v, nD)^m$ denote the convex hull in $K_v \otimes_K \Gamma(nmD)$ generated by m-fold products of elements of $\mathscr{B}(E_v, nD)$. By [33, Th. 15.1], for each sufficiently large n, and all sufficiently large m (depending on n),

$$\log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\,\mathrm{vol}_v(\mathscr{B}(E_v, nD)^m)\right) < \frac{\varepsilon}{3}\cdot(nm)^{d+1}. \tag{1.2}$$

Fix such an n, and let $\eta > 1$ be such that $C\cdot\log(\eta) < \varepsilon/3$. Then the set $\mathscr{B}_0 = \eta^{-n}\mathscr{B}(E_v, nD)$ satisfies

$$\begin{aligned}\log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nD)^m)/\,\mathrm{vol}_v((\mathscr{B}_0)^m)\right) &= \dim_K\left(\Gamma(nmD)\right)\log(\eta^{nm})\\ &< \frac{\varepsilon}{3}\cdot(nm)^{d+1}\end{aligned} \tag{1.3}$$

for each m. As $\mathscr{B}_0$ is contained in the interior of $\mathscr{B}(E_v, nD)$, there exist finitely many functions $h_1, \ldots, h_M$ in $\mathscr{B}(E_v, nD)$ such that $\mathscr{B}_0$ is contained in the interior of the convex hull of $h_1, \ldots, h_M$. For sufficiently small $T > 1$, $\mathscr{B}_0$ is also contained in the interior of the convex hull of $h_1/T, \ldots, h_M/T$. Fix such a T. For $1 \le R < T$, put

$$E_{v,R} = \left\{z \in X_v(\mathbb{C}) : |h_1(z)| \le R, \ldots, |h_M(z)| \le R\right\}.$$

Then $E_v \subseteq E_{v,R}$; $E_{v,R}$ is stable under $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$ and bounded away from $\mathrm{supp}(D)$; and $h_1/T, \ldots, h_M/T \in \mathscr{B}_v(E_{v,R}, nD)$. By (1.2) and (1.3), for all sufficiently large m,

$$\log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\,\mathrm{vol}_v(\mathscr{B}(E_{v,R}, nD)^m)\right) < \frac{2\varepsilon}{3}\cdot(nm)^{d+1},$$

so $E_{v,R}$ satisfies (1) in the lemma. Proposition 2.12 shows that for small enough $R > 1$, $G(z; E_{v,R}, D)$ is continuous. Hence we can take $E_v(\varepsilon) = E_{v,R}$ for an appropriate R. □

Proof of Theorem 1.1, assuming the results of Section 2

After replacing D by ND for a suitable N, we can assume, without loss, that D is very ample. Viewing $\mathbb{R} \otimes_{\mathbb{Q}} \Gamma(nD)$ as the archimedean component of $\mathbb{A}_K \otimes_K \Gamma(nD)$, and regarding $\Gamma(nD)$ as embedded in $\mathbb{A}_K \otimes_K \Gamma(nD)$ along the diagonal, put

$$S_n = \Big\{ f \in \Gamma(nD) : f \in \big(\mathbb{R} \otimes_{\mathbb{Q}} \Gamma(nD)\big) \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)\Big\}.$$

As $S_\gamma(\mathbb{E}, D) > 0$, our standard hypotheses imply that each nonarchimedean $\mathscr{B}(E_v, nD)$ is compact and open, and if $h_1, \ldots, h_M$ is a K-basis for $\Gamma(nD)$, then for all but finitely many v, $\mathscr{B}(E_v, nD)$ is the "trivial set" $\bigoplus O_v h_i$ (see [33, formula (4)] for the result over a field where a monic basis exists, and see [33, formula (22)] for the descent to K). It follows that S_n is a finitely generated O_K-module. Furthermore, by the weak approximation theorem, S_n is dense in $B(E_v, nD)$ for each v. Let $O_K[S_n]$ be the graded O_K-algebra generated in degree 1 by S_n, and put $\mathfrak{X}_n = \mathrm{Proj}(O_K[S_n])$. As $L^{\otimes n}$ extends to the line bundle $\mathscr{O}_{\mathfrak{X}_n}(1)$ on $\mathfrak{X}_n$, by definition there is a fractional line bundle $\mathscr{L}_n$ on $\mathfrak{X}_n$ which induces L on X.

Fix a sequence of numbers $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and a sequence of numbers $\eta_n > 1$ with $\lim_{n\to\infty} \eta_n = 1$.

For each archimedean v, Lemma 1.2 shows that there are sets $E_{v,n} := E_v(\varepsilon_n)$ containing E_v such that for each n,

(1) $\limsup_{m\to\infty} (1/(nm)^{d+1}) \log \big(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\mathrm{vol}_v(\mathscr{B}(E_{v,n}, nmD))\big)$ $< \varepsilon_n$;

(2) $G(z; E_{v,n}, D)$ is continuous.

Write $X_v = \mathbb{C}_v \times_K X$. Since $G(z; E_{v,n}, D)$ is continuous, Corollary 2.6 and Theorem 2.18 provide a smoothing $g_{v,n}$ of $G(z; E_{v,n}, D)$ such that

$$|g_{v,n}(z) - G(z; E_{v,n}, D)| < \log(\eta_n)$$

and such that $g_{v,n}$ induces a metric on $L_v = \mathbb{C}_v \otimes_K L$ on $X_v(\mathbb{C})$, with positive first Chern form, given on $X_v(\mathbb{C}) \setminus \mathrm{supp}(D)$ by

$$\{\!| f |\!\}_{g_{v,n}}(z) = \frac{|f(z)|}{\exp(g_{v,n}(z))}.$$

If $K_v \cong \mathbb{R}$, then $g_{v,n}$ can be taken to be stable under complex conjugation. Together, the metrics $\{\!| \ |\!\}_{g_{v,n}}$ determine a metric $\{\!| \ |\!\}_n$ on L over $\mathfrak{X}(\mathbb{C})$, stable under complex conjugation. We claim that the sequence of fractional metrized line bundles $(\mathfrak{X}_n, \mathscr{L}_n, \{\!| \ |\!\}_n)$ has the properties in the theorem.

Write $(S_n)^m$ for the degree m component of $O_K[S_n]$. Since O_K is a Nagata ring, for each n and all sufficiently large m (depending on n),

$$H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm}) = (S_n)^m \tag{1.4}$$

(see [16, p. 123]). For each nonarchimedean v, write $\mathscr{B}(E_v, nD)^m$ for the O_v-submodule of $K_v \otimes_K \Gamma(nmD)$ generated by m-fold products of elements of

$\mathscr{B}(E_v, nD)$. Because $B_v(E_v, nD)$ is closed and S_n is dense in $B_v(E_v, nD)$, for each m,

$$\mathscr{B}(E_v, nD)^m = O_v \otimes_{O_K} (S_n)^m. \tag{1.5}$$

From (1.4) and (1.5) we conclude that for all sufficiently large m (depending on n),

$$H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm}) = \Gamma(nmD) \cap \Big(\big(\mathbb{R} \otimes_{\mathbb{Q}} \Gamma(nmD)\big) \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)^m \Big), \tag{1.6}$$

where we view both sets on the right-hand side as subsets of $\mathbb{A}_K \otimes_K \Gamma(nmD)$.

Fix $\varepsilon > 0$. By [33, Th. 15.6(B)], for all sufficiently large n and all sufficiently large m (depending on n), if $\mathrm{vol}_{\mathbb{A}}$ is any Haar measure on $\mathbb{A}_K \otimes_K \Gamma(nmD)$, then

$$\log \Big(\mathrm{vol}_{\mathbb{A}} \big(\mathscr{B}(\mathbb{E}, nmD)\big) / \mathrm{vol}_{\mathbb{A}} \Big(\prod_{v|\infty} \mathscr{B}(E_v, nmD) \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)^m \Big) \Big)$$
$$< \varepsilon \cdot (nm)^{d+1}. \tag{1.7}$$

(As formulated in [33], Theorem 15.6(B) concerns the quantity

$$\log \Big(\mathrm{vol}_{\mathbb{A}} \big(\mathscr{B}(\mathbb{E}, nmD)\big) / \mathrm{vol}_{\mathbb{A}} \Big(\prod_{v} \mathscr{B}(E_v, nD)^m \Big) \Big);$$

since $\mathscr{B}(\mathbb{E}, nmD) = \prod_v \mathscr{B}(E_v, nmD)$ and $\mathscr{B}(E_v, nD)^m \subset \mathscr{B}(E_v, nmD)$ for each archimedean v, Theorem 15.6(B) implies (1.7).)

For each archimedean v, and each n and m, the metric $\{\!\| \ \|\!\}_{g_{v,n}}$ determines a metric on $L_v^{\otimes nm}$ which we denote by $\{\!\| \ \|\!\}_{g_{v,n}}^{\otimes nm}$. Put

$$\mathscr{B}_{g_{v,n}}(nm) = \Big\{ f \in K_v \otimes_K \Gamma(nmD) : \sup_{z \in X_v(\mathbb{C}_v)} \{\!\| f \|\!\}_{g_{v,n}}^{\otimes nm} \le 1 \Big\}.$$

Green's function $G(z; E_{v,n}, D)$ determines another metric $\{\!\| \ \|\!\}_{G_{v,n}}^{\otimes nm}$ on $L_v^{\otimes nm}$, given by $\{\!\| f \|\!\}_{G_{v,n}}^{\otimes nm}(z) = |f(z)| / \exp(nmG(z; E_{v,n}, D))$. Let $\| \ \|_{G_{v,n}}$ be the associated sup norm. Proposition 2.16 shows that the unit ball $\mathscr{B}(E_v, nmD)$ for the norm $\| \ \|_{E_v}$ and the unit ball $\mathscr{B}_{G_{v,n}}(nm)$ for $\| \ \|_{G_{v,n}}$ coincide:

$$\mathscr{B}(E_v, nmD) = \mathscr{B}_{G_{v,n}}(nm).$$

This and the fact that $|g_{v,n}(z) - G(z; E_{v,n}, D)| < \log(\eta_v)$ for all z imply

$$\eta_n^{-nm} \mathscr{B}_{g_{v,n}}(nm) \subseteq \mathscr{B}(E_{v,n}, nmD) \subseteq \eta_n^{nm} \mathscr{B}_{g_{v,n}}(nm).$$

Let vol_v be any Haar measure on $K_v \otimes_K \Gamma(nD)$. If C is a constant such that $\dim(\Gamma(L^{\otimes nm})) \le C \cdot (nm)^d$, we conclude that

$$\Big| \log \big(\mathrm{vol}_v(\mathscr{B}(E_{v,n}, nmD)) / \mathrm{vol}_v(\mathscr{B}_{g_{v,n}}(nm)) \big) \Big| \le C \log(\eta_n) \cdot (nm)^{d+1}.$$

On the other hand, by the construction of $E_{v,n}$, for all sufficiently large m,

$$\left|\log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\,\mathrm{vol}_v(\mathscr{B}(E_{v,n}, nmD))\right)\right| \leq \varepsilon_n \cdot (nm)^{d+1}.$$

Consequently, for all sufficiently large n and all sufficiently large m (depending on v and n),

$$\left|\log\left(\mathrm{vol}_v(\mathscr{B}(E_v, nmD))/\,\mathrm{vol}_v(\mathscr{B}_{g_{v,n}}(nm))\right)\right| \leq \varepsilon \cdot (nm)^{d+1}. \tag{1.8}$$

By (1.7), (1.8), and Lemma 1.2, if we put

$$\mathscr{B}_n(nm) = \prod_{v|\infty} \mathscr{B}_{g_{v,n}}(nm) \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)^m$$

and if r is the number of archimedean places of K, then for all sufficiently large n and all sufficiently large m (depending only on n),

$$\left|\log\left(\mathrm{vol}_{\mathbb{A}}(\mathscr{B}(\mathbb{E}, nmD))/\,\mathrm{vol}_{\mathbb{A}}(\mathscr{B}_n(nm))\right)\right| < (r+1) \cdot \varepsilon \cdot (nm)^{d+1}. \tag{1.9}$$

Using the strong approximation theorem and standard isomorphisms of group theory, it is easy to see that if T is a fundamental domain for $(\mathbb{R} \otimes_{\mathbb{Q}} \Gamma(nmD))/H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm})$, then $T \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)^m$ is a fundamental domain for $(\mathbb{A}_K \otimes_K \Gamma(nmD))/\,\Gamma(nmD)$.

Formula (1.9) holds for any choice of the Haar measure $\mathrm{vol}_{\mathbb{A}}$. If we put $\mathrm{vol}_{\mathbb{A}} = \prod_v \mathrm{vol}_v$ where the nonarchimedean local Haar measures are normalized so that each $\mathrm{vol}_v(\mathscr{B}(E_v, nD)^m) = 1$, and if we regard $\mathrm{vol}_\infty = \prod_{v|\infty} \mathrm{vol}_v$ as a Haar measure on $\mathbb{R} \otimes_{\mathbb{Q}} \Gamma(nmD)$, then we have

$$\begin{aligned}\mathrm{covol}_{\mathbb{A}}(nmD) &= \mathrm{vol}_{\mathbb{A}}\left(T \times \prod_{v \text{ finite}} \mathscr{B}(E_v, nD)^m\right)\\ &= \mathrm{vol}_\infty(T) = \mathrm{covol}_\infty\left(H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm})\right).\end{aligned}$$

Moreover, $\prod_{v|\infty} \mathscr{B}_{g_{v,n}}(nm) = \mathscr{B}_{\sup}(n, m)$ in the sense of Theorem 0.1. Thus (1.9) can be reformulated as

$$\begin{aligned}&\Big|\log\left(\mathrm{vol}_{\mathbb{A}}(\mathscr{B}(\mathbb{E}, nmD))/\,\mathrm{covol}_{\mathbb{A}}(nmD)\right)\\ &\quad - \log\left(\mathrm{vol}_\infty(\mathscr{B}_{\sup}(n, m))/\,\mathrm{covol}_\infty(H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm}))\right)\Big| < (r+1)\varepsilon \cdot (nm)^{d+1}.\end{aligned} \tag{1.10}$$

Now by the existence of the sectional capacity (see [33, Th. C]), for each n,

$$\lim_{m\to\infty} \frac{(d+1)!}{(nm)^{d+1}} \log\left(\mathrm{vol}_{\mathbb{A}}(\mathscr{B}(\mathbb{E}, nmD))/\,\mathrm{covol}_{\mathbb{A}}(nmD)\right) = -\log\left(S_\gamma(\mathbb{E}, D)\right).$$

On the other hand, by Gillet-Soulé's arithmetic amplitude theorem,

$$\lim_{m\to\infty} \frac{(d+1)!}{(nm)^{d+1}} \log\left(\mathrm{vol}_\infty(\mathscr{B}_{\sup}(n,m))/\,\mathrm{covol}_\infty(H^0(\mathfrak{X}_n, \mathscr{L}_n^{\otimes nm}))\right) = (\mathfrak{X}_n, \mathscr{L}_n, \{\!\| \ \|\!\}_n)^{d+1}.$$

Hence, for all sufficiently large n, (1.10) gives

$$\left| -\log\left(S_\gamma(\mathbb{E}, D)\right) - (\mathfrak{X}_n, \mathscr{L}_n, \{\!\| \ \|\!\}_n)^{d+1} \right| \le (d+1)! \cdot (r+1) \cdot \varepsilon.$$

Since ε is arbitrary,

$$-\log\left(S_\gamma(\mathbb{E}, D)\right) = \lim_{n\to\infty} (\mathfrak{X}_n, \mathscr{L}_n, \{\!\| \ \|\!\}_n)^{d+1}. \tag{1.11}$$

□

Remark. The key fact underlying the main theorem is the fact that the sectional capacity exists under very weak hypotheses. If the sets E_v are such that $G(z; E_v, D)$ is continuous for each archimedean v, it can be shown that the convergence of the metrics $\{\!\| \ \|\!\}_n$ is uniform; and in that case, the existence of the limit on the right-hand side of (1.11) is *equivalent* to the existence of the sectional capacity. However, in general, $G(z; E_v, D)$ is not continuous, and the $g_{v,n}$ converge pointwise but not uniformly to $G(z; E_v, D)$. In this case, the main theorem goes beyond what can be shown using the methods of uniform convergence used in [43].

Examples

We conclude this section with two examples illustrating the kinds of models that Theorem 1.1 produces.

Our first example concerns places v where E_v is the "trivial set" consisting of all $z \in X(\mathbb{C}_v)$ which specialize to points not in $\mathrm{supp}(D) \pmod v$. Recall that an effective relative Cartier divisor on a scheme $\mathfrak{X}/\,\mathrm{Spec}(A)$ is an effective Cartier divisor whose local equations define a subscheme flat over $\mathrm{Spec}(A)$.

PROPOSITION 1.3

Let v (resp., S) be a nonarchimedean place (resp., a finite set of nonarchimedean places) of K. Suppose that A is either O_v or the localization O_S of O_K at the primes in S. Let $\mathfrak{X}_A/\,\mathrm{Spec}(A)$ be a normal model of X whose closed fibres over $\mathrm{Spec}(A)$ are reduced. Suppose that D extends to an ample effective relative Cartier divisor $\mathscr{D}_A$ on $\mathfrak{X}_A$. Suppose further that for each finite place w of K corresponding to a closed point of $\mathrm{Spec}(A)$, E_w is trivial with respect to $\mathfrak{X}_w = \mathfrak{X} \otimes_A O_w$ and $\mathscr{D}_w = \mathscr{D}_A \otimes_A O_w$, in the sense that E_w consists of the set of all points of $X(\mathbb{C}_w)$ which do not specialize to a point on the closed fibre of $\mathfrak{X}_w$ belonging to the reduction of $\mathrm{supp}(\mathscr{D}_w)$. Then for all sufficiently large n, $\mathrm{Proj}(O_K[S_n]) \otimes_{O_K} A$ is isomorphic to $\mathfrak{X}_A$.

Proof
We give the proof only when $A = O_v$, as the global case is similar. Write $\mathscr{D}_v = \mathscr{D}_{O_v}$.

Let $i : \mathfrak{X}_v \to P^m_{O_v}$ be the projective embedding associated to $\mathscr{D}_v$. The argument of [16, Exer. II.5.14] shows that for all sufficiently large integers N, the map $\Gamma(\mathbb{P}^m_{O_v}, O_{\mathbb{P}^m}(N)) \to \Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(N\mathscr{D}_v))$ is surjective. Thus for all sufficiently large n, the graded O_v-algebra $R_v^{(n)}$ generated by $\Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$ in degree 1 defines the projective scheme $\operatorname{Proj}(R_v^{(n)})$ which is the image of $\mathfrak{X}_v$ under the composition of $i : \mathfrak{X}_v \to \mathbb{P}^m_{O_v}$ with the n-tuple embedding of $\mathbb{P}^m_{O_v} \to \mathbb{P}^{m'}_{O_v}$ for an appropriate m'. Hence $\operatorname{Proj}(R_v^{(n)})$ is isomorphic to $\mathfrak{X}_v$, and to prove the proposition, it suffices to show $O_v[S_n] = R_v^{(n)}$. For this, it is enough to prove that $\mathscr{B}(E_v, nD) = \Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$ for all $n > 0$.

If $f \in \Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$, then f is regular off $\mathscr{D}_v$. Since $z \in E_v$ implies that the Zariski closure of z is disjoint from $\mathscr{D}_v$, we conclude that $|f(z)|_v \le 1$, so $f \in \mathscr{B}(E_v, nD)$.

Since $\Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$ is a finitely generated O_v-module, it now suffices to show that when π_v is a uniformizer of O_v and $\pi_v^{-1} f$ is not in $\Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$, then $\pi_v^{-1} f$ is not in $\mathscr{B}(E_v, nD)$. In other words, we must show that there exists a $z \in E_v$ for which $|\pi_v^{-1} f(z)|_v > 1$. Because the special fibre of $\mathfrak{X}_v$ has been assumed to be reduced, and $n\mathscr{D}_v$ is a very ample effective relative Cartier divisor, the hypothesis $\pi_v^{-1} f \notin \Gamma(\mathfrak{X}_v, O_{\mathfrak{X}_v}(n\mathscr{D}_v))$ implies that f is a unit in the local ring of the generic point of some irreducible component T of the special fibre of $\mathfrak{X}_v$. Thus there is a smooth point x of T such that f is a unit in the local ring $O_{X_v,x}$. Again, using the fact that the special fibre of $\mathfrak{X}_v$ is reduced, we see that $\mathfrak{X}_v \to \operatorname{Spec}(O_v)$ is smooth in a neighborhood of x. The local lifting property for smooth morphisms (cf. [23, p. 30]) implies there is a point $z \in \mathfrak{X}_v(\mathbb{C}_v)$ which specializes to a point of special fibre of $\mathfrak{X}_v$ which is defined over the algebraic closure of the residue field of v and which lies above x. Hence $z \in E_v$ because x does not lie on D. Since f is a unit of $O_{X_v,x}$, we also have $|f(z)|_v = 1$, so $\pi_v^{-1} f$ is not in $\mathscr{B}(E_v, nD)$. By our earlier remarks, this completes the proof. □

Our second example concerns the particular variety $X = \mathbb{P}^1/\mathbb{Q}$. Fix a coordinate function z, and let $\{U_0, U_1\}$ be the covering of X given by the affine sets $U_0 = \mathbb{P}^1 \backslash \{z = \infty\}$ and $U_1 = \mathbb{P}^1 \backslash \{z = 0\}$. Let D be the Cartier divisor defined by the pairs $(U_0, 1)$ and $(U_1, 1/z)$ so that $\operatorname{supp}(D) = (\infty)$.

Fix a nonarchimedean place v of $K = \mathbb{Q}$, and let $q = q_v$ be the rational prime underlying v. Identifying $\mathbb{P}^1(\mathbb{C}_v)$ with $\mathbb{C}_v \cup \{\infty\}$ by means of the coordinate function z, take $E_v = \mathbb{Z}_q$, the ring of q-adic integers. We consider the local component $\mathfrak{X}_{n,v}/\operatorname{Spec}(\mathbb{Z}_q)$ of the model produced by our construction. Identifying $\mathbb{Q}_q \otimes \Gamma(nD)$

with the space of polynomials in $\mathbb{Q}_q[z]$ of degree less than or equal to n, write

$$S_{n,v} = \Big\{ f \in \mathbb{Q}_q \otimes \Gamma(nD) : \sup_{x \in \mathbb{Z}_q} (|f(x)|_v) \leq 1 \Big\}.$$

If $O_v[S_{n,v}]$ is the graded $\mathbb{Z}_q$-algebra generated in degree 1 by $S_{n,v}$, then $\mathfrak{X}_{n,v} = \mathrm{Proj}(O_v[S_{n,v}])$.

It is well known that a $\mathbb{Z}_q$-basis for $S_{n,v}$ is given by the binomial polynomials

$$g_k(z) = \binom{z}{k} = \frac{1}{k!} \prod_{\ell=0}^{k-1} (z - \ell)$$

for $k = 0, \ldots, n$. The Lagrange interpolation polynomials

$$h_{n,k}(z) = \prod_{\substack{\ell=0 \\ \ell \neq k}}^{n-1} \frac{z - \ell}{k - \ell}, \quad 0 \leq k \leq n-1,$$

also belong to $S_{n,v}$; indeed, $h_{n,k}(z) = g_k(z) \cdot (-1)^{n-k-1} g_{n-k-1}(z-k-1)$. Conversely, each $f \in S_{n,v}$ of degree less than or equal to $n-1$ can be written as $f(z) = \sum_{k=0}^{n-1} f(k) \cdot h_{n,k}(z)$, where $f(k) \in \mathbb{Z}_q$ by hypothesis. Thus another $\mathbb{Z}_q$-basis for $S_{n,v}$ is given by $h_{n,0}(z), \ldots, h_{n,n-1}(z), g_n(z)$.

Note that for each k,

$$\frac{g_n(z)}{h_{n,k}(z)} = \frac{z-k}{(-1)^{n-k-1} n \binom{n-1}{k}} = u_k \cdot \frac{z-k}{q^{m(k)}},$$

where $m(k) = \mathrm{ord}_h \left(n \binom{n-1}{k} \right)$ and u_k is a unit in $\mathbb{Z}_q$. Thus, dehomogenizing at $g_n(z)$ and at each of the $h_{n,k}(z)$ in turn, we find that $\mathfrak{X}_{n,v}$ is covered by the affine schemes

$$\mathrm{Spec}\left(\mathbb{Z}_q \Big[\frac{q^{m(0)}}{z-0}, \ldots, \frac{q^{m(n-1)}}{z-(n-1)} \Big] \right)$$

and

$$\mathrm{Spec}\left(\mathbb{Z}_q \Big[\frac{z-k}{q^{m(k)}}, \frac{z-k}{q^{m(k)}} \cdot \frac{q^{m(0)}}{z-0}, \ldots, \frac{z-k}{q^{m(k)}} \cdot \frac{q^{m(n-1)}}{z-(n-1)} \Big] \right)$$

for $k = 0, \ldots, n-1$.

Using this, one can work out the structure of $\mathfrak{X}_{n,v}$ for any given n. When $n = q^m$, it is particularly simple. In that case, $m(k) = \mathrm{ord}_q \left(n \binom{n-1}{k} \right) = m$ for each k. Put $y_k = q^m/(z-k)$. Then for each $k \neq \ell$,

$$y_k y_\ell = \frac{q^m}{k-\ell} \cdot (y_k - y_\ell) = u_{k,\ell} \cdot q^{m - \mathrm{ord}_q(k-\ell)} \cdot (y_k - y_\ell),$$

where $u_{k,\ell}$ is a unit in $\mathbb{Z}_q$ and $\mathrm{ord}_q(k-\ell) < m$. It follows that

$$\mathrm{Spec}\left(\mathbb{Z}_q\Big[\frac{q^{m(0)}}{z-0}, \ldots, \frac{q^{m(n-1)}}{z-n+1}\Big]\right) \cong \mathrm{Spec}\left(\mathbb{Z}_q[y_0, \ldots, y_{n-1}]/(y_k y_\ell = u_{k,\ell} q^{m-\mathrm{ord}_q(k-\ell)}(y_k - y_\ell))_{k \neq \ell}\right)$$

has reduced special fibre $\mathrm{Spec}(\mathbb{F}_q[y_0, \ldots, y_{n-1}]/(y_k y_\ell = 0)_{k\neq\ell})$ consisting of the $n = q^m$ coordinate axes. The special fibre of each of the other coordinate patches consists of a single line, corresponding to the local coordinate function $z_k = (z-k)/q^m$ on the ball $\{z \in \mathbb{C}_v : |z-k|_q \leq 1/q^m\}$. Each of these glues onto one of the coordinate axes above, with $z_k y_k = 1$. Thus the special fibre of $\mathfrak{X}_{q^m,v}$ consists of q^m lines $\mathbb{P}^1/\mathbb{F}_q$, meeting as a divisor with normal crossings at a common point ∞.

2. Extremal Green's functions

In this section we study the archimedean Green's functions $G(z; E_v, D)$. We show that if E_v is not pluripolar, then $G(z; E_v, D)$ is finite and defines a Weil function for D and hence an associated metric on $\mathcal{O}_X(D)$. We prove that E_v can be approximated by sets with continuous Green's functions, and if $G(z; E_v, D)$ is continuous, then the associated metric can be smoothed to a $\mathscr{C}^\infty$-positive metric within ε of the original metric, for any $\varepsilon > 0$.

Our plan is to identify $G(z; E_v, D)$ with the "plurisubharmonic extremal function $\Phi(z; E_v, D)$ of minimal growth along D," which has been extensively studied in analysis. The key results are known for plurisubharmonic functions on Stein spaces, and the only novelty in our approach consists in carrying them over in an arithmetic context to Weil functions.

To simplify notation, in this section we drop the subscript v, and we let $X = X_v(\mathbb{C})$ be a smooth connected projective complex manifold, D an effective, ample Cartier divisor on X, and $E \subset X \backslash \mathrm{supp}(D)$ a compact subset. We write Y for the affine subset $X \backslash \mathrm{supp}(D)$.

Plurisubharmonic functions

Recall the following definitions and facts.

(A) Given an open set $U \subseteq \mathbb{C}^N$, a function $v : U \to [-\infty, \infty)$ is called *plurisubharmonic* (cf. [18, p. 62], [19, Th. 3]) if

(1) $v(z)$ is not identically $-\infty$ on any connected component of U;

(2) $v(z)$ is upper semicontinuous (i.e., $\limsup_{x\to z} v(x) = v(z)$ for each $z \in W$, or, equivalently, $v^{-1}([-\infty, a))$ is open for each $a \in \mathbb{R}$);

(3) for any $\vec{a}, \vec{b} \in \mathbb{C}^N$, the function $\hat{v} : \mathbb{C} \to [-\infty, \infty)$ given by $\hat{v}(\lambda) = v(\vec{a}+\lambda\vec{b})$ is either subharmonic, or identically $-\infty$, on each component of $\{\lambda \in \mathbb{C} : \vec{a} + \lambda\vec{b} \in U\}$.

(Recall that $u(z) : \mathbb{C} \to [-\infty, \infty)$ is subharmonic if it is upper semicontinuous, is not identically $-\infty$, and has the integral average property

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta$$

for each $z \in \mathbb{C}$ and each sufficiently small $r > 0$.)

(B) If $v \in \mathscr{C}^2(U)$, then v is plurisubharmonic if and only if

$$dd^c v = \frac{i}{2\pi} \sum_{j,k} \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\, dz_j \wedge d\bar{z}_k \tag{2.1}$$

is positive semidefinite. Even when $v \notin \mathscr{C}^2$, if $v(z)$ is plurisubharmonic, then $dd^c v$ is a positive semidefinite (1, 1)-current with measure coefficients (see [18, p. 110]).

(C) Let $\mathscr{T}(U)$ be the space of test functions, that is, the space of $\mathscr{C}^\infty$-functions on U with compact support. A plurisubharmonic function $v(z)$ on $U \subseteq \mathbb{C}^N$ is called *strongly plurisubharmonic* if for each $w \in \mathscr{T}(U)$ there is a $t_0 > 0$ such that for all $t \in \mathbb{R}$ with $|t| < t_0$ the function $v(z) + tw(z)$ is plurisubharmonic (see [27, Def. 2]). If $v \in \mathscr{C}^2$, then $v(z)$ is strongly plurisubharmonic if and only if $dd^c v$ is positive definite.

(D) Let X be a complex analytic space, and let $U \subseteq X$ be an open subset. A function $v : U \to [-\infty, \infty)$ is called plurisubharmonic if, for each $a \in U$ and each analytic embedding $h : V \to \mathbb{C}^N$ of a sufficiently small neighborhood V of a, there is a plurisubharmonic function $\tilde{v}$ on a neighborhood $\tilde{V}$ of $h(a)$ such that $v(z)$ coincides with $\tilde{v}(h(z))$ on V (see [40, Déf. 1.1]). Similarly, v is called strongly plurisubharmonic if $\tilde{v}$ can be taken to be strongly plurisubharmonic. A basic theorem of J. Fornaess and R. Narasimhan [13] asserts that an upper semicontinuous function $v : U \to [-\infty, \infty)$ is plurisubharmonic if and only if for each holomorphic $h : D(0,1) \to U$ the function $v(h(z))$ is either subharmonic or is identically $-\infty$.

Weil functions

The divisor D can be described by giving a finite open covering $\{U_\alpha\}$ of X and a family $\{\varphi_\alpha\}$ of meromorphic functions on $\{U_\alpha\}$ such that on $U_\alpha \cap U_\beta$, $\varphi_{\alpha\beta} := \varphi_\alpha/\varphi_\beta \in \mathscr{O}^\times(U_\alpha \cap U_\beta)$. Fix such a family $\{(U_\alpha, \varphi_\alpha)\}$. For the associated line bundle $\mathscr{O}_X(D)$, the transition function on $U_\alpha \cap U_\beta$ is $\varphi_{\alpha\beta}$, and the canonical meromorphic section "**1**" is given by $\{(U_\alpha, \varphi_\alpha)\}$.

A function $f : Y = X \backslash \operatorname{supp}(D) \to \mathbb{R}$ is called a *Weil function* relative to D if on each U_α there exists a bounded function $\gamma_\alpha : U_\alpha \to \mathbb{R}$ so that

$$f + \log(|\varphi_\alpha|) = \gamma_\alpha \tag{2.2}$$

on $U_\alpha \backslash \operatorname{supp}(D)$. Note that each γ_α is defined on the entire set U_α, not just on $U_\alpha \backslash \operatorname{supp}(D)$. We call a Weil function f continuous, or upper or lower semicontinuous, or $\mathscr{C}^\infty$, or plurisubharmonic, or strongly plurisubharmonic, if there exist γ_α's for f with these properties. It is clear that if f_1 and f_2 are Weil functions relative to the divisors D_1 and D_2, then $f_1 + f_2$ is a Weil function relative to $D_1 + D_2$.

LEMMA 2.1
If a Weil function f relative to D is continuous or plurisubharmonic, then the γ_α's with these properties are uniquely determined. If f and g are Weil functions relative to D, then $f - g$ is bounded on $X \backslash \operatorname{supp}(D)$; moreover, if f is continuous or plurisubharmonic and g is continuous or plurisubharmonic, there is a canonical extension of $f - g$ from $X \backslash \operatorname{supp}(D)$ to all of X, which is continuous (resp., $\mathscr{C}^\infty$) if f and g are.

Proof
The boundedness of $f - g$ follows from the compactness of X.

Suppose that $f + \log(|\varphi_\alpha|) = \gamma_\alpha$ on $U_\alpha \backslash \operatorname{supp}(D)$. If f is continuous, the requirement of continuity uniquely determines the γ_α on U_α; so does the requirement of plurisubharmonicity (see [18, Cor. 2.9.8]).

Now suppose that g is another Weil function relative to D, with $g + \log(|\varphi_\alpha|) = \tau_\alpha$ on $U_\alpha \backslash \operatorname{supp}(D)$. If f and g are continuous or plurisubharmonic, we can extend $f - g$ to X by putting $(f - g)(z) = \gamma_\alpha(z) - \tau_\alpha(z)$ on U_α. This extension is well defined because on $(U_\alpha \cap U_\beta) \backslash \operatorname{supp}(D)$ we have

$$\gamma_\alpha - \gamma_\beta = \log(|\varphi_{\alpha\beta}|) = \tau_\alpha - \tau_\beta,$$

so the uniqueness of the γ_α and τ_α implies that $\gamma_\alpha - \tau_\alpha = \gamma_\beta - \tau_\beta$ on $U_\alpha \cap U_\beta$. The extension is clearly $\mathscr{C}^\infty$ if f and g are. □

In the future, if f and g are continuous or plurisubharmonic, whenever we write $f - g$ we understand this to mean the natural extension to X given by Lemma 2.1.

PROPOSITION 2.2
If D is an ample, effective divisor on X, then there exists a $\mathscr{C}^\infty$, strongly plurisubharmonic Weil function $W(z)$ relative to D.

Proof
Without loss, we can assume that D is very ample, with $X \hookrightarrow \mathbb{P}^n$ and D defined by $z_0 = 0$. Then we can take

$$W(z_0 : \cdots : z_n) = \log\left(\sqrt{\frac{|z_0|^2 + \cdots + |z_n|^2}{|z_0|^2}}\right) \tag{2.3}$$

on Y. A standard computation (see [26, p. 87]) shows that $dd^c W(z)$ is positive definite on Y, and it is positive definite in the sense of Weil functions on all of X. □

Note that if D is ample and $W(z)$ is any $\mathscr{C}^0$ Weil function for D, then for any $r \in \mathbb{R}$ the set

$$\{z \in X : W(z) \le r\} \tag{2.4}$$

is a compact subset of $Y = X \setminus \operatorname{supp}(D)$, and Y is the union of these sets for all r. A function $W(z)$ satisfying these conditions is called an *exhaustion function* for Y.

A function with the properties in Proposition 2.2 is called a *strong Weil exhaustion function* relative to D.

Suppose that D is an ample, effective divisor on X, and suppose that $W(z)$ is a strong Weil exhaustion function relative to D. Let us introduce the following classes of functions:

$$\begin{aligned}
\mathrm{PSH}(D) = \big\{&\text{plurisubharmonic } f : X\setminus \operatorname{supp}(D) \to \mathbb{R} : \\
&\text{for some } C,\ f(z) \le W(z) + C \text{ for all } z\big\},\\
\mathrm{PSH}_+(D) = \big\{&\text{plurisubharmonic } f : X\setminus \operatorname{supp}(D) \to \mathbb{R} : \\
&f \text{ is a Weil function relative to } D\big\},\\
\mathrm{PSH}_+^c(D) = \big\{&f : X\setminus \operatorname{supp}(D) \to \mathbb{R} : f \text{ is a continuous}\\
&\text{plurisubharmonic Weil function relative to } D\big\}.
\end{aligned}$$

Trivially, $\mathrm{PSH}_+^c(D) \subset \mathrm{PSH}_+(D)$. Also, $\mathrm{PSH}_+(D) \subset \mathrm{PSH}(D)$ since $W(z)$ is a Weil function relative to D, and any two such functions have bounded difference. For the same reason, these families depend only on D and not on the choice of $W(z)$.

Loosely, $\mathrm{PSH}(D)$ is the family of plurisubharmonic functions with order of growth no greater than $W(z)$ as z approaches $\operatorname{supp}(D)$, and $\mathrm{PSH}_+(D)$ is the family of plurisubharmonic functions such that $|f(z) - W(z)|$ is bounded on Y. $\mathrm{PSH}_+^c(D)$ is the family of such functions that are continuous on Y and such that, for each $q \in \operatorname{supp}(D)$, if $\varphi_q(z)$ is a local equation for D at q, then in a sufficiently small neighborhood U of q there is an extension $\gamma(z)$ of $f(z) + \log(|\varphi_q(z)|)$ which is continuous and plurisubharmonic on U.

The extremal function $\Phi(z; E, D)$

Definition 2.1

Let $E \subset Y$ be compact and nonempty. Define the extremal function

$$\Phi(z; E, D) = \sup\big\{f(z) : f \in \mathrm{PSH}, \text{ and } f|_E \le 0\big\}.$$

More generally, for any function $b : E \to \mathbb{R}$, put

$$\Phi_b(z; E, D) = \sup\big\{f(z) : f \in \mathrm{PSH}(D), \text{ and } f|_E \le b\big\}.$$

Recall that for any function $F : Y \to \mathbb{R} \cup \{-\infty\}$, the *upper semicontinuous regularization* $F^\star$ of F is

$$F^\star(z) := \limsup_{x \to z} F(x).$$

Plurisubharmonic functions are by definition upper semicontinuous. The extremal function $\Phi_b(z; E, D)$ is not, in general, plurisubharmonic, but if b is bounded and E is not pluripolar, then $\Phi_b^\star(z; E, D)$ is locally bounded and plurisubharmonic on Y (see [40, Th. 4.2]). Furthermore, under these hypotheses, $\Phi_b^\star(z; E, D)$ coincides with $\Phi_b(z; E, D)$ outside a pluripolar set. (This is a consequence of a general fact due to E. Bedford and B. Taylor [4, Th. 7.1] about so-called negligible sets; alternately, see [18, p. 166, and Th. 4.7.6].)

LEMMA 2.3

The following properties of a compact set $E \subset X \backslash \operatorname{supp}(D)$ are equivalent:

(i) *E is pluripolar;*

(ii) *there exists some $v \in \operatorname{PSH}(D)$ with $v|_E \equiv -\infty$;*

(iii) *$\Phi^\star(z; E, D) \equiv \infty$;*

(iv) *for some $z_0 \in X \backslash \operatorname{supp}(D)$, $\Phi^\star(z_0; E, D) = \infty$;*

(v) *for some bounded $b : E \to \mathbb{R}$ and $z_0 \in X \backslash \operatorname{supp}(D)$, $\Phi_b^\star(z_0; E, D) = \infty$.*

Proof

The equivalence of (i), (ii), and (iii) is shown in [40, Th. 3.9]. The equivalence of (ii) and (iv) follows from [40, Lem. 3.10]. The equivalence of (iv) and (v) follows from the fact that for each bounded $b : E \to \mathbb{R}$, we have $\Phi^\star(z_0, E, D) = \infty$ if and only if $\Phi_b^\star(z_0; E, D) = \infty$. □

Remark. Another condition can almost certainly be added to this equivalence, namely, that E is pluripolar if and only if its local sectional capacity $S_\gamma(E, D)$ (introduced in [33, Sec. 14]) is zero. This is known to be true when $X = \mathbb{P}^n$, $D = H_\infty$, as noted in [32, p. 535].

We next show that if $b : E \to \mathbb{R}$ is continuous, then $\Phi_b(z; E, D)$ is not only a function on $X \backslash \operatorname{supp}(D)$ but is in fact a Weil function relative to D.

THEOREM 2.4

If E is not pluripolar, and $b : E \to \mathbb{R}$ is continuous, then

(i) *$\Phi_b^\star(z; E, D)$ is a plurisubharmonic Weil function relative to D; in particular, $\Phi_b^\star(z; E, D) \in \operatorname{PSH}_+(D)$;*

(ii) *on $Y\backslash E$, $\Phi^\star_b(z; E, D)$ satisfies the complex Monge-Ampère equation*

$$\left(dd^c\Phi^\star_b(z; E, D)\right)^d = 0,$$

where $d = \dim(X)$. This holds, in the sense of Weil functions, on all of $X\backslash E$: For each $q \in \operatorname{supp}(D)$, if $\varphi_q(z)$ is a local equation for D at q, then in a sufficiently small neighborhood of q, the natural extension $\gamma(z)$ of $\Phi^\star_b(z; E, D) + \log(|\varphi_q(z)|)$ satisfies $(dd^c\gamma(z))^d = 0$.

Proof
In [40, Th. 4.2] it is shown that when b is continuous, $\Phi^\star_b(z; E, D)$ is plurisubharmonic on Y with $dd^c\Phi^\star_b(z; E, D) = 0$ on $Y\backslash E$. Moreover, if $W(z)$ is a strong Weil exhaustion function for D, then $\Phi^\star_b(z; E, D) - W(z)$ is bounded on Y. Thus $\Phi^\star_b(z; E, D) \in \mathscr{PSH}_+(D)$.

To see that $\Phi^\star_b(z; E, D)$ is plurisubharmonic on $\operatorname{supp}(D)$ in the sense of Weil functions, suppose $q \in \operatorname{supp}(D)$, and let $\varphi_q(z)$ be a local equation for D in a neighborhood U of q. By construction, $\Phi^\star_b(z; E, D) - W(z)$ is bounded on U, and $W(z) + \log(|\varphi_q(z)|)$ is bounded near q. Both are plurisubharmonic on $U\backslash \operatorname{supp}(D)$. By a general fact about plurisubharmonic functions, the natural extension of $\Phi^\star_b(z; E, D) + \log(|\varphi_q(z)|)$ to U given by

$$\gamma(z) = \limsup_{\substack{x\to z \\ x\notin \operatorname{supp}(D)}} \Phi^\star_b(x; E, D) + \log\left(|\varphi_q(x)|\right)$$

is plurisubharmonic (see [18, Th. 2.9.22]).

Lastly, we show that $(dd^c\gamma(z))^d = 0$ near q. Without loss, we can assume that D is very ample. Choose $h(z) \in K(X)^\times$ so that $D' := D - \operatorname{div}(h)$ is effective, $q \notin \operatorname{supp}(D')$, and $E \cap \operatorname{supp}(D') = \phi$. Now consider the situation above relative to D', with $b' := b + \log(|h(z)|)$ on E. By the definitions, one sees that $\Phi'_b(z; E, D') = \Phi_b(z, E, D) + \log(|h(z)|)$. Using what has been shown above, we conclude that $dd^c\Phi_{b'}(z, E, D')^d = 0$ on $Y'\backslash E$, where $Y' = X\backslash \operatorname{supp}(D')$. Transferring back, we conclude that $(dd^c\gamma(z))^d = 0$ in a neighborhood of q. □

Identification of $\Phi(z; E, D)$ and $G(z; E, D)$
We now show that Green's function $G(z; E, D)$, defined in (0.2), coincides with $\Phi(z; E, D)$. When E is not pluripolar, it follows that $G(z; E, D)$ is a plurisubharmonic Weil function on X; in particular, it is finite on $Y = X\backslash \operatorname{supp}(D)$, and for each $q \in \operatorname{supp}(D)$, if $\varphi_q(z)$ is a local equation for D at q, then $G(z; E, D) + \log(|\varphi_q(z)|)$ extends to a function plurisubharmonic along D.

The fact that $G(z; E, D)$ coincides with $\Phi(z; E, D)$ on Y is a simple consequence of [40, Th. 5.1].

PROPOSITION 2.5 (A. Zériahi)
Suppose that Y is embedded in $\mathbb{C}^m$ for some m; let $A_n(Y)$ be the vector space of functions on Y given by polynomials in $\mathbb{C}[x_1, \ldots, x_m]$ of total degree less than or equal to n. Then for each compact $E \subset Y$ and each $z \in Y$,

$$\Phi(z; E, D) = \sup_{n \geq 1} \sup_{\substack{f \in A_n(Y) \\ \|f\|_E \leq 1}} \frac{1}{n} \log\big(|f(z)|\big).$$

Recall that

$$G(z; E, D) = \sup_{n \geq 1} \sup_{\substack{f \in \Gamma(nD) \\ \|f\|_E \leq 1}} \Big(\frac{1}{n} \log\big(|f(z)|\big)\Big). \tag{2.5}$$

COROLLARY 2.6
For any compact $E \subset Y$,

$$G(z; E, D) = \Phi(z; E, D).$$

Proof
For each integer $N \geq 1$, if $f \in \Gamma(nD)$, then $f^N \in \Gamma(NnD)$. It follows that the outer sup in (2.5) can be taken over n that are multiples of N, so $G(z; E, ND) = NG(z; E, ND)$. Similarly, $h \in \text{PSH}(D)$ if and only if $N \cdot h \in \text{PSH}(ND)$, so $\Phi(z; E, ND) = N\Phi(z; E, D)$.

Thus we can assume, without loss, that D is very ample. After embedding X in an appropriate $\mathbb{P}^m$ using the sections of $H^0(X, \mathscr{O}_X(D))$, we can assume that D is cut out by the hyperplane $z_0 = 0$. For sufficiently large n, $H^0(X, \mathscr{O}_X(nD))$ coincides with $H^0(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(n))|_X$ (see [16, p. 126]). Dehomogenizing at z_0, this says that $\Gamma(nD) = A_n(Y)$. □

COROLLARY 2.7
If E is not pluripolar, then $G^\star(z; E, D)$ is finite on Y and is a plurisubharmonic Weil function for D on X.

Proof
This follows from Theorem 2.4 and Corollary 2.6. □

Continuity
For the proof of Theorem 1.1, we need to smooth $G(z; E, D) = \Phi(z; E, D)$. The first step is to find conditions under which $\Phi(z; E, D)$ is continuous. We show that $\Phi(z; E, D)$ is continuous as a Weil function on X if and only if it is continuous

as a function on E, and then we give geometric conditions on E which assure that $\Phi(z; E, D)$ is continuous on E.

As usual, identify $H^0(X, \mathscr{O}_X(nD))$ with

$$\{f \in K(X) : \mathrm{div}(f) + nD \geq 0\}.$$

We claim that for any continuous $b : E \to \mathbb{R}$, the sup defining $\Phi_b(z; E, D)$ can be taken over the class $\mathrm{PSH}_+^c(D)$.

LEMMA 2.8

Let $E \subset Y$ be compact and nonempty, and let $b : E \to \mathbb{R}$ be continuous. Suppose $f \in \mathrm{PSH}(D)$ and $f|_E \leq b$. Then for any $q \in Y$ with $f(q) > -\infty$, and any $\varepsilon > 0$, there exists $g \in \mathrm{PSH}_+^c(D)$ such that $g|_E \leq b$ and $g(q) \geq f(q) - \varepsilon$.

Proof

Without loss of generality, we can assume $b \geq 0$; otherwise, subtract $\inf_{z\in E} b(z)$ from b and f. After replacing D (and b and f) with an appropriate multiple, we can assume that D is very ample. We can also assume that X is projectively embedded via the sections of $\mathscr{O}_X(D)$ and that $W(z)$ is the strong Weil exhaustion function defined by (2.3).

After these reductions, the result is essentially given by [40, Lem. 4.1]. That lemma produces a function g_0 which is continuous and plurisubharmonic on Y and such that $|g_0(z) - W(z)|$ is uniformly bounded on Y, with $g_0|_E \leq b$ and $g_0(q) \geq f(q) - \varepsilon/2$. By definition, such a function belongs to $\mathrm{PSH}_+(D)$.

However, g_0 need not a priori belong to $\mathrm{PSH}_+^c(D)$ because $g_0(z) - W(z)$ may not have a continuous extension to $\mathrm{supp}(D)$. To remedy this, we use the following device. For any $R \in \mathbb{R}$, write $B_R = \{z \in Y : W(z) \leq R\}$. Let C be a constant such that $|g_0(z) - W(z)| \leq C$ on Y. Choose $R_0 > 0$ large enough that $q \in B_{R_0}$, and let $0 < \delta < 1$ be such that $\delta < \varepsilon/(2(R_0 + C))$. Fix $R_1 > R_0$ with $R_1 > (1-\delta)C + \delta R_0$, and take $R_2 > R_1$ large enough that $R - R_1 \geq (1-\delta)(R + C)$ for all $R \geq R_2$. Define

$$g(z) = \max\big((1-\delta)g_0(z), W(z) - R_1\big). \tag{2.6}$$

By construction, $g(z)$ is continuous and plurisubharmonic on Y. For $z \in B_{R_0}$,

$$\begin{aligned} (1-\delta)g_0(z) &\geq (1-\delta)\big(W(z) - C\big) \\ &= W(z) - (1-\delta)C - \delta\big(W(z)\big) \\ &\geq W(z) - [(1-\delta)C + \delta R_0] \geq W(z) - R_1, \end{aligned} \tag{2.7}$$

so $g(z)$ coincides with $(1-\delta)g_0(z)$ on B_{R_0}. For $z \notin B_{R_2}$, we have $W(z) > R_2$, and so

$$(1-\delta)g_0(z) \leq (1-\delta)\big(W(z) + C\big) \leq W(z) - R_1. \tag{2.8}$$

Thus $g(z)$ coincides with $W(z) - R_1$ on $Y \backslash B_{R_2}$. Together, (2.6), (2.7), and (2.8) show that $g \in \mathrm{PSH}_+^c(D)$. Using the fact that $g_0(q) \le W(q) + C \le R_0 + C$, we also have

$$\begin{aligned} g(q) - f(q) &= g_0(q) - f(q) - \delta \cdot g_0(q) \\ &\ge -\frac{\varepsilon}{2} - \delta \cdot (R_0 + C) \ge -\varepsilon. \end{aligned}$$

□

It follows from Lemma 2.8 that

$$\Phi_b(z; E, D) = \sup\left\{ f(z) : f \in \mathrm{PSH}_+^c(D), \text{ and } f|_E \le b \right\}. \tag{2.9}$$

COROLLARY 2.9

Let $E \subset Y$ be compact and nonempty. Then for each continuous $b : E \to \mathbb{R}$, $\Phi_b(z; E, D)$ is lower semicontinuous as a Weil function on X.

Proof

It is clear from (2.9) that $\Phi_b(z; E, D)$ is lower semicontinuous on Y. Its lower semicontinuity on $\mathrm{supp}(D)$ as a Weil function follows by an argument like the one in the proof of Theorem 2.4. Without loss, we can assume that D is very ample. Fix $q \in \mathrm{supp}(D)$. Choose $h(z) \in K(X)^\times$ so that $D' := D + \mathrm{div}(h)$ is effective, $q \notin \mathrm{supp}(D')$, and $E \cap \mathrm{supp}(D') = \phi$. Put $Y' := X \backslash \mathrm{supp}(D')$ and $b' := b - \log(|h|)$ on E. Then

$$\Phi_{b'}(z; E, D') = \sup\left\{ f(z) : f \in \mathscr{PSH}_+^c(D'), \; f|_E \le b' \right\}$$

is lower semicontinuous at q. There is a one-to-one correspondence between functions $f \in \mathrm{PSH}_+^c(D)$ and functions $f' \in \mathrm{PSH}_+^c(D')$, given by $f' := f - \log(|h|)$. Furthermore, $f \le b$ on E if and only if $f' \le b'$. Thus for $z \notin \mathrm{supp}(D \cup D')$,

$$\Phi_b(z; E, D) - \log\left(|h(z)|\right) = \Phi_{b'}(z; E, D').$$

Since $1/h(z)$ is a local equation for D at q, $\Phi_b(z; E, D)$ is lower semicontinuous at q in the sense of Weil functions. □

COROLLARY 2.10

Let $E \subset Y$ be compact, and let $b : E \to \mathbb{R}$ be continuous. If $\Phi_b(z; E, D)$ is continuous (as a function on Y) at each point of E, then it is a continuous plurisubharmonic Weil function on X relative to D, and

$$\Phi_b(z; E, D) = \Phi_b^\star(z; E, D).$$

Proof
By Lemma 2.3, we can assume, without loss, that E is not pluripolar. By Theorem 2.4, $\Phi_b^\star(z; E, D)$ is a plurisubharmonic Weil function on X. In particular, it is upper semicontinuous on Y (and on X, in the sense of Weil functions).

Because of our hypothesis, for each $x \in E$,

$$\Phi_b^\star(x; E, D) := \limsup_{z \to x} \Phi_b(z; E, D) = \Phi_b(x; E, D) \leq b(x);$$

by Theorem 2.4, $\Phi_b^\star(z; E, D)$ belongs to $\mathrm{PSH}_+(D)$. Thus $\Phi_b^\star(z; E, D)$ is a competitor in the sup defining $\Phi_b(z; E, D)$. Hence $\Phi_b^\star(z; E, D) \leq \Phi_b(z; E, D)$. The reverse inequality is trivial, so $\Phi_b^\star(z; E, D) = \Phi_b(z; E, D)$.

By Corollary 2.9, $\Phi_b(z; E, D)$ is lower semicontinuous on Y (and on X, as a Weil function). Combining these shows that $\Phi_b(z; E, D)$ is continuous on Y and continuous on X as a Weil function. □

We also note the following fact for motivation in the nonarchimedean case.

COROLLARY 2.11
If $f \in \mathrm{PSH}_+^c(D)$, then for any $\varepsilon > 0$, there are an $m \in \mathbb{N}$ and $f_1, \ldots, f_t \in H^0(X, \mathcal{O}_X(mD))$ such that for all $z \in Y$,

$$0 \leq f(z) - \max_{1 \leq j \leq t} \left(\frac{1}{m} \log \left(|f_j(z)| \right) \right) \leq \varepsilon.$$

Moreover, the natural extension of $f(z) - \max_{1 \leq j \leq t} \left((1/m) \log(|f_j(z)|) \right)$ to X satisfies these inequalities for all $z \in \mathrm{supp}(D)$.

Proof
If, instead of requiring that the conclusion holds for all $z \in Y$, we ask only that it hold on a compact subset $K \subset Y$, then the first assertion is [40, Lem. 5.2]. For a compact set K meeting $\mathrm{supp}(D)$ but contained in an affine subset of X, the extension to Weil functions follows by an argument like that in Theorem 2.4.

The full result follows from this since X is compact, and we can cover it by a finite number of open sets with compact closure, contained in affine subsets. □

Next, we give geometric conditions that imply the hypotheses of Corollary 2.10.

PROPOSITION 2.12
Suppose that E is compact and nonempty, and suppose that there exist sections $f_1, \ldots, f_M \in H^0(X, \mathcal{O}_X(nD))$ such that

$$E = \left\{ z \in Y : |f_j(z)| \leq 1 \text{ for } j = 1 \ldots, M \right\}$$

(necessarily $M \geq \dim(X)$). For each sufficiently small $R > 1$, if

$$E_R = \{z \in Y : |f_j(z)| \leq R \text{ for } j = 1, \ldots, M\},$$

then $\Phi(z; E_R, D)$ is a continuous Weil function and $\Phi_b(z; E_R, D)$ is a continuous Weil function for any continuous $b : E_R \to \mathbb{R}$.

If E is the closure of its interior, the same assertions hold for $\Phi(z; E, D)$ and $\Phi_b(z; E, D)$.

Proof

By Corollary 2.10, it suffices to show that $\Phi_b(z; E_R, D)$ is continuous on E_R for each sufficiently small R. After replacing D with an appropriate multiple of itself, and the $f_j(z)$ with powers of themselves, we can assume that D is very ample and $n = 1$.

After embedding X in $\mathbb{P}^N$ via the sections of $H^0(X, \mathcal{O}_X(D))$, we can assume that D is the hyperplane section $z_0 = 0$. Let $g_1, \ldots, g_N$ be the remaining coordinate functions, dehomogenized at z_0. Without loss, we can assume that each g_i has been scaled so that $\|g_i\|_E \leq 1/2$. Then

$$F = (f_1, \ldots, f_M, g_1, \ldots, g_N)$$

defines an embedding $Y \hookrightarrow \mathbb{C}^{M+N}$ for which

$$E = \{z \in Y : |f_j(z)| \leq 1, |g_i| \leq 1\}.$$

The proof of [40, Th. 3.16] now shows that for all sufficiently small $R > 1$, E_R is "locally L-regular" at each $z \in E_R$. In outline, the argument is as follows:

(1) for sufficiently small $R > 1$, E_R is the closure of its interior E_R^0;

(2) for such R, trivially, $\Phi^\star(z; E_R, D) = 0$ on E_R^0;

(3) by the Łojasiewicz selection lemma for analytic curves (see [20]), for any boundary point z_0 of E_R there is an analytic curve $\gamma : [0, 1] \to Y$ such that $\gamma(0) = z_0$ and, for all $t \in (0, 1]$, $\gamma(t)$ is in E_R^0;

(4) for small enough $\delta > 0$, the function $f(t) = \Phi^\star(\gamma(t); E_R, D)$ extends to a subharmonic function on the disc $D(0, \delta)$ with $f(0) = \Phi^\star(z_0; E_R, D)$;

(5) because the interval $[0, 1]$ is not "thin" at zero (see, e.g., [18, Th. 2.7.2]), it follows that $\lim_{t \to 0^+} f(t) = f(0)$ and hence

$$\Phi^\star(z_0, E_R; D) = f(0) = \lim_{t \to 0^+} f(t) = 0.$$

Thus $0 = \Phi^\star(z_0; E_R, D) \geq \Phi(z_0; E_R, D) \geq 0$, and so $\Phi(z; E_R, D)$ is continuous at z_0. Just as in [40, Th. 3.16], it follows that E_R is locally L-regular at z_0.

Using the local L-regularity for all $z \in E$, [40, Th. 4.2] now tells us that for any continuous $b : E_R \to \mathbb{R}$ the function $\Phi_b(z; E_R, D)$ is continuous. If E is the closure of its interior, the same arguments apply to $\Phi_b(z; E, D)$. □

Smoothing

When $\Phi_b(z; E_v, D)$ is continuous, it can be smoothed to a $\mathscr{C}^\infty$ plurisubharmonic Weil function. For the convenience of the reader, we give the proof below, although the referee has pointed out that this is already known from the work of V. Maillot [21, Th. 4.6.1].

THEOREM 2.13

Let X be a smooth, projective complex manifold, and let D be an ample effective divisor on X. Suppose that $f : X \setminus \operatorname{supp}(D) \to \mathbb{R}$ is a continuous plurisubharmonic Weil function for D. Then for any $\varepsilon > 0$, there is a $\mathscr{C}^\infty$, strongly plurisubharmonic Weil function g for D such that $|g - f|(z) < \varepsilon$ (in the sense of Weil functions) for all $z \in X$.

Proof

The proof consists of two steps. The first achieves strong plurisubharmonicity, and the second gets smoothness.

The first step is trivial. We claim that there is a continuous, strongly plurisubharmonic Weil function h for D such that $|h - f|(z) < \varepsilon/2$ for all $z \in X$. Indeed, if $W(z)$ is the strong Weil exhaustion function for D given by (2.3), then for any $0 < \delta < 1$,

$$h(z) = (1 - \delta) f(z) + \delta \cdot W(z)$$

is strongly plurisubharmonic; while if δ is small enough, then $|h - f|(z) = \delta \cdot |f - W|(z)$ is uniformly small.

Replace f by h. The second step is an application of a lemma of R. Richberg [27, Satz 4.1], which says that a continuous, strongly plurisubharmonic function on $\mathbb{C}^d$ can be smoothed on any compact subset of a given open set, while remaining unchanged outside the open set. Recall that $V \Subset U$ means that the closure $\overline{V}$ of V is compact and contained in U.

LEMMA 2.14 (Richberg)

Let A be an analytic set in a region $\Omega \subseteq \mathbb{C}^d$, and let F be a continuous, strongly plurisubharmonic function on A. Let B be a (possibly empty) open subset of A such that φ is $\mathscr{C}^\infty$ on a neighborhood of $\overline{B}$. Then, for given open subsets $V \subset U \subseteq A$ with $\phi \neq V \Subset U$, and for a given $\varepsilon > 0$, there exists a continuous, strongly plurisubharmonic function $\tilde{F}$ on A with the following properties:

(1) $\tilde{F}|_{A\setminus U} = F|_{A\setminus U}$,

(2) $F \le \tilde{F} \le F + \varepsilon$ *on* U,

(3) $\tilde{F}$ *is* $\mathscr{C}^\infty$ *on a neighborhood of* $\overline{B} \cup \overline{V}$.

Using this, we can complete the proof of Theorem 2.13. Compare [27, Satz 4.2]; here, instead of smoothing an "honest" function, we are smoothing a Weil function.

Take a finite open cover $\{A_1, \ldots, A_r\}$ of X such that each A_k is isomorphic to an open set $A'_k \subset \mathbb{C}^d$ through a biholomorphic map $\tau_k : A_k \to \mathbb{C}^d$, where $d = \dim(X)$; using this isomorphism, identify A_k with a subset of $\mathbb{C}^d$. On A_k, let D be defined by the holomorphic function φ_k. In addition, choose open sets $V_1, \ldots, V_r$ and $U_1, \ldots, U_r$ such that each $V_k \Subset U_k \Subset A_k$ for each k, and such that $\{V_1, \ldots, V_r\}$ is also an open cover of X.

Put $f_0 = f$. We inductively construct a sequence of Weil functions $f_1, \ldots, f_r$ for D such that for each $k = 1, \ldots, r$,

(1) $f_k|_{X \backslash U_k} = f_{k-1}|_{X \backslash U_k}$;

(2) $0 \leq f_k - f_{k-1} \leq \varepsilon/r$ for all $z \in X$ (in the sense of Weil functions);

(3) f_k is a continuous, strongly plurisubharmonic Weil function that is $\mathscr{C}^\infty$ (in the sense of Weil functions) on a neighborhood of $\overline{V_1 \cup \cdots \cup V_k}$.

Assuming that f_{k-1} has been constructed, we define f_k as follows. By the definition of a continuous, strongly plurisubharmonic Weil function, there is a continuous, strongly plurisubharmonic function F_k on A_k such that $F_k = f_{k-1} + \log(|\varphi_k|)$ on $A_k \backslash \operatorname{supp}(D)$. Put $B_k = (V_1 \cup \cdots \cup V_{k-1}) \cap U_k$. By induction, F_k is $\mathscr{C}^\infty$ on a neighborhood of $\overline{B_k}$. Applying Richberg's lemma, we obtain a continuous, strongly plurisubharmonic function $\tilde{F}_k$ on A_k which is $\mathscr{C}^\infty$ on a neighborhood of $\overline{B_k} \cup \overline{V_k}$, coincides with F_k on $A_k \backslash U_k$, and satisfies $0 \leq \tilde{F}(z) - F(z) \leq \varepsilon/2r$ for all $z \in A_k$. Put

$$f_k = \begin{cases} \tilde{F}_k - \log(|\varphi_k|) & \text{on } A_k, \\ f_{k-1} & \text{on } X \backslash A_k. \end{cases}$$

It is easy to see that f_k satisfies (a), (b), and (c) above. Taking $g = f_r$, we obtain the theorem. □

Metrics attached to Green's functions

If D is a Cartier divisor defined by a family $\{(U_\alpha, \varphi_\alpha)\}$, then to give a metric on $\mathscr{O}_X(D)$ is the same as giving a family of functions $\rho_\alpha : U_\alpha \to \mathbb{R}_{>0}$ such that

$$\rho_\alpha = |\varphi_{\alpha\beta}| \cdot \rho_\beta \quad \text{on } U_\alpha \cap U_\beta. \tag{2.10}$$

The corresponding metric $\lVert\ \rVert$ on $\mathscr{O}_X(D)$ has the property that for any $z \in X$, and any meromorphic section $\Lambda = \{\lambda_\alpha\}$ of $\mathscr{O}_X(D)$ which is defined and nonzero at z,

$$\lVert \Lambda \rVert(z) := \frac{|\lambda_\alpha(z)|}{\rho_\alpha(z)} \quad \text{if } z \in U_\alpha.$$

We say that $\lVert\ \rVert$ is continuous (resp., $\mathscr{C}^\infty$) if each ρ_α is continuous (resp., $\mathscr{C}^\infty$).

The following result is well known.

PROPOSITION 2.15
There is a natural correspondence between Weil functions on $X \setminus \operatorname{supp}(D)$ *relative to* D *and metrics on* $\mathcal{O}_X(D)$. *This correspondence preserves the properties of continuity and of being* $\mathcal{C}^\infty$.

Proof
If $\{\!\|\ \|\!\}$ is a metric on $\mathcal{O}_X(D)$, we can define a Weil function f relative to D by

$$f(z) = -\log\left(\{\!\|\mathbf{1}\|\!\}(z)\right) \quad \text{on } U_\alpha.$$

For this f we have $\gamma_\alpha = \log(\rho_\alpha)$ on U_α.

Conversely, if f is a Weil function relative to D, with $\log(|\varphi_\alpha|) = \gamma_\alpha$ on U_α, then

$$\begin{cases} f + \log(|\varphi_\alpha|) = \gamma_\alpha & \text{on } U_\alpha, \\ f + \log(|\varphi_\beta|) = \gamma_\beta & \text{on } U_\beta \end{cases}$$

yields $\log(|\varphi_{\alpha\beta}|) = \log(|\varphi_\alpha|) - \log(|\varphi_\beta|) = \gamma_\alpha - \gamma_\beta$ on $U_\alpha \cap U_\beta$, so that

$$e^{\gamma_\alpha} = |\varphi_{\alpha\beta}| \cdot e^{\gamma_\beta}.$$

Thus $\rho_\alpha = e^{\gamma_\alpha}$ defines a metric $\{\!\|\ \|\!\}_f$ on $\mathcal{O}_X(D)$, such that for a section $\Lambda = \{\lambda_\alpha\}$,

$$\{\!\|\Lambda\|\!\}_f(z) = \frac{|\lambda_\alpha(z)|}{e^{\gamma_\alpha(z)}} \quad \text{on } U_\alpha.$$

Note that for the canonical meromorphic section $\mathbf{1}$ we have

$$\{\!\|\mathbf{1}\|\!\}_f(z) = e^{\log(|\varphi_\alpha(z)|) - \gamma_\alpha(z)} = e^{-f(z)}$$

on $X \setminus \operatorname{supp}(D)$. Clearly, this correspondence between metrics and Weil functions preserves the properties of being continuous or $\mathcal{C}^\infty$. □

If a metric $\{\!\|\ \|\!\}$ on $\mathcal{O}_X(D)$ is induced by the family $\{\rho_\alpha\}$, then the first Chern class of the metric is the $(1, 1)$-form $dd^c \log \rho_\alpha$ on X, in the sense of distributions. By (2.10), this form is well defined on the whole of X. We say that the metric is nonnegative (resp., positive) if its first Chern class is nonnegative (resp., positive) (cf. [19]).

In particular, the metric is induced by a plurisubharmonic (resp., strongly plurisubharmonic) Weil function if and only if its first Chern class is nonnegative (resp., positive).

Assume that E is not pluripolar. Since $G(z; E, D)$ is a Weil function relative to D, we can define a metric $\{\!\|\ \|\!\}_G^{\otimes n}$ on $\mathcal{O}_X(nD)$:

$$\{\!\|\mathbf{1}\|\!\}_G^{\otimes n}(z) = \frac{1}{\exp(nG(z; E, D))}.$$

PROPOSITION 2.16
Put

$$B(E, nD) := \left\{ f \in H^0(X, \mathscr{O}_X(nD)) : |f(z)| \le 1 \text{ for all } z \in E \right\},$$
$$B_G(n) := \left\{ f \in H^0(X, \mathscr{O}_X(nD)) : \{\!\!| f(z) |\!\!\}_G^{\otimes n} \le 1 \text{ for all } z \in X \right\}.$$

Then $B(E, nD) = B_G(n)$.

Proof
If $f \in B_G(n)$, then $\{\!\!| f |\!\!\}_G(z) \le 1$ for all $z \in X$. In particular, since $G(z; E, D) = 0$ on E,

$$|f(z)| = \frac{|f(z)|}{\exp(nG(z; E, D))} \le 1$$

for all $z \in E$. Thus $f \in B(E, nD)$, which shows $B_G(n) \subseteq B(E, nD)$. On the other hand, if $B_G(n) \ne B(E, nD)$, take any $g \in B(E, nD) \backslash B_G(n)$. Then $\|g\|_E \le 1$ and there exists $w \in X$ such that $\{\!\!| g |\!\!\}_G^{\otimes n}(w) > 1$. Since the metric $\{\!\!| \ |\!\!\}_G^{\otimes n}$ is upper semicontinuous, we can assume $w \in Y$. Since g is a competitor in the sup defining $G(z; E, D)$, necessarily $G(w; E, D) \ge (1/n) \log(|g(w)|)$. This implies

$$1 \ge \frac{|g(w)|}{\exp(nG(w; E, D))} = \{\!\!| g |\!\!\}_G^{\otimes mn}(w) > 1,$$

a contradiction. So we must have $B_G(n) = B(E, nD)$. □

Lastly, we introduce a measure for the distance between metrics on $\mathscr{O}_X(D)$. Given metrics $\{\!\!| \ |\!\!\}_1$ and $\{\!\!| \ |\!\!\}_2$, note that for each $z \in X$ the expression $\{\!\!| \ |\!\!\}_1(z)/\{\!\!| \ |\!\!\}_2(z)$ makes sense: if Λ is a meromorphic section of $\mathscr{O}_X(D)$ with nonzero value at z, then one can define

$$\frac{\{\!\!| \ |\!\!\}_1(z)}{\{\!\!| \ |\!\!\}_2(z)} = \frac{\{\!\!| \Lambda |\!\!\}_1(z)}{\{\!\!| \Lambda |\!\!\}_2(z)}$$

since the right-hand side is independent of the choice of Λ. We define the distance between $\{\!\!| \ |\!\!\}_1$ and $\{\!\!| \ |\!\!\}_2$ to be

$$\delta(\{\!\!| \ |\!\!\}_1, \{\!\!| \ |\!\!\}_2) = \sup_{z \in X} \left(\left| \log \left(\frac{\{\!\!| \ |\!\!\}_1(z)}{\{\!\!| \ |\!\!\}_2(z)} \right) \right| \right).$$

It is easily checked that $\delta(\{\!\!| \ |\!\!\}_1, \{\!\!| \ |\!\!\}_2)$ is symmetric, transitive, and nonnegative, and it is zero if and only if $\{\!\!| \ |\!\!\}_1 = \{\!\!| \ |\!\!\}_2$. Thus δ is a distance function on the set of metrics on $\mathscr{O}_X(D)$.

LEMMA 2.17
Let f and g be continuous or plurisubharmonic Weil functions on X relative to D,

with associated metrics $\| \ \|_f$ *and* $\| \ \|_g$ *on* $\mathcal{O}_X(D)$. *Then*

$$\frac{\| \ \|_f(z)}{\| \ \|_g(z)} = e^{(g-f)(z)}$$

relative to the natural extension of $f - g$, *so*

$$\delta(\| \ \|_1, \| \ \|_2) = \sup_{z \in X} |(f-g)(z)|.$$

Proof

Let $f + \log(|\varphi_\alpha|) = \gamma_\alpha$, $g + \log(|\varphi_\alpha|) = \tau_\alpha$ on $U_\alpha \setminus \operatorname{supp}(D)$. Then $g - f = \tau_\alpha - \gamma_\alpha$, and for a local section λ_α on U_α,

$$\|\lambda_\alpha\|_f(z) = \frac{|\lambda_\alpha(z)|}{e^{\gamma_\alpha(z)}}, \qquad \|\lambda_\alpha\|_g(z) = \frac{|\lambda_\alpha(z)|}{e^{\tau_\alpha(z)}}.$$

Thus

$$\frac{\| \ \|_f(z)}{\| \ \|_g(z)} = e^{\tau_\alpha(z)-\gamma_\alpha(z)} = e^{(g-f)(z)}.$$ □

We can now give the main approximation theorem for metrics defined by Green's functions.

THEOREM 2.18

Let X *be a smooth, projective complex manifold, let* D *be an effective, ample Cartier divisor on* X, *and let* $E \subset X \setminus \operatorname{supp}(D)$ *be a compact set that is not pluripolar. Assume that* $G(z; E, D)$ *is a continuous Weil function for* D, *and let* $\| \ \|_G$ *be the metric on* $\mathcal{O}_X(D)$ *induced by* $G(z; E, D)$. *Then for any* $\varepsilon > 0$, *there is a smooth, positive metric* $\| \ \|_g$ *on* $\mathcal{O}_X(D)$ *such that*

$$\delta(\| \ \|_g, \| \ \|_G) < \varepsilon.$$

If X *and* D *are induced by varieties defined over* $\mathbb{R}$, *and* E *is stable under complex conjugation, then we can require the metric* $\| \ \|_g$ *to be stable under complex conjugation as well.*

Proof

The first part follows immediately from Theorem 2.13 and Corollary 2.6. The assertion concerning stability under complex conjugation is trivial, by symmetrization. □

3. Functoriality properties of Green's functions

In this section we establish the behavior of the extremal Green's function under pullbacks by finite surjective morphisms and by products.

THEOREM 3.1 (Pullback formula)
Let Z and X be smooth, projective complex manifolds of the same dimension d. Assume that D is an effective ample divisor on X, $E \subset X \setminus \operatorname{supp}(D)$ is compact and not pluripolar, and $\pi : Z \to X$ is a finite, surjective morphism. Then

$$G\big(\pi(w); E, D\big) = G\big(w; \pi^{-1}(E), \pi^*(D)\big).$$

To prove Theorem 3.1, we need a lemma. Given $N \in \mathbb{N}$, let $s_k(\zeta_1, \ldots, \zeta_N)$ be the kth elementary symmetric function in $\zeta_1, \ldots, \zeta_N$.

LEMMA 3.2
The map $\operatorname{Sym} : \mathbb{C}^N \to \mathbb{C}^N$ *given by*

$$\operatorname{Sym}(\zeta_1, \ldots, \zeta_N) = \big(s_1(\zeta_1, \ldots, \zeta_N), \ldots, s_N(\zeta_1, \ldots, \zeta_N)\big)$$

is finite and hence proper. In particular, $\operatorname{Sym}^{-1}(\{(w_1, \ldots, w_N) : |w_j| \le 1\})$ *is compact, so there is a constant $B < \infty$ such that*

$$\operatorname{Sym}^{-1}\big(\{(w_1, \ldots, w_N) : |w_j| \le 1\}\big) \subseteq \big\{(\zeta_1, \ldots, \zeta_N) : |\zeta_j| \le B\big\}.$$

Proof
Each ζ_i is a root of $z^N + \sum_{k=1}^N (-1)^k s_k(\zeta_1, \ldots, \zeta_N) z^{N-k} = 0$. □

Proof of Theorem 3.1
By Proposition I.4.4 of R. Hartshorne [16], $\pi^*(D)$ is ample on Z. First, we claim that $G(\pi(w); E, D) \le G(w; \pi^{-1}(E), \pi^*(D))$.

If $f \in H^0(X, \mathscr{O}_X(nD))$ satisfies $\|f\|_E \le 1$, then

$$f \circ \pi = \pi^*(f) \in H^0\big(Z, \mathscr{O}_Z(n\pi^*(D))\big)$$

with $\|\pi^*(f)\|_{\pi^{-1}(E)} = \|f\|_E \le 1$, so

$$\frac{1}{n}\log\big(|f(\pi(w))|\big) = \frac{1}{n}\log\big(|\pi^*(f)(w)|\big) \le G\big(w, \pi^{-1}(E), \pi^*(D)\big).$$

Taking the sup over such f's yields the inequality.

Next, we claim that $G(w; \pi^{-1}(E), \pi^*(D)) \le G(\pi(w); E, D)$.

Let $\mathscr{U} = \{U_i\}$ be an affine open cover of X, and suppose that D is represented by φ_i on U_i. Put $V_i = \pi^{-1}(U_i)$; then $\mathscr{V} = \{V_i\}$ is an open cover of Z, and on V_i, $\pi^*(D)$ is represented by $\psi_i := \pi^*(\varphi_i) = \varphi_i \circ \pi$. Since U_i is affine and π is finite, V_i is also affine. By [16, Lem. III.4.1], we can compute cohomology groups as the Čech groups:

$$H^0\big(X, \mathscr{O}_X(nD)\big) = \check{H}^0\big(\mathscr{U}, \mathscr{O}_X(nD)\big),$$
$$H^0\big(Z, \mathscr{O}_Z(n\pi^*(D))\big) = \check{H}^0\big(\mathscr{V}, \mathscr{O}_Z(n\pi^*(D))\big).$$

Suppose that $g \in H^0\big(Z, \mathscr{O}_Z(n\pi^*(D))\big)$ is such that $\|g\|_{\pi^{-1}(E)} \leq 1$. We can assume that g is represented by $g_i \in \mathscr{O}(V_i)$ with $g_i = \psi_i^n g$ on V_i; consequently,

$$g = \frac{g_i}{\psi_i^n} = \frac{g_j}{\psi_j^n} \quad \text{on } V_i \cap V_j. \tag{3.1}$$

Write $N = \deg(\pi)$, and for each $z \in X$, let $\pi^{-1}(z) = \{w_1, \ldots, w_N\}$, counting multiplicities. On U_i, for $1 \leq k \leq N$, put

$$S_i^{(k)}(z) = s_k\big(g_i(w_1), \ldots, g_i(w_N)\big).$$

We readily see that $S_i^{(k)}(z) \in \mathscr{O}(U_i)$, noting that π is a covering map outside the branch locus and using the Riemann extension theorem. If $z \in U_i \cap U_j$, then using (3.1) and the fact that $\psi_i = \varphi_i \circ \pi$, one finds that

$$\frac{S_i^{(k)}(z)}{\varphi_i^{nk}(z)} = \frac{S_j^{(k)}(z)}{\varphi_j^{nk}(z)}.$$

Thus $\{S_i^{(k)}\}$ is a global section of $\mathscr{O}_X(nkD)$, and it corresponds to a global meromorphic function $G^{(k)}$ with

$$G^{(k)} = \frac{S_i^{(k)}}{\varphi_i^{nk}} \quad \text{on } U_i.$$

Put $C = 2^N$. Since $\|g\|_{\pi^{-1}(E)} \leq 1$, $\|G^{(k)}\|_E \leq \binom{N}{k} < C$ for $1 \leq k \leq N$. Consequently, by the extremal property of $G(z; E, D)$, for each k we have

$$\frac{1}{nk} \log\left(\left|\frac{G^{(k)}(z)}{C}\right|\right) \leq G(z; E, D),$$

which can be reformulated as

$$\left|s_k\left(\frac{g_i(w_1)}{C^{1/k}\varphi_i^n(z)\exp(nG(z; E, D))}, \ldots, \frac{g_i(w_N)}{C^{1/k}\varphi_i^n(z)\exp(nG(z; E, D))}\right)\right| \leq 1$$

for $1 \leq k \leq N$. Since $\varphi_i(z) = \psi_i(w_j)$ for each j, we can apply Lemma 3.2 and conclude that for each $w \in Z \backslash \operatorname{supp}(\pi^*(D))$,

$$\left|\frac{g(w)}{C^{1/k}\exp(nG(\pi(w); E, D))}\right| \leq B.$$

Transposing terms, taking logarithms, and dividing by n, we get

$$\frac{1}{n}\log\big(|g(w)|\big) \leq G\big(\pi(w); E, D\big) + \frac{1}{n}\log(BC^{1/k}). \tag{3.2}$$

However, for each $M \in \mathbb{N}$, we have $g^M \in H^0(Z, \mathscr{O}_Z(nM\pi^*(D)))$ and $\|g^M\|_{\pi^{-1}(E)} \leq 1$. Replacing n by nM and g by g^M in (3.2) and then letting $M \to \infty$ gives

$$\frac{1}{n} \log\big(|g(w)|\big) \leq G\big(\pi(w); E, D\big).$$

Finally, taking the sup over all functions $(1/n)\log(|g(w)|)$ yields

$$G\big(w; \pi^{-1}(E), \pi^*(D)\big) \leq G\big(\pi(w); E, D\big).$$ □

THEOREM 3.3 (Product formula)
Let X_1 and X_2 be smooth, projective complex manifolds, let D_1 on X_1 and D_2 on X_2 be ample effective divisors, and let $E_i \subset X_i \backslash \operatorname{supp}(D_i)$ be compact and not pluripolar, for $i = 1, 2$. Put $X = X_1 \times X_2$, let $\pi_i : X \to X_i$ be the projection, set $D = \pi_1^*(D_1) + \pi_2^*(D_2)$, and take $E = E_1 \times E_2$. For $z \in X \backslash \operatorname{supp}(D)$, write $z = (z_1, z_2)$ where $z_i \in X_i \backslash \operatorname{supp}(D_i)$. Then

$$G(z; E, D) = G(z_1; E_1, D_1) + G(z_2; E_2, D_2). \tag{3.3}$$

Proof
It is well known that in this context, D is ample.

In proving the theorem, we use the equality of Green's functions and the plurisubharmonic extremal functions, and we work with the latter. For simplicity of notation, write $\Phi(z)$ for the function on the right-hand side of (3.3). We want to show that $\Phi(z) = \Phi(z; E, D)$.

Let us first assume that $\Phi(z_1; E_1, D_1)$ and $\Phi(z_2; E_2, D_2)$ are continuous plurisubharmonic Weil functions relative to D_1 and D_2, respectively. This means that $\Phi(z)$ is continuous as well. Under this assumption, it is clear that $\Phi(z) \leq \Phi(z; E, D)$ because $\Phi(z) \in \operatorname{PSH}(D)$ and $\Phi|_E \leq 0$. To prove the reverse inequality, we need the following version of the maximum principle for plurisubharmonic functions, due to Bedford and Taylor. Write $d_i = \dim(X_i)$, and put $d = \dim(X) = d_1 + d_2$.

LEMMA 3.4
If $\Omega \Subset X \backslash \operatorname{supp}(D)$, and u, v are continuous plurisubharmonic functions on $\overline{\Omega}$, with $(dd^c u)^d = 0$ and $u \geq v$ on $\partial\Omega$, then $u \geq v$ on Ω.

Proof
See [4, Cor. 4.5] or [40, Th. 1.9]. □

By Theorem 2.4, in the complement of E_i, each $\Phi(z_i; E_i, D_i)$ satisfies the complex Monge-Ampère equation

$$\big(dd^c \Phi(z_i; E_i, D_i)\big)^{d_i} = 0.$$

It follows from this and the binomial expansion that

$$\left(dd^c\Phi(z; E, D)\right)^d = 0$$

in the complement of E.

Suppose that $f \in \mathrm{PSH}^c_+(D)$ satisfies $f|_E \leq 0$. Since $\Phi(z)$ is a Weil function for D, there is a constant $C > 0$ such that

$$f(z) \leq \Phi(z) + C \tag{3.4}$$

for all $z \in Y = X \setminus \mathrm{supp}(D)$. Given $r > 0$, put $B_r := \{z \in Y : \Phi(z) < r\}$. By (3.4), for any $\varepsilon > 0$ there is an $r_0 = r_0(\varepsilon) > 0$ such that $E \subset B_{r_0}$ and, for any z with $\Phi(z) \geq r_0$,

$$f(z) \leq (1+\varepsilon)\Phi(z). \tag{3.5}$$

Fix $r > r_0$, and take $\Omega = B_r \setminus E$. By assumption, $f(z) \leq 0 = (1+\varepsilon)\Phi(z)$ on E, and by (3.5), $f(z) \leq (1+\varepsilon)\Phi(z)$ on ∂B_r. We have already seen that $(dd^c\Phi)^d = 0$ in the complement of E. Hence, by Theorem 3.4, $f(z) \leq (1+\varepsilon)\Phi(z)$ in Ω.

Since $\bigcup_{r>r_0} B_r = Y$, and $\varepsilon > 0$ is arbitrary, it follows that $f \leq \Phi(z)$ for all $z \in Y$. Taking the supremum over all $f \in \mathrm{PSH}^c_+(D)$, we obtain $\Phi(z; E, D) \leq \Phi(z)$. Thus the theorem is proved when the $\Phi(z_i; E_i, D_i)$ are continuous.

The general case follows from [18, Cor. 5.1.5] and the following simple property. Suppose $F = \bigcap_{k=1}^\infty F_k$, where $\{F_k\}_k$ is a decreasing sequence of compact sets in Y. Then

$$\Phi(z; F, D) = \sup_{k>0} \Phi(z; F_k, D).$$

This property follows immediately from the definition (cf. [18, Cor. 5.1.2]). □

4. Adelic intersection theory and capacity theory

Let X be a smooth, connected projective variety of dimension d over a number field K. In this section we discuss the problem of developing an adelic arithmetic intersection theory along the lines of Gillet and Soulé's work. One goal of such a theory is to compute sectional capacities via analytic methods at all places, without having to take limits of intersections as in Theorem 1.1.

The model we have in mind is the theory developed by R. Rumely in [31] when $d = 1$, that is, when X is a curve. Here we outline some properties that a generalization of [31] to arbitrary dimension should have. This generalization is far from being realized. We, however, discuss an example when $d = 2$ in which the formal properties we require for an adelic intersection pairing lead to the correct value for the sectional capacity of a particular set.

The desired adelic intersection theory should be analytic in character and rich enough that not only "smooth" Green's functions but also certain types of singular

Green's functions are permitted. Divisorial objects of the theory should be pairs consisting of a K-rational divisor D on the variety X, and a family of Green's functions for all places v of K:

$$\mathscr{D} = (D, \{G_v\}_{v\in\mathscr{M}_K}). \tag{4.1}$$

Here, for each v, the "v-adic Green's function"

$$G_v : X(\mathbb{C}_v)\setminus \operatorname{supp}(D) \to \mathbb{R}$$

should be a Weil function for D, stable under $\operatorname{Gal}^c(\mathbb{C}_v/K_v)$.

We propose that the Weil functions allowed should be those that are linear combinations of "v-adic plurisubharmonic functions." In the archimedean case, v-adic plurisubharmonic functions are just the classical plurisubharmonic functions. In the nonarchimedean case the space of v-adic plurisubharmonic functions should be the smallest class containing functions of the form

$$G_v(z) = \max_{1\le i\le n} \left(\log(|g_i(z)|_v)\right) \tag{4.2}$$

with each $g_i(z) \in \mathbb{C}_v(X)$, and which is stable under the operations satisfied by archimedean plurisubharmonic functions: taking positive linear combinations, taking the sup of finitely many functions, taking uniform limits, taking limits of arbitrary decreasing sequences which do not identically approach $-\infty$, and taking the upper semicontinuous regularization of the sup of a family locally bounded from above (see [18, p. 69]).

Such $G_v(z)$ should satisfy an analogue of the Poincaré-Lelong formula

$$dd^c G_v = \omega_{G_v} - \delta_D,$$

where ω_{G_v} is a "(1, 1)-current with positive measure coefficients" and δ_D is the Dirac current, integration over $D(\mathbb{C}_v)$. There should also be an exterior product for the ω_v. In the archimedean case, the dd^c-operator and exterior product are those given by Bedford and Taylor [4]. In the nonarchimedean case, the corresponding theory is purely conjectural. However, we offer the following speculations about how it might be obtained.

Previous investigations by various authors (see [7], [17], [31], [41], [42], [43]) suggest that the spaces of "forms" and "currents" should be related to intersection theory and should be limits of spaces of functionals attached to the tower of all models of X_v. In our analytic context, probably the theory of v-adic plurisubharmonic functions should first be developed in the setting of rigid analytic geometry, for affinoid domains. Limits should be taken over the tower of all analytic reductions of the space, and objects attached to a given analytic reduction should be considered "smooth." Perhaps V. Berkovich's formulation of rigid geometry (see [5]), in which the underlying spaces are compact, may be useful. The Berkovich space attached to an irreducible affinoid algebra $A/\mathbb{C}_v$ is the set of all bounded multiplicative seminorms on A. It is a path-connected, compact Hausdorff space in which the Tate space $\max(A)$ is dense. The Berkovich space associated to an arbitrary rigid analytic space is constructed by gluing affinoid ones. Let X_v^{an} be the rigid analytic space attached to X_v. In addition to points coming from $X_v^{an}(\mathbb{C}_v)$, the Berkovich space associated to X_v^{an} has a point corresponding to each irreducible component of the special fibre of each analytic reduction of X_v^{an}.

If $\dim(X) = d$, and $G(z)$ is v-adic plurisubharmonic on a subdomain of X_v^{an}, it seems plausible that $(dd^c(G))^d$ should be a nonnegative measure on the corresponding Berkovich space. Let us call a function $G = G_v$ of the form (4.2) "good" if the g_i have no common zeros and for each x there are at most $d + 1$ functions $g_i(z)$ for which $|g_i(x)|_v$ is maximal. Suppose that $G(z)$ is good, and suppose that $|g_1(z)|_v, \ldots, |g_{d+1}(z)|_v$ are maximal at some x. If $g_1(z) - g_{d+1}(z), \ldots, g_d(z) - g_{d+1}(z)$ have k common zeros in the set where $|g_1(z)|_v = \cdots = |g_{d+1}(z)|_v$ is maximal, it seems likely that $(dd^c(G))^d$ should assign weight $d! \cdot k$ to that set.

We expect each plurisubharmonic function $H(z)$ to have a "generic value" on the set of points reducing to each component of the special fibre of each analytic reduction of X_v^{an}. (By this, we mean a value λ such that for each $\varepsilon > 0$, "almost all" the values of $H(z)$ on points reducing to that component should satisfy $|H(z) - \lambda| < \varepsilon$.) If U is affinoid of dimension d, and $H, G_1, \ldots, G_d$ are v-adic plurisubharmonic functions, there should be an "integral"

$$\int_U H(z)\, dd^c G_1 \wedge \cdots \wedge dd^c G_d$$

which is a weighted average of the generic values of $H(z)$. The integral is expected to be multilinear (not alternating) in $G_1, \ldots, G_d$ since $dd^c G_k$ should be analogous to a $(1, 1)$-form. Note that if an appropriate multilinear integral could be constructed, it could be used to define the "dd^c" operator; namely, if G is plurisubharmonic on U, then $dd^c G$ should be the functional that for each 1-dimensional $W \subset U$ takes each plurisubharmonic H on W to $\int_{W(\mathbb{C}_v)} H\, dd^c G|_W$.

For the purposes of capacity theory, it would be sufficient to construct a $(d + 1)$-fold intersection product, or height pairing. The intersection product should satisfy a moving lemma for adelic principal divisors $\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), \{-\log_v(|f(z)|_v)\}_{v \in M_K}\big)$:

$$\big\langle \mathscr{D}_1 + \widehat{\mathrm{div}}(f_1), \ldots, \mathscr{D}_{d+1} + \widehat{\mathrm{div}}(f_{d+1})\big\rangle = \langle \mathscr{D}_1, \ldots, \mathscr{D}_{d+1}\rangle. \tag{4.3}$$

When the underlying algebraic divisors are effective and meet properly, the intersection product should also have a decomposition into local terms:

$$\langle \mathscr{D}_1, \ldots, \mathscr{D}_{d+1} \rangle = \sum_v \langle \mathscr{D}_1, \ldots, \mathscr{D}_{d+1} \rangle_v \log(q_v), \tag{4.4}$$

where $\langle \mathscr{D}_1, \ldots, \mathscr{D}_{d+1} \rangle_v$ is defined by a formula analogous to the one in Gillet-Soulé's theory; if $\mathscr{D}_i = (D_i, \{G_{v,i}\}_{v \in \mathscr{M}_K})$ and $dd^c G_{v,i} = \omega_{v,i} - \delta_{D_i}$, then

$$\begin{aligned} \langle \mathscr{D}_1, \ldots, \mathscr{D}_{d+1} \rangle_v &= \sum_{k=1}^{d+1} \int_{(D_1 \cdot \ldots \cdot D_{k-1})(\mathbb{C}_v)} G_{v,k}\, \omega_{v,k+1} \wedge \cdots \wedge \omega_{v,d+1} \\ &= \int_{X(\mathbb{C}_v)} G_{v,1}\, \omega_{v,2} \wedge \cdots \wedge \omega_{v,d+1} \\ &\quad + \int_{D_1(\mathbb{C}_v)} G_{v,2}\, \omega_{v,3} \wedge \cdots \wedge \omega_{v,d+1} \\ &\quad + \cdots + G_{v,d+1}(D_1 \cdot \ldots \cdot D_d). \end{aligned} \tag{4.5}$$

We propose the following normalization for the terms in (4.3), (4.4), and (4.5). For archimedean v, let $q_v = e^{1/2}$ if $K_v \cong \mathbb{R}$, and let $q_v = e$ if $K_v \cong \mathbb{C}$. For nonarchimedean v, let q_v be the order of the residue field $k_v = O_v/\pi_v O_v$. Write $\log_v(x)$ for the logarithm to the base q_v. Let $|x|_v$ be the canonical absolute value on K_v given by the modulus of additive Haar measure. Also, write $|x|_v$ for the unique extension of this absolute value to $\mathbb{C}_v$. For archimedean v, $|\ |_v = |\ |$ if $K_v \cong \mathbb{R}$, while $|\ |_v = |\ |^2$ if $K_v \cong \mathbb{C}$. In either case, if $f(z)$ is a rational function, then $\log_v(|f(z)|_v) = \log(|f(z)|^2)$, as required for the Poincaré-Lelong formula $dd^c \log_v(|f(z)|_v) + \delta_{\mathrm{div}(f)} = 0$, while the factor $\log(q_v)$ in (4.4) corresponds to the $1/2$ in Gillet-Soulé's isomorphism (0.3). For nonzero $\kappa \in K$, the product formula reads

$$\sum_{v \in \mathscr{M}_K} \log_v(|\kappa|_v) \log(q_v) = 0.$$

Suppose that there exists an adelic intersection theory along the lines suggested above. Because of the stability properties of v-adic plurisubharmonic functions (and the identity $\min(x, y) = x + y - \max(x, y)$), for each v the classical Weil distributions

$$G_v(z) = \min_{1 \le i \le m} \max_{1 \le j \le n} \big(\log_v(|g_{ij}(z)|_v) \big), \tag{4.6}$$

where $g_{ij}(z) \in \mathbb{C}_v(X)$, are among the Green's functions allowed; so is the upper semicontinuous regularization $G^\star(z; E_v, D)$ of Green's function $G(z; E_v, D)$ attached to a set $E_v \subset X_v(\mathbb{C}_v)$ and ample divisor D:

$$G(z; E_v, D) = \sup_{n \ge 1} \sup_{\substack{f \in K_v \otimes_K \Gamma(nD) \\ \|f\|_{E_v} \le 1}} \Big(\frac{1}{n} \log_v(|f(z)|_v) \Big). \tag{4.7}$$

In the archimedean case we have seen that if E_v is not pluripolar, then $G(z; E_v, D)$ is finite for all $z \in X(\mathbb{C}_v) \backslash \operatorname{supp}(D)$ and is a Weil function for D. For nonarchimedean v, we expect this to hold if and only if the local sectional capacity $S_\gamma(E_v, D)_v$ introduced in [33, Sec. 14] is positive.

We now come to our main reason for desiring such a theory. Let D be an effective, ample, K-rational divisor on X, and let $\mathbb{E} = \prod_v E_v$ be an adelic set such that $\mathbb{E}$ and D satisfy the standard hypotheses stated in the introduction, and $S_\gamma(\mathbb{E}, D) > 0$. Then

$$\mathscr{D} = \big(D, \{G^\star(z; E_v, D)\}_{v \in \mathscr{M}_K}\big) \tag{4.8}$$

is an arithmetic divisor in the theory.

CONJECTURE 4.1

For an adelic intersection theory with the formal properties described above, and for the divisor (4.8),

$$-\log\big(S_\gamma(\mathbb{E}, D)\big) = \mathscr{D}^{d+1}. \tag{4.9}$$

The philosophy behind the conjecture is that just as in classical electrostatics, each E_v should determine a “field,” its Green’s function $G(z; E_v, D)$, which represents its potential-theoretic size. A function $f \in K_v \otimes_K \Gamma(nD)$ which contributes to Green’s function should contribute asymptotically to the volume measured by the sectional capacity, and conversely. Theorem 1.1 suggests that arithmetic intersection theory gives the right way to extract the capacity from the field.

For curves, the conjecture is true; formula (4.9) is the final assertion in [31] for the adelic intersection pairing constructed there. Note that in the theory developed in [31], the sets E_v considered need not have continuous Green’s functions. A crucial point is that the *upper semicontinuous* (subharmonic) Green’s functions $G^\star(z; E_v, D)$ must be used in order for the pairing to respect linear equivalence (see [31, p. 342]).

In higher dimensions, an adelic intersection theory sufficient for the needs of capacity theory remains to be developed. In some cases, though, one can propose natural interpretations of the terms in (4.5) which lead to correct values for sectional capacities, as in the example in Section (A). In Section (B) we discuss work on adelic intersection theory by other authors.

(A) A numerical example in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $K = \mathbb{Q}$, and take $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let D be a Cartier divisor with associated Weil divisor $r_1 H_1 + r_2 H_2$, where $0 < r_1, r_2 \in \mathbb{Z}$ and $H_i = \pi_i^{-1}(\infty)$ for $i = 1, 2$, so that $\mathscr{O}_X(D) \cong \mathscr{O}_X(r_1, r_2)$. Put $Y = X \backslash \operatorname{supp}(D) \cong \mathbb{A}^1 \times \mathbb{A}^1$. Fixing coordinate functions on the two copies of $\mathbb{A}^1$, we have $Y(\mathbb{C}_v) \cong \mathbb{C}_v \times \mathbb{C}_v$ for each v. Let

$$\mathbb{E} = \prod_v E_v$$

be as follows. At the archimedean place, fix $1 \le p < \infty$, and let E_∞ be the L^p unit ball in $\mathbb{C} \times \mathbb{C}$,

$$E_\infty = \left\{(u, w) \in \mathbb{C} \times \mathbb{C} : |u|^p + |w|^p \le 1\right\}.$$

For each nonarchimedean v, write $\widehat{O}_v$ for the ring of integers of $\mathbb{C}_v$, and let $q = q_v$ be the rational prime underlying v. Fix disjoint finite sets S_1, S_2 of nonarchimedean places, and for each $v \in S_1$, let $E_v = \mathbb{Z}_q \times \mathbb{Z}_q$. For each $v \in S_2$, let $E_v = q^{m_{v1}} \widehat{O}_v \times q^{m_{v2}} \widehat{O}_v$, where $m_{v1}, m_{v2} \in \mathbb{Z}$. Finally, for each $v \notin S_1 \cup S_2 \cup \{\infty\}$, let $E_v = \widehat{O}_v \times \widehat{O}_v$.

PROPOSITION 4.2
For all $0 < r_1, r_2 \in \mathbb{Z}$,

$$\begin{aligned} -\log\left(S_\gamma(\mathbb{E}, D)\right) = \frac{1}{p}\Big((r_1 + r_2)^3 \log(r_1 + r_2) - r_1^2 r_2 - r_1 r_2^2 \\ - (r_1^3 + 3r_1^2 r_2)\log(r_1) - (r_2^3 + 3r_1 r_2^2)\log(r_2)\Big) \\ + \sum_{v \in S_1} (3r_1^2 r_2 + 3r_1 r_2^2)\frac{1}{q_v - 1}\log(q_v) \\ + \sum_{v \in S_2} (3m_{v1} \cdot r_1^2 r_2 + 3m_{v2} \cdot r_1 r_2^2)\log(q_v). \end{aligned} \tag{4.10}$$

The proof, which is purely computational, is given in the appendix. In the appendix we also determine Green's functions $G(z; E_v, D)$ for all v. We then compute $\mathscr{D}^3$ for the divisor $\mathscr{D} = (D, \{G^\star(z; E_v, D)\}_{v \in \mathscr{M}_K})$ using the heuristics for an adelic intersection theory proposed above. The result is the same quantity (4.10) as in Proposition 4.2, verifying Conjecture 4.1 in this case:

$$-\log\left(S_\gamma(\mathbb{E}, D)\right) = \mathscr{D}^3. \tag{4.11}$$

Remark. For curves, the main theorem of [30] asserts that if $D = r_1(\zeta_1) + \cdots + r_m(\zeta_m)$, then $-\log(S_\gamma(\mathbb{E}, D))$ is a quadratic form in $r_1, \ldots, r_m$ whose matrix is the "Green's matrix" in Cantor's capacity theory (see [29]). In higher dimensions, it had been thought that $-\log(S_\gamma(\mathbb{E}, D))$ might be a homogeneous form of degree $d + 1$ in the coefficients of the components of D. Proposition 4.2 shows that this is not always the case. In terms of Conjecture 4.1, the reason for the breakdown is nonlinearity of $G(z; E_v, D)$ in the divisor D, for dimensions $d \ge 2$ (see formula (A.9)).

Remark. In the appendix, after choosing appropriate divisors $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ linearly equivalent to $\mathscr{D}$, we represent $\mathscr{D}^3$ as a sum of local intersection terms $\sum_v \langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_v \log(q_v)$.

The referee has pointed out that when $S_1 = S_2 = \phi$ in the example above, so that all the nonarchimedean E_v are trivial, the equality

$$-\log\big(S_\gamma(\mathbb{E}, D)\big) = \langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_\infty \log(q_\infty) \tag{4.12}$$

can be derived from general principles. This yields (4.11), provided one assumes that each nonarchimedean local intersection index $\langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_v$ is zero, as follows from the heuristics in the appendix when E_v is trivial. More generally, (4.12) holds for any $\mathbb{E} = \prod_v E_v$ for which all the nonarchimedean E_v are trivial, provided $G(z; E_\infty, D)$ is continuous (and $\mathbb{E}$ and D satisfy the standard hypotheses).

This comes out as follows. Let $\mathfrak{X} = \mathbb{P}^1 \times \mathbb{P}^1/\operatorname{Spec}(O_K)$, let $\mathscr{D}$ be the natural extension of D to $\mathfrak{X}$, and let $\mathscr{L} = \mathscr{O}_{\mathfrak{X}}(\mathscr{D})$. For our set E_∞, Green's function $G(z; E_\infty, D)$ is continuous. (This can be seen by direct computation, as in the appendix, or by applying the argument in the proof of Proposition 2.12.) By Theorem 2.18 (or by Maillot [21, Th. 4.6.1]), the metric $\{\!\|\ \|\!\}$ on $\mathscr{L}(\mathbb{C})$ associated to $G(z; E_\infty, D)$ can be uniformly approximated by a sequence of smooth, positive metrics $\{\!\|\ \|\!\}_n$. Hence the metrized line bundle $\overline{\mathscr{L}}$ is "admissible" in the sense of Zhang [43]. Zhang's formalism extending the arithmetic Hilbert-Samuel formula to such bundles yields the equality $-\log(S_\gamma(\mathbb{E}, D)) = \overline{\mathscr{L}}^3$. On the other hand, Maillot's extension of Gillet-Soulé's intersection pairing to the generalized Chow groups $\widehat{\mathrm{CH}}^*_{\mathrm{int}}(\mathfrak{X})$ enables us to compute $\overline{\mathscr{L}}^3$. Unwinding the definitions in [21, Sec. 5.5.2] for the choice of sections used in the appendix and comparing the result with (4.5) gives $\overline{\mathscr{L}}^3 = \langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_\infty \log(q_\infty)$.

(B) Other works

The following papers may bear on the development of an adelic intersection theory sufficient for the needs of capacity theory.

For curves, E. Kani [17] has developed a nonarchimedean potential theory in the context of rigid analysis. He constructs a finitely additive theory of integration of "generic values" of real-valued functions on affinoid spaces, and he interprets measures on the boolean algebra generated by rational subdomains as analogous to 1-forms. (We would consider them (1, 1)-forms.) He creates a dictionary between objects in potential theory and intersection theory, interpreting Green's functions and harmonic measures in terms of heights and intersection pairings. He associates a "potential theory" to each analytic reduction of an affinoid space, establishes an energy-minimization principle, and solves the corresponding Dirichlet and Neumann problems. However, his theory concerns only “smooth” objects in the context of the present paper, and it does not consider passage to a limit over all analytic reductions.

First for curves, and then for varieties of arbitrary dimension, Zhang [41], [42], [43] has constructed an adelic intersection theory. He works in the context of metrized line bundles and shows that if a sequence of metrics converge uniformly, then the associated intersection numbers also converge. In this way, he is able to define intersection numbers for metrized line bundles with continuous, but not necessarily $\mathscr{C}^\infty$, metrics at archimedean places and with uniform limits of “algebraic” metrics at finite places. Zhang gives a number of striking applications, including a Riemann-Roch theorem in the case of curves, and a Nakai-Moishezon theorem. However, his theory is not adequate for capacity theory, as the metrics associated to extremal Green’s functions are not in general uniform limits but rather arise from upper semicontinuous envelopes of the Green’s functions which Zhang considers.

For varieties of arbitrary dimension, S. Bloch, Gillet, and Soulé [7] have developed a nonarchimedean Arakelov theory, using intersection theory and taking limits over the tower of all models of the variety. Their theory is conditional on resolution of singularities for models, but under that hypothesis, it establishes a complete dictionary between objects in differential geometry and intersection theory. In particular, for each p with $0 \leq p \leq \dim(X)$, they define analogues of the four groups: “closed (p, p)-forms,” “(p, p)-forms modulo the image of ∂ and $\overline{\partial}$,” “closed (p, p)-currents,” and “(p, p)-currents modulo the image of ∂ and $\overline{\partial}$.” These groups are identified with appropriate direct and inverse limits of the Chow cycle groups and operational Chow groups. They define a dd^c-operator, which corresponds to a pushforward from the special fibre to the general fibre, followed by Poincaré duality, then a pullback from the general fibre to the special fibre. They show that these groups fit into natural exact sequences analogous to the ones that occur in the archimedean case. However, in addition to being conditional, their theory defines only Green’s functions associated to “smooth” (p, p)-forms: those arising from a single model. Green’s functions $G(z; E_v, D)$ are not in general smooth in this sense.

For curves, J.-B. Bost [8] has developed an extension of Arakelov theory which incorporates singular Green’s functions at archimedean places. Bost’s main purpose was to prove an arithmetic analogue of the Lefschetz theorem for projective surfaces. To do so, he needed to embed objects from classical potential theory, including Green’s functions and equilibrium potentials, into Arakelov theory. He introduces the class of “L_1^2-Green’s functions” (roughly, the class of generalized functions $g(z)$ on a Riemann surface for which, locally, $g(z) = \varphi - \log(|f(z)|^2)$, where $f(z)$ is meromorphic and where φ and the current $\partial\varphi$ are L^2). This class is somewhat broader than the class of Green’s functions considered in [31], and it has good functoriality properties under pullbacks and pushforwards by finite maps. Bost extends Gillet-Soulé’s

$*$-product to such Green's functions, studies the resulting intersection pairing, and proves an analogue of the Hodge index theorem for it.

More recently, P. Autissier [3] has further developed Bost's theory using ideas from probabilistic potential theory and has proved analogues of Moishezon's criterion, the arithmetic amplitude theorem, the Szpiro-Ullmo-Zhang small points theorem, and Y. Bilu's equidistribution theorem.

For varieties of arbitrary dimension, Maillot [21] has developed an extension of Gillet-Soulé's arithmetic intersection theory in which *continuous* plurisubharmonic Green's functions are admitted. Making use of the work of Bedford, Taylor, and J.-P. Demailly, he constructs the theory of cycles corresponding to Zhang's "integrable" metrized line bundles. He then shows that Gillet-Soulé's intersection product extends to the arithmetic Chow groups of integrable cycles $\widehat{\mathrm{CH}}^*_{\mathrm{int}}(X) \otimes \mathbb{Q}$, in particular, constructing a height pairing for elements of $\widehat{\mathrm{CH}}^1_{\mathrm{int}}(X) \otimes \mathbb{Q}$.

Any attack on Conjecture 4.1 must deal with the problem of relating volumes of balls of adelic sections with intersection indices, and it probably involves two steps: first, developing an adelic intersection theory incorporating v-adic plurisubharmonic functions; and second, showing that the same asymptotics are obtained for the volumes of the unit adelic balls for the metrics associated to Green's functions $G(z; E_v, D)$ and $G^\star(z; E_v, D)$. In this connection we mention the work of A. Abbes and T. Bouche [2], which provides an "elementary" proof of the arithmetic Hilbert-Samuel theorem (Gillet-Soulé's Theorem 0.1 in the introduction) relating volumes of balls of archimedean sections to Gillet-Soulé's arithmetic intersection numbers. We also note the work of Rumely, C. Lau, and R. Varley [33] on the existence of the sectional capacity, which introduces the so-called monic basis as a tool for normalizing volumes of spaces of adelic sections in a coherent way, in order to deduce asymptotics for volumes of balls.

Recently, A. Werner [39] has found an interpretation of nonarchimedean intersection indices for linear spaces in $\mathbb{P}^n$ in terms of combinatorial data from the Bruhat-Tits building associated to $\mathrm{PGL}(n+1)$. It is interesting to see how far her result can be generalized although it seems not as directly related to the present problem as the preceding works, it opens an analytic approach to nonarchimedean intersection theory.

Finally, one can ask about potential applications of an intersection theory of the kind proposed here. For example, we could cite almost any of the applications of Arakelov theory from the past two decades; but we refer the reader specifically to the beautiful papers of Zhang, Bilu, E. Ullmo, and L. Szpiro on small points (see [6], [36], [37], [43]), and the expositions by Abbes in [1] and Zhang in [45]. We note the work of Bost on the arithmetic Lefschetz theorem (see [8]), and the work of Autissier [3]. We also note the Fekete-Szegö theorem on curves (see [29]) and the existence theorem for algebraic integer points as developed by L. Moret-Bailly and others (see

[12], [22], [24], [25], [28]). Lastly, we note the possibility of an adelic Nevanlinna theory along the lines of P. Vojta's program (see [38]).

A. Appendix: Computation of the numerical example

Proof of Proposition 4.2
Introduce homogeneous coordinates $(u_0 : u_1, w_0 : w_1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The monomials $u_0^{k_0} u_1^{k_1} w_0^{\ell_0} w_1^{\ell_1}$ with $k_0 + k_1 = nr_1$, $\ell_0 + \ell_1 = nr_2$ form a K-basis for $H^0(X, \mathscr{O}_X(nD))$. Such monomials for all n give a basis for the graded algebra

$$R = \sum_{n \geq 0} H^0\big(X, \mathscr{O}_X(nD)\big),$$

which is closed under multiplication. Moreover, this basis carries a natural order (the lexicographic order, graded by the degree) compatible with multiplication. In this setting, the methods of [32] and [33] show that the sectional capacity $S_\gamma(\mathbb{E}, D)$ can be decomposed as a product of *local* sectional capacities,

$$S_\gamma(\mathbb{E}, D) = \prod_v S_\gamma(E_v, D)_v,$$

where the $S_\gamma(E_v, D)_v$ depend on the choice of the multiplicative ordered basis. For each v, the local sectional capacity is defined as follows. Let vol_v be the canonical Haar measure on K_v for which $\mathrm{vol}_v(O_v) = 1$ for finite v, and let vol_v be the Lebesgue measure on $K_v \cong \mathbb{R}$ for $v = \infty$. Using the monomial basis, we can identify $K_v \otimes_K H^0(X, \mathscr{O}_X(nD))$ with $K_v^{(nr_1+1)(nr_2+1)}$. By transport of structure, we obtain a Haar measure on $K_v \otimes_K H^0(X, \mathscr{O}_X(nD))$, also denoted vol_v. Then

$$-\log\big(S_\gamma(E_v, D)_v\big) = \lim_{n\to\infty} \frac{(d+1)!}{n^{d+1}} \log\Big(\mathrm{vol}_v(\mathscr{B}(E_v, nD))\Big), \qquad \text{(A.1)}$$

where the limit on the right-hand side exists by arguments similar to those in [32].

After dehomogenizing at u_0, w_0, a basis for $H^0(X, \mathscr{O}_X(nD))$ is given by the monomials $u^k w^\ell$ with $0 \leq k \leq nr_1$, $0 \leq \ell \leq nr_2$. For $v \notin S_1 \cup S_2 \cup \{\infty\}$, we have $E_v = \widehat{O}_v \times \widehat{O}_v$, and it is easy to see that

$$\mathscr{B}(E_v, nD) = \bigoplus_{\substack{0 \leq k \leq nr_1 \\ 0 \leq \ell \leq nr_2}} O_v\, u^k w^\ell$$

for each n. Thus $\mathrm{vol}_v(\mathscr{B}(E_v, nD)) = 1$ for each n. By (A.1), $S_\gamma(E_v, D)_v = 1$ for each $v \notin S_1 \cup S_2 \cup \{\infty\}$. Consequently,

$$-\log\big(S_\gamma(\mathbb{E}, D)\big) = \sum_{v \in S_1 \cup S_2 \cup \{\infty\}} -\log_v\big(S_\gamma(E_v, D)_v\big)\log(q_v).$$

The local capacity in the nonarchimedean compact case

Fix $v \in S_1$, and write $q = q_v$. Then $E_v = \mathbb{Z}_q \times \mathbb{Z}_q$. For each integer $k \geq 0$, let $g_k(z)$ denote the binomial polynomial $\binom{z}{k}$. As noted previously, these polynomials form a $\mathbb{Z}_q$-basis for the set of polynomials $f \in \mathbb{Q}_q[z]$ satisfying $\| f\|_{\mathbb{Z}_q} \leq 1$.

We claim that for each n, the polynomials

$$g_j(u)g_k(w) \in \mathbb{Q}_q[u, w], \quad 0 \leq j \leq r_1 n,\ 0 \leq k \leq r_2 n,$$

form a $\mathbb{Z}_q$-basis for $\mathscr{B}(E_v, nD)$. Clearly, they belong to $\mathscr{B}(E_v, nD)$ and form a basis for $H^0(X, \mathscr{O}_X(nD))$ over $\mathbb{Q}_q$. Suppose that $h(u, w) = \sum c_{jk} g_j(u) g_k(w)$ satisfies $\|h\|_{E_v} \leq 1$. Taking $u = 0$ and noting that $g_0(0) = 1$ while $g_j(0) = 0$ for all $j \geq 1$, it follows that $\| \sum_k c_{0k} g_k(w)\|_{\mathbb{Z}_q} \leq 1$. By the one-variable case, $c_{0k} \in \mathbb{Z}_q$ for all k. Subtracting $\sum_k c_{0k} g_k(w)$ from $h(u, w)$, we can assume that $c_{0k} = 0$ for all k. Inductively, suppose that $c_{\ell k} = 0$ for all $\ell < i$ and all k. Taking $u = i$ and using the fact that $g_i(i) = 1$ while $g_j(i) = 0$ for all $j > i$, we see that $\| \sum_k c_{ik} g_k(w)\|_{\mathbb{Z}_q} \leq 1$. As before, it follows that $c_{ik} \in \mathbb{Z}_q$ for all k. Subtracting $\sum_k c_{ik} g_i(u) g_k(w)$ from $h(u, w)$, we can continue the induction.

From this it follows, as in [32, Lem. 2.5], that

$$\mathrm{vol}_v\left(B(E_v, nD)\right) = \prod_{j=0}^{nr_1} \prod_{k=0}^{nr_2} \left|\frac{1}{j!k!}\right|_v.$$

Thus

$$\begin{aligned}\log_v\left(\mathrm{vol}_v(\mathscr{B}(E_v, nD))\right) &= (n+1)r_1 \sum_{k=0}^{nr_2} \mathrm{ord}_q(k!) + (n+1)r_2 \sum_{j=0}^{nr_1} \mathrm{ord}_q(j!) \\ &= \frac{n^3}{6}(3r_1^2 r_2 + 3r_1 r_2^2)\frac{1}{q_v - 1} + O_{r_1,r_2}\left(n^2 \log(n)\right),\end{aligned}$$

and so

$$-\log_v\left(S_\gamma(E_v, D)_v\right) = (3r_1^2 r_2 + 3r_2 r_2^2)\frac{1}{q_v - 1}.$$

The local capacity in the nonarchimedean noncompact case

For each nonarchimedean $v \notin S_1$, writing $q = q_v$, we have $E_v = q^{m_{v1}} \widehat{O}_v \times q^{m_{v2}} \widehat{O}_v$ for certain integers m_{v1}, m_{v2}. (Take $m_{v1} = m_{v2} = 0$ if $v \notin S_1 \cup S_2$.) It is easy to see that

$$\mathscr{B}(E_v, nD) = \bigoplus_{\substack{0 \leq k \leq nr_1 \\ 0 \leq \ell \leq nr_2}} O_v \cdot \left(\frac{u}{q^{m_{v1}}}\right)^k \left(\frac{w}{q^{m_{v2}}}\right)^\ell = \bigoplus_{\substack{0 \leq k \leq nr_1 \\ 0 \leq \ell \leq nr_2}} \left(\frac{1}{q^{km_{v1} + \ell m_{v2}}}\right) O_v \cdot u^k w^\ell.$$

Hence $\mathrm{vol}_v(\mathscr{B}(E_v, nD)) = \prod_{k,\ell} q^{km_{v1}+\ell m_{v2}}$ and

$$-\log_v\left(S_\gamma(E_v, D)_v\right) = \lim_{n\to\infty}\left(\frac{3!}{n^3}\sum_{\substack{0\le k\le nr_1\\ 0\le \ell\le nr_2}} km_{v1} + \ell m_{v2}\right)$$
$$= 3m_{v1}\cdot r_1^2 r_2 + 3m_{v2}\cdot r_1 r_2^2.$$

The local capacity in the archimedean case

We now compute $S_\gamma(E_\infty, D)_\infty$. To do so, we first determine lower and upper bounds for $\mathrm{vol}_\infty(\mathscr{B}(E_\infty, nD))$. Upon passage to the limit as $n\to\infty$, these bounds yield the same expression for $-\log(S_\gamma(E_\infty, D)_\infty)$, which can be interpreted as a limit of Riemann sums for a certain integral.

Recall that a set $E\subset\mathbb{C}\times\mathbb{C}$ is called *fully circled* if $(e^{i\theta}u, e^{i\varphi}w)\in E$ for all $(u,w)\in E$ and $\theta,\varphi\in\mathbb{R}$. Since E_∞ is fully circled, for each monomial $u^k w^\ell$ the maximum value of $|u^k w^\ell|$ on E_∞ is taken on for nonnegative real u, w. By calculus, if $k,\ell>0$, this maximum value is

$$\left(\frac{k^k\ell^\ell}{(k+\ell)^{k+\ell}}\right)^{1/p}. \tag{A.2}$$

Otherwise, the maximum is 1, and for convenience we interpret (A.2) as 1 if $k=0$ or $\ell=0$.

To obtain a lower bound for $\mathrm{vol}_\infty(\mathscr{B}(E_v, nD))$, note that if $h(u,w) = \sum_{\substack{0\le k\le nr_1\\ 0\le \ell\le nr_2}} a_{k\ell}u^k w^\ell \in \mathbb{R}\otimes_{\mathbb{Q}} H^0(X, \mathscr{O}_X(nD))$ with

$$|a_{k\ell}| \le \frac{1}{(n+1)^2 r_1 r_2}\left(\frac{(k+\ell)^{k+\ell}}{k^k\ell^\ell}\right)^{1/p}$$

for each k,ℓ, then $\|h(u,w)\|_{E_\infty}\le 1$. Consequently,

$$\mathrm{vol}_\infty\left(\mathscr{B}(E_\infty, nD)_\infty\right) \ge \prod_{\substack{0\le k\le nr_1\\ 0\le \ell\le nr_2}} \frac{1}{(n+1)^2 r_1 r_2}\left(\frac{(k+\ell)^{k+\ell}}{k^k\ell^\ell}\right)^{1/p}.$$

For an upper bound, suppose that $h(u,w)$ as above satisfies $\|h\|_{E_\infty}\le 1$. For each $R_1, R_2>0$ with $R_1^p + R_2^p = 1$, Cauchy's formula gives

$$|a_{k\ell}| = \left|\left(\frac{1}{2\pi i}\right)^2 \int_{C(0,R_1)}\int_{C(0,R_2)} \frac{h(u,w)}{u^{k+1}w^{\ell+1}}\,du\,dw\right| \le \frac{1}{R_1^k R_2^\ell}.$$

Letting $R_1^k R_2^\ell$ approach its maximum, (A.2) shows that

$$|a_{k\ell}| \le \left(\frac{(k+\ell)^{k+\ell}}{k^k\ell^\ell}\right)^{1/p}.$$

Hence

$$\mathrm{vol}_\infty\left(\mathscr{B}(E_\infty, nD)\right) \le \prod_{\substack{0\le k\le r_1 n\\ 0\le \ell\le r_2 n}} 2\Big(\frac{(k+\ell)^{k+\ell}}{k^k\ell^\ell}\Big)^{1/p}.$$

Combining these bounds yields

$$\begin{aligned}
&-\log\big(S_\gamma(\mathbb{E}, D)_\infty\big)\\
&\quad= \lim_{n\to\infty}\frac{3!}{n^3}\log\left(\mathrm{vol}_\infty(\mathscr{B}(E_\infty, nD))\right)\\
&\quad= \lim_{n\to\infty}\frac{6}{p}\sum_{\substack{0<k\le nr_1\\ 0<\ell\le nr_2}}\left(\left(\frac{k}{n}+\frac{\ell}{n}\right)\log\left(\frac{k}{n}+\frac{\ell}{n}\right)-\frac{k}{n}\log\left(\frac{k}{n}\right)-\frac{\ell}{n}\log\left(\frac{\ell}{n}\right)\right)\cdot\frac{1}{n^2}\\
&\quad= \frac{6}{p}\int_0^{r_1}\int_0^{r_2}(x+y)\log(x+y)-x\log(x)-y\log(y)\,dx\,dy\\
&\quad= \frac{1}{p}\big((r_1+r_2)^3\log(r_1+r_2)-r_1^2r_2-r_1r_2^2\\
&\qquad\qquad -(r_1^3+3r_1^2r_2)\log(r_1)-(r_2^3+3r_1r_2^2)\log(r_2)\big).
\end{aligned}\tag{A.3}$$

This proves Proposition 4.2. □

Determination of the extremal Green's functions

For $(x, y)\in\mathbb{R}^2$, put

$$|(x,y)|_D = \begin{cases}\max\left(\frac{|x|}{r_1}, \frac{|y|}{r_2}\right) & \text{if } (x,y)\ne(0,0),\\ 1 & \text{if } (x,y)=(0,0),\end{cases}$$

and for $h(u, w)\in K[u, w]$, put

$$\deg_D\left(h(u,w)\right) = \left|\left(\deg_u(h(u,w)), \deg_w(h(u,w))\right)\right|_D.$$

Thus, for $n\ge 1$, $h(u, w)\in H^0(X, \mathscr{O}_X(nD))$ if and only if $\deg_D(h)\le n$, and for a nonconstant monomial $u^k w^\ell$, $\deg_D(u^k w^\ell) = |(k,\ell)|_D = \max(k/r_1, \ell/r_2)$.

The least $n\ge 1$ for which $h(u, w)$ occurs in $K_v\otimes_K H^0(X, \mathscr{O}_X(nD))$ is $n = \lceil\deg_D(h)\rceil$, and $\deg_D(h^N) = N\deg_D(h)$ is an integer for some N, so

$$\begin{aligned}
G\big((u,w); E_v, D\big) &:= \sup_{n\ge 1}\sup_{\substack{h\in K_v\otimes_K\Gamma(nD)\\ \|h\|_{E_v}\le 1}}\left(\frac{1}{n}\log_v\left(|h(u,w)|_v\right)\right)\\
&= \sup_{\|h\|_{E_v}\le 1}\left(\frac{1}{\deg_D(h)}\log_v\left(|h(u,w)|_v\right)\right).
\end{aligned}\tag{A.4}$$

Green's function in the nonarchimedean compact case

Fix $v \in S_1$, and write $q = q_v$. Then $E_v = \mathbb{Z}_q \times \mathbb{Z}_q$. As before, write $g_k(z) = \binom{z}{k}$. In the one-dimensional case, Green's function $G(z;\mathbb{Z}_q,(\infty)) = \sup_{n\ge 1}\sup_{0\le k\le n}(1/n)\log_v(|g_k(z)|_v)$ of the set $\mathbb{Z}_q \subset \mathbb{P}^1(\mathbb{C}_v)$ is given by

$$\begin{aligned} G\bigl(z;\mathbb{Z}_q,(\infty)\bigr) &= \frac{1}{q-1} - \int_{\mathbb{Z}_q} -\log_v(|z-x|_v)\,d\mu_q(x) \\ &= \begin{cases} 0 & \text{if } z \in \mathbb{Z}_q, \\ \frac{1}{q-1} + \log_v(|z|_v) & \text{if } |z|_v > 1, \end{cases} \end{aligned}$$

for $z \in \mathbb{C}_v$, where μ_q is the additive Haar measure on $\mathbb{Z}_q$ with $\mu_q(\mathbb{Z}_q) = 1$ (see [29, p. 354]).

We have already seen that for each n, the polynomials $g_j(u)g_k(w)$ for $0 \le j \le nr_1$, $0 \le k \le nr_2$ form a $\mathbb{Z}_q$-basis for $\mathscr{B}(E_v, nD)$. By the ultrametric inequality, it follows that

$$\begin{aligned} G\bigl((u,w);E_v,D\bigr) &= \sup_{n\ge 1} \sup_{\substack{h\in K_v\otimes_K\Gamma(nD) \\ \|h\|_{E_v}\le 1}} \Bigl(\frac{1}{n}\log_v\bigl(|h(u,w)|_v\bigr)\Bigr) \\ &= \sup_{n\ge 1}\Bigl(\sup_{0\le j\le nr_1}\frac{1}{n}\log_v\bigl(|g_j(u)|_v\bigr) + \sup_{0\le k\le nr_2}\frac{1}{n}\log_v\bigl(|g_k(u)|_v\bigr)\Bigr) \\ &= r_1 G\bigl(u;\mathbb{Z}_q,(\infty)\bigr) + r_2 G\bigl(w;\mathbb{Z}_q,(\infty)\bigr). \end{aligned}$$

Green's function in the nonarchimedean noncompact case

For each nonarchimedean $v \notin S_1$, again writing $q = q_v$, formula (A.4) gives

$$\begin{aligned} &G\bigl((u,w),E_v,D\bigr) \\ &\quad = r_1\max\bigl(0,\log_v(|u/q^{m_{v1}}|_v)\bigr) + r_2\max\bigl(0,\log_v(|w/q^{m_{v2}}|_v)\bigr). \end{aligned} \tag{A.5}$$

This is because for a polynomial $h(u,w) = \sum_{\substack{0\le k\le nr_1 \\ 0\le \ell\le nr_2}} a_{k\ell}(u/q^{m_{v1}})^k(w/q^{m_{v2}})^\ell \in K_v \otimes_K H^0(X,\mathscr{O}_X(nD))$ we have $\|h(u,w)\|_{E_v} \le 1$ if and only if each $a_{k\ell} \in O_v$. By the ultrametric inequality, $|h(u,w)|_v \le \max_{\substack{0\le k\le nr_1 \\ 0\le \ell\le nr_2}} |(u/q^{m_{v1}})^k(w/q^{m_{v2}})^\ell|_v$, and so

$$\begin{aligned} &\frac{1}{\deg_D(h)}\log_v\bigl(|h(u,w)|_v\bigr) \\ &\quad \le r_1\max\bigl(0,\log_v(|u/q^{m_{v1}}|_v)\bigr) + r_2\max\bigl(0,\log_v(|w/q^{m_{v2}}|_v)\bigr). \end{aligned} \tag{A.6}$$

Moreover, for $(u,w) \in E_v$, equality is achieved in (A.6) when $h(u,w) = 1$; for $u \in q^{m_{v1}}\widehat{O}_v$, $w \notin q^{m_{v2}}\widehat{O}_v$, it is achieved when $h(u,w) = (w/q^{m_{v2}})^{nr_2}$; for $u \notin q^{m_{v1}}\widehat{O}_v$, $w \in q^{m_{v2}}\widehat{O}_v$, it is achieved when $h(u,w) = (u/q^{m_{v1}})^{nr_1}$; and for $u \notin q^{m_{v1}}\widehat{O}_v$, $w \notin q^{m_{v2}}\widehat{O}_v$, it is achieved when $h(u,w) = (u/q^{m_{v1}})^{nr_1}(w/q^{m_{v2}})^{nr_2}$.

Green's function in the archimedean case

In the archimedean case, to compute $G(z; E_\infty, D)$ we begin with a proposition modeled on [35, Prop. 5.4] which gives the extremal Green's function of any fully circled set. Put

$$\mathscr{M}_E(u, w) = \sup_{k,\ell \geq 0} \left(\left(\frac{|u^k w^\ell|}{\|u^k w^\ell\|_E} \right)^{1/|(k,\ell)|_D} \right).$$

PROPOSITION A.1

If $E \subset \mathbb{C}^2$ is a fully circled set, and if $D = r_1 H_1 + r_2 H_2$ as above, then

$$G\big((u, w); E, D\big) = \log_\infty \big(\mathscr{M}_E(u, w)\big) = \sup_{k,\ell \geq 0} \frac{1}{|(k, \ell)|_D} \log_\infty \Big(\frac{|u^k w^\ell|}{\|u^k w^\ell\|_E} \Big).$$

Proof

It is trivial that $\log_\infty(\mathscr{M}_E(u, w)) \leq G((u, w); E, D)$. For the inequality in the other direction, suppose that

$$h(u, w) = \sum_{k,\ell} a_{k\ell}\, u^k w^\ell \in \mathbb{C}[u, w]$$

satisfies $\|h\|_E \leq 1$. Without loss, we can assume that $h(u, w)$ is nonconstant.

Taking $(R_1, R_2) \in E \cap \mathbb{R}^2_{\geq 0}$ so that $R_1^k R_2^\ell = \|u^k w^\ell\|_E$, and using Cauchy's formula and the fact that E is fully circled, we see that

$$\begin{aligned}|a_{k\ell}| &= \frac{1}{(2\pi)^2} \Big| \int_0^{2\pi} \int_0^{2\pi} \frac{h(R_1 e^{i\theta_1}, R_2 e^{i\theta_2})}{R_1^{k+1} e^{i(k+1)\theta_1} R_2^{\ell+1} e^{i(\ell+1)\theta_2}} R_1 i e^{i\theta_1} R_2 i e^{i\theta_2}\, d\theta_1\, d\theta_2 \Big| \\ &\leq \frac{1}{R_1^k R_2^\ell} = \frac{1}{\|u^k w^\ell\|_E}.\end{aligned}$$

Consequently,

$$\begin{aligned}|h(u, w)| &\leq \sum_{|(k,\ell)|_D \leq \deg_D(h)} \frac{|u^k w^\ell|}{\|u^k w^\ell\|_E} \\ &\leq \big(\deg_D(h) r_1 + 1\big)\big(\deg_D(h) r_2 + 1\big) \mathscr{M}_E(u, w)^{\deg_D(h)}.\end{aligned}$$

This yields

$$\frac{1}{\deg_D(h)} \log_\infty \big(|h(u, w)|\big) \;\leq\; \log_\infty \big(\mathscr{M}_E(u, w)\big) + O\Big(\frac{\log(\deg_D(h))}{\deg_D(h)}\Big).$$

Replacing $h(u, w)$ by a power $h(u, w)^N$ and letting $N \to \infty$ gives

$$\frac{1}{\deg_D(h)} \log_\infty \big(|h(u, w)|\big) \leq \log_\infty \big(\mathscr{M}_E(u, w)\big). \qquad \square$$

Now specialize to the case

$$E = E_\infty = \{(u, w) \in \mathbb{C}^2 : |u|^p + |w|^p \le 1\}.$$

Earlier we saw that

$$\|u^k w^\ell\|_{E_\infty} = \Big(\frac{k^k \ell^\ell}{(k+\ell)^{k+\ell}}\Big)^{1/p}.$$

For $(u, w) \in E_\infty$, we have $G((u, w); E_\infty, D) = 0$ as $1 \in K_v \otimes_K H^0(X, \mathscr{O}_X(nD))$ for all n. To determine $G((u, w); E_\infty, D)$ for $(u, w) \notin E_\infty$, first note that by homogeneity, if $x = |u|$, $y = |w|$, then $\mathscr{M}_{E_\infty}(u, w) = \mathscr{M}_{E_\infty}(x, y)$. So, for fixed $x, y \ge 0$ with $x^p + y^p > 1$, we want the value of

$$\log_\infty\big(\mathscr{M}_{E_\infty}(x, y)\big) = \max\Bigg(0, \max_{(k,\ell)\neq(0,0)} \frac{1}{|(k,\ell)|_D} \log_\infty\bigg(\bigg|\Big(\frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell}\Big)^{1/p} x^k y^\ell\bigg|\bigg)\Bigg). \tag{A.7}$$

For any $0 < t \in \mathbb{R}$, the right-hand side is unchanged when (k, ℓ) is replaced by $(tk, t\ell)$, and it is continuous on the set

$$\{(\kappa, \lambda) \in \mathbb{R}^2 : \kappa, \lambda \ge 0, |(\kappa, \lambda)|_D = 1\}. \tag{A.8}$$

Thus we can take the inner max in (A.7) over the set (A.8). Recall that $\log_\infty$ is the logarithm to the base $e^{1/2}$ since $K_\infty \cong \mathbb{R}$, so $\log_\infty(x) = 2\log(x)$. Put

$$\begin{aligned} f(\kappa) &= \frac{2}{p}\big((\kappa + r_2)\log(\kappa + r_2) - \kappa\log(\kappa) - r_2\log(r_2)\big) \\ &\quad + 2\kappa\log(x) + 2r_2\log(y), \\ g(\lambda) &= \frac{2}{p}\big((r_1 + \lambda)\log(r_1 + \lambda) - r_1\log(r_1) - \lambda\log(\lambda)\big) \\ &\quad + 2r_1\log(x) + 2\lambda\log(y). \end{aligned}$$

Since $|(\kappa, \lambda)|_D = 1$ means $\max(\kappa/r_1, \lambda/r_2) = 1$, we find that

$$G\big((u, w); E_\infty, D\big) = \max\big(0, \max_{0\le\kappa\le r_1} f(\kappa), \max_{0\le\lambda\le r_2} g(\lambda)\big).$$

The two expressions on the right-hand side are similar in form. The unique point where $f'(\kappa) = 0$ occurs at $\kappa_0 = r_2 x^p/(1 - x^p)$, and this belongs to the interval $0 \le \kappa \le r_1$ if and only if

$$0 \le x \le \Big(\frac{r_1}{r_1 + r_2}\Big)^{1/p}.$$

From this, one sees that if $0 \le x \le (r_1/(r_1 + r_2))^{1/p}$, then the maximum of $f(\kappa)$ for $0 \le \kappa \le r_1$ is

$$f(\kappa_0) = 2r_2\log(y) - \frac{2r_2}{p}\log(1 - x^p),$$

while if $x > (r_1/(r_1+r_2))^{1/p}$, the maximum is

$$f(r_1) = 2r_1 \log(x) + 2r_2 \log(y) + \frac{2}{p}\big((r_1+r_2)\log(r_1+r_2) - r_1\log(r_1) - r_2\log(r_2)\big).$$

Similar formulas hold for $g(\lambda)$. Partition $\mathbb{C}^2 \backslash E_\infty$ into three regions:

$$\begin{cases} \text{region I:} & \left\{(u,w) \in \mathbb{C}^2 : |u| \le \left(\frac{r_1}{r_1+r_2}\right)^{1/p},\ |u|^p + |w|^p > 1\right\}, \\ \text{region II:} & \left\{(u,w) \in \mathbb{C}^2 : |w| \le \left(\frac{r_2}{r_1+r_2}\right)^{1/p},\ |u|^p + |w|^p > 1\right\}, \\ \text{region III:} & \left\{(u,w) \in \mathbb{C}^2 : |u| > \left(\frac{r_1}{r_1+r_2}\right)^{1/p} \text{ and } |w| > \left(\frac{r_2}{r_1+r_2}\right)^{1/p}\right\}. \end{cases}$$

Then $G((u,w); E_\infty, D)$ is given by

$$\begin{cases} 0 & \text{on } E_\infty, \\ 2r_2\log(|w|) - \frac{2r_2}{p}\log(1-|u|^p) & \text{on region I}, \\ 2r_1\log(|u|) - \frac{2r_1}{p}\log(1-|w|^p) & \text{on region II}, \\ 2r_1\log(|u|) + 2r_2\log(|w|) & \\ \quad + \frac{2}{p}\big[(r_1+r_2)\log(r_1+r_2) - r_1\log(r_1) - r_2\log(r_2)\big] & \text{on region III}. \end{cases} \tag{A.9}$$

By routine but tedious computations, one finds that

(1) $G((u,w); E_\infty, D)$ is continuous everywhere, and it is $\mathscr{C}^1$ except on ∂E_∞;

(2) writing $u = re^{i\theta}$ and $w = \rho e^{i\varphi}$,

$$\begin{cases} dd^c G\big((u,w); E_\infty, D\big) = 0 & \text{on the interior of } E_\infty, \\ dd^c G\big((u,w); E_\infty, D\big) = r_2 \frac{p}{2\pi} \frac{r^{p-1}}{(1-r^p)^2}\, dr \wedge d\theta & \text{on region I}, \\ dd^c G\big((u,w); E_\infty, D\big) = r_1 \frac{p}{2\pi} \frac{\rho^{p-1}}{(1-\rho^p)^2}\, d\rho \wedge d\varphi & \text{on region II}, \\ dd^c G\big((u,w); E_\infty, D\big) = 0 & \text{on region III}; \end{cases} \tag{A.10}$$

(3) $dd^c G((u,w); E_\infty, D)$ has no residues except on ∂E_∞; on ∂E_∞, the residue is given by the 1-form $d^c G((u,w); E_\infty, D)$, computed using the expression from region I (resp., region II) in (A).

Computation of the (conjectural) intersection product

Associate to the set $\mathbb{E}$ and the divisor $D = r_1H_1 + r_2H_2$ the generalized arithmetic divisor

$$\mathscr{D} = \big\langle D, \{G((u,w); E_v, D)\}_{v\in\mathscr{M}_K}\big\rangle.$$

$\big($Note that each $G((u,w);E_v,D)$ is continuous, so here $G^\star((u,w);E_v,D) = G((u,w);E_v,D).\big)$ Assuming that there is an intersection theory for such objects which extends Gillet-Soulé's theory, which respects linear equivalence by principal arithmetic divisors $\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), \{-\log_v(|f(z)|_v)\}_{v\in\mathscr{M}_K}\big)$, and which has a decomposition into local terms as in (4.4) and (4.5), we formally compute the self-intersection number $\mathscr{D}^3$. Put

$$\begin{aligned}
\mathscr{D}_1 &= \big\langle D_1, \{G_{v1}\}_{v\in\mathscr{M}_K}\big\rangle := \mathscr{D},\\
\mathscr{D}_2 &= \big\langle D_2, \{G_{v2}\}_{v\in\mathscr{M}_K}\big\rangle := \mathscr{D} + r_1\,\widehat{\mathrm{div}}(u-1) + r_2\,\widehat{\mathrm{div}}(w-1),\\
\mathscr{D}_3 &= \big\langle D_3, \{G_{v3}\}_{v\in\mathscr{M}_K}\big\rangle := \mathscr{D} + r_1\,\widehat{\mathrm{div}}(u) + r_2\,\widehat{\mathrm{div}}(w),
\end{aligned}$$

in which $D_1 = r_1(u=\infty) + r_2(w=\infty)$, $D_2 = r_1(u=1)+r_2(w=1)$, $D_3 = r_1(u=0)+r_2(w=0)$, and

$$\begin{aligned}
G_{v1} &= G\big((u,w);E_v,D\big),\\
G_{v2} &= G\big((u,w);E_v,D\big) - r_1\log_v(|u-1|_v) - r_2\log_v(|w-1|_v),\\
G_{v3} &= G\big((u,w);E_v,D\big) - r_1\log_v(|u|_v) - r_2\log_v(|w|_v).
\end{aligned} \tag{A.11}$$

Since D_1, D_2, and D_3 meet transversely, with $D_1\cdot D_2\cdot D_3 = 0$, according to (4.5) we expect

$$\mathscr{D}^3 = \langle \mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3\rangle = \sum_v \langle \mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3\rangle_v \log(q_v).$$

If $dd^cG_{vi} = \omega_{vi} - \delta_{D_i}$, then each local intersection term is given by

$$\langle \mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3\rangle_v = G_{v3}(D_1\cdot D_2) + \int_{D_1(\mathbb{C}_v)} G_{v2}\,\omega_{v3} + \int_{X(\mathbb{C}_v)} G_{v1}\,\omega_{v2}\wedge\omega_{v3}, \tag{A.12}$$

where $D_1\cdot D_2 = r_1r_2(1,\infty) + r_1r_2(\infty,1)$.

The intersection product in the nonarchimedean compact case

For nonarchimedean $v\in S_1$, writing $q = q_v$, Green's functions are given by

$$\begin{aligned}
G_{v1}(u,v) &= r_1G\big(u;\mathbb{Z}_q,(\infty)\big) + r_2G\big(w;\mathbb{Z}_q,(\infty)\big),\\
G_{v2}(u,v) &= r_1\big(G(u;\mathbb{Z}_q,(\infty)) - \log_v(|u-1|_v)\big)\\
&\quad + r_2\big(G(w;\mathbb{Z}_q,(\infty)) - \log_v(|w-1|_v)\big),\\
G_{v3}(u,v) &= r_1\big(G(u;\mathbb{Z}_q,(\infty)) - \log_v(|u|_v)\big)\\
&\quad + r_2\big(G(w;\mathbb{Z}_q,(\infty)) - \log_v(|w|_v)\big).
\end{aligned}$$

The first term in the local intersection product $\langle \mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3\rangle_v$ is

$$\begin{aligned}
G_{v3}(D_1\cdot D_2) &= r_1r_2G_{v3}\big((1,\infty)\big) + r_1r_2G_{v3}\big((\infty,1)\big)\\
&= r_1r_2\Big(r_1\frac{1}{q_v-1} + r_2\frac{1}{q_v-1}\Big).
\end{aligned} \tag{A.13}$$

The (conjectural) second term is

$$\int_{D_1(\mathbb{C}_v)} G_{v2}\omega_{v3} = r_1 \int_{\{u=\infty\}} G_{v2}\,\omega_{v3} + r_2 \int_{\{w=\infty\}} G_{v2}\,\omega_{v3}.$$

The theory in [31] tells us that if $d\mu_q(z)$ is the additive Haar measure on $\mathbb{Z}_q$ with total mass 1, then $\omega_{v3}|_{\{u=\infty\}}$ should be $r_2 d\mu_q(w)$ on $\mathbb{Z}_q$, and $\omega_{v3}|_{\{w=\infty\}}$ should be $r_1 d\mu_q(u)$ on $\mathbb{Z}_q$. (These are the only reasonable possibilities in light of the translation invariance of $\mathbb{Z}_q$). Here $G_{v2}((\infty, w)) = r_1(1/(q_v - 1)) - r_2 \log_q(|w-1|_q))$, so

$$\begin{aligned} r_1 \int_{\{u=\infty\}} G_{v2}\,\omega_{v3} &= r_1 r_2 \int_{\mathbb{Z}_q} r_1 \frac{1}{q_v - 1} - r_2 \log_q(|w-1|_q)\, d\mu_q(w) \\ &= r_1^2 r_2 \frac{1}{q_v - 1} + r_1 r_2^2 \frac{1}{q_v - 1}. \end{aligned} \tag{A.14}$$

Similarly,

$$r_2 \int_{\{w=\infty\}} G_{v2}\,\omega_{v3} = r_1^2 r_2 \frac{1}{q_v - 1} + r_1 r_2^2 \frac{1}{q_v - 1}. \tag{A.15}$$

Finally, we expect that $\omega_{v2} \wedge \omega_{v3}$ should be $2! r_1 r_2 \cdot d\mu_q(u)\, d\mu_q(w)$, where, as before, $d\mu_q(z)$ is the additive Haar measure on $\mathbb{Z}_q$ with $\mu_q(\mathbb{Z}_q) = 1$. Since $G_{v1}(u, w) = 0$ on $\mathbb{Z}_q \times \mathbb{Z}_q$, we expect that

$$\int_{X(\mathbb{C}_v)} G_{v1}\,\omega_{v2} \wedge \omega_{v3} = 0, \tag{A.16}$$

though at present we must take this as a definition in our case. Thus, summing (A.13), (A.14), (A.15), and (A.16), we expect that for each $v \in S_1$,

$$\langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_v \log(q_v) = (3r_1^2 r_2 + 3r_1 r_2^2) \frac{1}{q_v - 1} \log(q_v). \tag{A.17}$$

The intersection product in the nonarchimedean noncompact case

By formulas (A.5) and (A.11), for each $v \notin S_1 \cup \infty$,

$$\begin{aligned} G_{v1}(u, v) &= r_1 \max\big(0, \log_v(|u|_v) + m_{v1}\big) + r_2 \max\big(0, \log_v(|w|_v) + m_{v2}\big), \\ G_{v2}(u, v) &= r_1\big(\max(0, \log_v(|u|_v) + m_{v1}) - \log_v(|u-1|_v)\big) \\ &\quad + r_2\big(\max(0, \log_v(|w|_v) + m_{v2}) - \log_v(|w-1|_v)\big), \\ G_{v3}(u, v) &= r_1\big(\max(0, \log_v(|u|_v) + m_{v1}) - \log_v(|u|_v)\big) \\ &\quad + r_2\big(\max(0, \log_v(|w|_v) + m_{v2}) - \log_v(|w|_v)\big). \end{aligned}$$

(If u or v is ∞, these are to be understood as appropriate limits.) As before, write $q = q_v$ for the prime underlying v.

The first term in (A.12) is

$$\begin{aligned} G_{v3}(D_1 \cdot D_2) &= r_1 r_2 G_{v3}\big((1,\infty)\big) + r_1 r_2 G_{v3}\big((\infty,1)\big) \\ &= r_1 r_2\big(r_1 \max(0, m_{v1}) + r_2 m_{v1}\big) + r_1 r_2\big(r_2 \max(0, m_{v2}) + r_1 m_{v2}\big) \\ &= r_1^2 r_2\big(\max(0, m_{v1}) + m_{v2}\big) + r_1 r_2^2\big(m_{v1} + \max(0, m_{v2})\big). \end{aligned} \tag{A.18}$$

The second term in (A.12) is

$$\int_{D_1(\mathbb{C}_v)} G_{v2}\omega_{v3} = r_1 \int_{\{u=\infty\}} G_{v2}\,\omega_{v3} + r_2 \int_{\{w=\infty\}} G_{v2}\,\omega_{v3}.$$

Based on our discussion concerning adelic intersection theory, we expect ω_{v3} to be a "(1, 1)-form" $r_1 d\mu_1(u) + r_2 d\mu_2(w)$, where $d\mu_1(u)$ assigns mass 1 to $q^{m_{v1}}\widehat{O}_v$ and $d\mu_2(w)$ assigns mass 1 to $q^{m_{v2}}\widehat{O}_v$. Thus we should have

$$\begin{aligned} \int_{D_1(\mathbb{C}_v)} G_{v2}\omega_{v3} = r_1 &\int_{q^{m_{v2}}\widehat{O}_v} G_{v2}(\infty, w)\, r_2\, d\mu_2(w) \\ &+ r_2 \int_{q^{m_{v1}}\widehat{O}_v} G_{v2}(u,\infty)\, r_1\, d\mu_1(u). \end{aligned}$$

The integrals on the right-hand side are expected to pick out the "generic value" of $G_{v2}(u, w)$ on the sets in question. The value of $G_{v2}(\infty, w)$ is $r_1 m_{v1}$ for all $w \in q^{m_{v2}}\widehat{O}_v$ if $m_{v2} > 0$, and it is $r_1 m_{v1} + r_2 m_{v2}$ for all $w \in q^{m_{v2}}\widehat{O}_v$ with $|w-1|_v = |q^{m_{v2}}|_v$, if $m_{v2} \le 0$. Thus the generic value of $G_{v2}(\infty, w)$ for $w \in q^{m_{v2}}\widehat{O}_v$ is $r_1 m_{v1} + r_2 \min(0, m_{v2})$. Similarly, the generic value of $G_{v2}(u,\infty)$ on $q^{m_{v1}}\widehat{O}_v$ is $r_1 \min(0, m_{v1}) + r_2 m_{v2}$. Hence we expect that

$$\int_{D_1(\mathbb{C}_v)} G_{v2}\omega_{v3} = r_1^2 r_2\big(m_{v1} + \min(0, m_{v1})\big) + r_1 r_2^2\big(\min(0, m_{v2}) + m_{v2}\big). \tag{A.19}$$

This equality can at present be taken only to be a definition in this case, as we have not made an independent definition of the integrals occurring in it.

Finally, we expect that

$$\int_{X(\mathbb{C}_v)} G_{v1}\,\omega_{v2} \wedge \omega_{v3} = 0 \tag{A.20}$$

since G_{v1} vanishes on E_v, regardless of what interpretation is given to $\omega_{v2} \wedge \omega_{v3}$. (Based on our discussion of adelic intersection theory, we expect that $\omega_{v1} = \omega_{v2} = \omega_{v3}$ and that $\omega_{v2} \wedge \omega_{v3} = \omega_{v1} \wedge \omega_{v1}$ should assign mass $2! \cdot r_1 r_2$ to the set $E_v = q^{m_{v1}}\widehat{O}_v \times q^{m_{v2}}\widehat{O}_v$.) However, at present this equality can only be taken as a definition.

Summing (A.18), (A.19), and (A.20), we expect that

$$\begin{aligned} \langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3\rangle_v \log(q_v) = r_1^2 r_2 &\cdot \big(2m_{v1} + \max(0, m_{v1}) + \min(0, m_{v1})\big)\log(q_v) \\ &+ r_1 r_2^2 \cdot \big(2m_{v2} + \max(0, m_{v2}) + \min(0, m_{v2})\big)\log(q_v) \\ &= (3m_{v1} \cdot r_1^2 r_2 + 3m_{v2} \cdot r_1 r_2^2)\log(q_v). \end{aligned} \tag{A.21}$$

Note that (A.21) is zero for all nonarchimedean $v \notin S_1 \cup S_2$.

It may be worth noting that $\langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_v \log(q_v)$ can be evaluated with different orderings of the $\mathscr{D}_i$. When this is done for $\langle \mathscr{D}_3, \mathscr{D}_2, \mathscr{D}_1 \rangle_v \log(q_v)$, the term $\int_{X(\mathbb{C}_v)} G_{v3}\, \omega_{v2} \wedge \omega_{v1}$ is nonzero, and to obtain (A.21), it is necessary to assume that $\omega_{v2} \wedge \omega_{v1}$ gives E_v the mass $2! \cdot r_1 r_2$ indicated above.

The intersection product in the archimedean case

In the archimedean case, formulas for $G_{\infty 1}$, $G_{\infty 2}$, and $G_{\infty 3}$ are given by (A) and (A.11). We have $dd^c G_{\infty i} = \omega_\infty - \delta_{D_i}$ for each i, where ω_∞ is as in (A.10) and δ_{D_i} is the Dirac current supported on D_i. We now evaluate each of the terms appearing on the right-hand side of (A.12) for $\langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_\infty \log(q_\infty)$. Since $q_\infty = e^{1/2}$, this is $(1/2)\langle \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3 \rangle_\infty$.

For the first term, we have $D_1 \cdot D_2 = r_1 r_2((1, \infty)) + r_1 r_2((\infty, 1))$. Since $(1, \infty)$ and $(\infty, 1)$ belong to the closure of region III, formula (A) gives

$$\begin{aligned}\frac{1}{2} G_{\infty 3}\big((1, \infty)\big) &= \frac{1}{2} G_{\infty 3}\big((\infty, 1)\big) \\ &= \frac{1}{p}\big((r_1 + r_2)\log(r_1 + r_2) - r_1 \log(r_1) - r_2 \log(r_2)\big).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{2} G_{\infty 3}(D_1 \cdot D_2) \\ &\quad = \frac{2}{p} r_1 r_2 (r_1 + r_2)\log(r_1 + r_2) - \frac{2}{p} r_1^2 r_2 \log(r_1) - \frac{2}{p} r_1 r_2^2 \log(r_2).\end{aligned} \tag{A.22}$$

The second term in the local intersection product is

$$\frac{1}{2}\int_{D_1(\mathbb{C})} G_{\infty 2}\, d\omega_\infty = \frac{1}{2} r_1 \int_{\{u=\infty\}} G_{\infty 2}\, \omega_\infty + \frac{1}{2} r_2 \int_{\{w=\infty\}} G_{\infty 2}\, \omega_\infty.$$

Consider the first integral. The hyperplane $\{u = \infty\}$ does not intersect E_∞ or the tube $|u| \le (r_1/(r_1 + r_2))^{1/p}$; it does meet the tube $|w| \le (r_2/(r_1 + r_2))^{1/p}$, and on that tube (writing $w = \rho e^{i\varphi}$), by (A.10),

$$\omega_\infty = dd^c\big(G((u, w); E_\infty, D)\big) = \frac{r_1 p}{2\pi} \frac{\rho^{p-1}}{(1 - \rho^p)^2}\, d\rho \wedge d\varphi.$$

Similarly, (A) shows that $G_{\infty 2}(\infty, w)$ is given by

$$\begin{aligned}&\lim_{u \to \infty}\left(2r_1 \log\left(\frac{|u|}{(1 - |w|^p)^{1/p}}\right) - 2r_1 \log(|u - 1|) - 2r_2 \log(|w - 1)\right) \\ &\qquad = -\frac{2r_1}{p} \log(1 - \rho^p) - 2r_2 \log(|w - 1|).\end{aligned}$$

Writing $R = (r_2/(r_1+r_2))^{1/p}$,

$$\begin{aligned}\frac{1}{2}r_1\int_{\{u=\infty\}} G_{\infty 2}\,\omega_\infty &= -r_1\int_{|w|\le R}\Big(\frac{r_1}{p}\log(1-\rho^p)\Big)\frac{pr_1}{2\pi}\frac{\rho^{p-1}}{(1-\rho^p)^2}\,d\rho\wedge d\varphi \\ &\quad - r_1\int_{|w|\le R}\big(r_2\log(|w-1|)\big)\frac{pr_1}{2\pi}\frac{\rho^{p-1}}{(1-\rho^p)^2}\,d\rho\wedge d\varphi \\ &= -\frac{r_1^3}{2\pi}\int_0^R\int_0^{2\pi}\log(1-\rho^p)\frac{\rho^{p-1}}{(1-\rho^p)^2}\,d\varphi\,d\rho \\ &\quad - pr_1^2r_2\int_0^R\Big(\frac{1}{2\pi}\int_0^{2\pi}\log(|\rho e^{i\varphi}-1|)\,d\varphi\Big)\frac{\rho^{p-1}}{(1-\rho^p)^2}\,d\rho.\end{aligned}$$

The second integral evaluates to zero by the mean value theorem for harmonic functions, while the first is

$$\begin{aligned}&= -r_1^3\int_0^R\log(1-\rho^p)\frac{\rho^{p-1}}{(1-\rho^p)^2}\,d\rho = \frac{r_1^3}{p}\int_1^{1-R^p}\log(t)\frac{1}{t^2}\,dt \\ &= \frac{1}{p}\big((r_1^3+r_1^2r_2)\log(r_1+r_2) - r_1^2r_2 - (r_1^3+r_1^2r_2)\log(r_1)\big). \qquad \text{(A.23)}\end{aligned}$$

Similarly, $(1/2)r_2\int_{\{w=\infty\}} G_{\infty 2}\,\omega_\infty$ evaluates to

$$\frac{1}{p}\big((r_2^3+r_1r_2^2)\log(r_1+r_2) - r_1r_2^2 - (r_2^3+r_1r_2^2)\log(r_2)\big). \qquad \text{(A.24)}$$

The third term in the local intersection product is $(1/2)\int_{\mathbb{P}^1\times\mathbb{P}^1} G_{\infty 1}\,\omega_\infty\wedge\omega_\infty$. By formulas (A.10), we see that $\omega_\infty\wedge\omega_\infty = 0$ except on ∂E_∞. However, $G_{\infty 1}(u,w) = 0$ on E_∞, so this term vanishes.

Conjecturally, the total intersection product should be the sum of (A.17), (A.21), (A.22), (A.23), and (A.24):

$$\begin{aligned}\mathscr{D}^3 &= \frac{1}{p}\big((r_1+r_2)^3\log(r_1+r_2) - r_1^2r_2 - r_1r_2^2 \\ &\qquad - (r_1^3+3r_1^2r_2)\log(r_1) - (r_2^3+3r_1r_2^2)\log(r_2)\big) \\ &\quad + \sum_{v\in S_1}(3r_1^2r_2+3r_1r_2^2)\frac{1}{q_v-1}\log(q_v) \\ &\quad + \sum_{v\in S_2}(3m_{v1}\cdot r_1^2r_2 + 3m_{v2}\cdot r_1r_2^2)\log(q_v). \qquad \text{(A.25)}\end{aligned}$$

This is the same as the formula for $-\log(S_\gamma(\mathbb{E}, D))$ in Proposition 4.2, and we see that (4.9) holds, subject to the assumptions in (A.14), (A.15), (A.16), (A.19), and (A.20).

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References

[1] A. ABBES, *Hauteurs et discrétude (d'après L. Szpiro, E. Ullmo et S. Zhang)*, Astérisque **245** (1997), 141–166, Séminaire Bourbaki 1996/97, exp. 825. MR 99h:14029 269

[2] A. ABBES and T. BOUCHE, *Théorème de Hilbert-Samuel "arithmétique,"* Ann. Inst. Fourier (Grenoble) **45** (1995), 345–401. MR 96e:14024 269

[3] P. AUTISSIER, *Points entiers sur les surfaces arithmétiques*, J. Reine Angew. Math. **531** (2001), 201–235. MR 2002a:11066 269

[4] E. BEDFORD and B. A. TAYLOR, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40. MR 84d:32024 246, 260, 262

[5] V. G. BERKOVICH, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Math. Surveys Monogr. **33**, Amer. Math. Soc., Providence, 1990. MR 91k:32038 262

[6] Y. BILU, *Limit distribution of small points on algebraic tori*, Duke Math. J. **89** (1997), 465–476. MR 98m:11067 269

[7] S. BLOCH, H. GILLET, and C. SOULÉ, *Non-Archimedean Arakelov theory*, J. Algebraic Geom. **4** (1995), 427–485. MR 96g:14019 262, 268

[8] J.-B. BOST, *Potential theory and Lefschetz theorems for arithmetic surfaces*, Ann. Sci. École Norm. Sup. (4) **32** (1999), 241–312. MR 2000c:14033 268, 269

[9] J.-B. BOST, H. GILLET, and C. SOULÉ, *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc. **7** (1994), 903–1027. MR 95j:14025

[10] T. CHINBURG, *Capacity theory on varieties*, Compositio Math. **80** (1991), 75–84. MR 93d:14039 230

[11] J.-P. DEMAILLY, "Monge-Ampère operators, Lelong numbers and intersection theory" in *Complex Analysis and Geometry*, Univ. Ser. Math., Plenum, New York, 1993, 115–193. MR 94k:32009

[12] R. ERNÉ, *On the degree of integral points of a projective space minus a horizontal hypersurface*, J. Reine Angew. Math. **532** (2001), 151–177. MR 2002d:14034 270

[13] J. E. FORNAESS and R. NARASIMHAN, *The Levi problem on complex spaces with singularities*, Math. Ann. **248** (1980), 47–72. MR 81f:32020 243

[14] H. GILLET and C. SOULÉ, *Amplitude Arithmétique*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 887–890. MR 90a:14028 232, 233

[15] ———, *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992), 473–543. MR 94f:14019 232, 233

[16] R. HARTSHORNE, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer, New York, 1977. MR 57:3116 236, 240, 248, 258

[17] E. KANI, "Potential theory on curves" in *Théorie des nombres (Quebec, 1987)*, de Gruyter, Berlin, 1989, 475–543. MR 91e:14020 262, 267

[18] M. KLIMEK, *Pluripotential Theory*, London Math. Soc. Monogr. (N.S.) **6**, Oxford Univ. Press, New York, 1991. MR 93h:32021 231, 242, 243, 244, 246, 247, 252, 261, 262

[19] P. LELONG, *Plurisubharmonic Functions and Positive Differential Forms*, trans. M. A. Dostal, Notes on Math. and Its Appl., Gordon and Breach, New York, 1969. 242, 255

[20] S. LOJASIEWICZ, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449 – 474. MR 30:3478 252

[21] V. MAILLOT, *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, Mém. Soc. Math. Fr. (N.S.) **80**, Soc. Math. France, Marseille, 2000. MR 2001j:14037 253, 267, 269

[22] P. MIKKELSON, *Effective bounds for integral points on arithmetic surfaces*, Ph.D. dissertation, Columbia University, New York, 1995. 270

[23] J. S. MILNE, *Étale Cohomology*, Princeton Math. Ser. **33**, Princeton Univ. Press, Princeton, 1980. MR 81j:14002 240

[24] L. MORET-BAILLY, *Groupes de Picard et problèmes de Skolem, I, II*, Ann. Sci. École Norm. Sup. (4) **22** (1989), 161 – 179, 181 – 194. MR 90i:11065 270

[25] ———, "Applications of local-global principles to arithmetic and geometry" in *Hilbert's Tenth Problem: Relations with Arithmetic and Algebraic Geometry (Ghent, Belgium, 1999)*, Contemp. Math. **270**, Amer. Math. Soc., Providence, 2000, 169 – 186. MR 2002f:11077 270

[26] D. MUMFORD, *Algebraic Geometry, I: Complex Projective Varieties*, Grundlehren Math. Wiss. **221**, Springer, Berlin, 1976. MR 56:11992 245

[27] R. RICHBERG, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1968), 257 – 286. MR 36:5386 243, 253, 254

[28] R. S. RUMELY, *Arithmetic over the ring of all algebraic integers*, J. Reine Angew. Math. **368** (1986), 127 – 133. MR 87i:11041 270

[29] ———, *Capacity Theory on Algebraic Curves*, Lecture Notes in Math. **1378**, Springer, Berlin, 1989. MR 91b:14018 266, 269, 274

[30] ———, *On the relation between Cantor's capacity and the sectional capacity*, Duke Math. J. **70** (1993), 517 – 574. MR 95f:14044 266

[31] ———, "An intersection pairing for curves, with analytic contributions from non-Archimedean places" in *Number Theory (Halifax, Nova Scotia, 1994)*, CMS Conf. Proc. **15**, Amer. Math. Soc., Providence, 1995, 325 – 357. MR 96m:14029 261, 262, 265, 268, 279

[32] R. RUMELY and C. F. LAU, *Arithmetic capacities on* $\mathbb{P}^n$, Math. Z. **215** (1994), 533 – 560. MR 95g:14028 246, 270, 271

[33] R. RUMELY, C. F. Lau, and R. Varley, *Existence of the Sectional Capacity*, Mem. Amer. Math. Soc. **145** (2000), no. 690. MR 2000j:14039 231, 234, 235, 236, 237, 238, 246, 265, 269, 270

[34] J. SICIAK, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc **105** (1962), 322 – 357. MR 26:1495 231

[35] ———, *Extremal plurisubharmonic functions in* $\mathbb{C}^n$, Ann. Polon. Math. **39** (1981), 175 – 211. MR 83e:32018 231, 275

[36] L. SZPIRO, E. ULLMO, and S. ZHANG, *Équirépartition des petits points*, Invent. Math. **127** (1997), 337 – 347. MR 98i:14027 269

[37] E. ULLMO, *Positivité et discrétion des points algébriques des courbes*, Ann. of Math. (2) **147** (1998), 167 – 179. MR 99e:14031 269

[38] P. VOJTA, *Diophantine Approximations and Value Distribution Theory*, Lecture Notes in Math. **1239**, Springer, Berlin, 1987. MR 91k:11049 270

[39] A. WERNER, *Non-Archimedean intersection indices on projective spaces and the Bruhat-Tits building for* PGL, Ann. Inst. Fourier (Grenoble) **51** (2001), 1483 – 1505. MR 2002h:14038 269

[40] A. ZÉRIAHI, *Fonction de Green pluricomplexe à pôle à l'infini sur un espace de Stein parabolique et applications*, Math. Scand. **69** (1991), 89 – 126. MR 93a:32021 231, 243, 246, 247, 249, 251, 252, 260

[41] S. ZHANG, *Positive line bundles on arithmetic surfaces*, Ann. of Math. (2) **136** (1992), 569 – 587. MR 93j:14024 262, 268

[42] ———, *Admissible pairing on a curve*, Invent. Math. **112** (1993), 171 – 193. MR 94h:14023 262, 268

[43] ———, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc **8** (1995), 187 – 221. MR 95c:14020 239, 262, 267, 268, 269

[44] ———, *Small points and adelic metrics*, J. Algebraic Geom. **4** (1995), 281 – 300. MR 96e:14025

[45] ———, "Small points and Arakelov theory" in *Proceedings of the International Congress of Mathematicians (Berlin, 1998), Vol. II*, Doc. Math. **1998**, extra vol. II, 217 – 225. MR 99i:14030 269

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