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A Robin formula for the Fekete-Leja transfinite diameter

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Abstract We generalize the classical Robin formula to higher dimensions.

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The purpose of this note is to give a formula for the Fekete-Leja transfinite diameter on $\mathbb{C}^N$, generalizing the classical Robin formula

$$d_\infty(E) = e^{-V(E)}$$

for the usual transfinite diameter. We will disengage this from a formula for the sectional capacity proved in arithmetic intersection theory ([5], Theorem 1.1, p. 233).

First recall the definition of the Fekete-Leja transfinite diameter for a compact set $E \subset \mathbb{C}^N$ (see [1,16]). Consider the set of monomials $z^k = z_1^{k_1} \cdots z_N^{k_N}$ in the polynomial ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_N]$. Let $\Gamma(n) \subset \mathbb{C}[z]$ be the space of polynomials of total degree at most n, and let $\mathcal{K}(n) = \{k \in \mathbb{Z}^N : k_i \geq 0, k_1 + \cdots + k_N \leq n\}$ be the index set for the monomial basis of $\Gamma(n)$. Put $q_n = \#(\mathcal{K}(n)) = \binom{n+N}{n}$.

Fixing n, take q_n independent vector variables $z_i = (z_{i1}, \ldots, z_{iN}) \in \mathbb{C}^N$, $i = 1, \ldots, q_n$. Let $k_1, \ldots, k_{q_n}$ be the indices in $\mathcal{K}(n)$. The Vandermonde determinant

$$Q_n(z_1, \ldots, z_{q_n}) := \det(z_i^{k_j})_{i,j=1}^{q_n}$$

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is a homogeneous polynomial in the z_{ij} of total degree $T_n = N \cdot \binom{n+N}{N+1}$. For each n, put

$$d_n(E) = \max_{z_1,\ldots,z_{q_n} \in E} |Q_n(z_1,\ldots,z_{q_n})|^{1/T_n}.$$

The Fekete-Leja transfinite diameter is defined by

$$d_\infty(E) = \lim_{n\to\infty} d_n(E);$$

the existence of the limit is due to Zaharjuta [16].

Henceforth, assume that $d_\infty(E) > 0$. This is equivalent to E being non-pluripolar [10].

For $f \in \mathbb{C}[z_1,\ldots,z_N]$ write $\|f\|_E = \sup_{z\in E}|f(z)|$. The Green's function $G^*(z,E)$ is the upper semicontinuous regularization of the Siciak extremal function

$$G(z,E) := \lim_{n\to\infty} \max_{\substack{f\in\Gamma(n)\\ \|f\|_E\le 1}} \frac{1}{n}\log(|f(z)|)\ .$$

Since E is not pluripolar, $G^*(z,E)$ is finite for all $z \in \mathbb{C}^N$ and is plurisubharmonic [9,10,17]. Write $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$ and let

$$\omega = dd^c(2G^*(z,E)) \tag{1}$$

be the associated positive $(1,1)$-current. (The factor 2 is needed for compatibility with the Poincaré-Lelong formula.)

Let $Z_0,\ldots,Z_N$ be homogeneous coordinates on $\mathbb{P}^N(\mathbb{C})$. For $k = 0,\ldots,N$ write H_k for the hyperplane $\{Z_k = 0\}$ and let $\mathbb{A}_k^N = \mathbb{P}^N(\mathbb{C})\backslash H_k$ be the corresponding affine patch. Identify $\mathbb{C}^N$ with $\mathbb{A}_0^N \subset \mathbb{P}^N(\mathbb{C})$ via the embedding $(z_1,\ldots,z_N) \hookrightarrow (1 : z_1 : \cdots : z_N)$. If $\|z\| = (|z_1|^2 + \cdots + |z_N|^2)^{1/2}$, then $|G^*(z,E) - \max(0,\log(\|z\|)|$ is uniformly bounded on $\mathbb{C}^N$ (for boundedness from above, see [9], Corollary 5.2.2, p. 193; for boundedness from below, note that since E is compact, it is contained in a ball $B(0,R) = \{z \in \mathbb{C}^N : \|z\| \le R\}$ for some large R, so $G^*(z,E) \ge G^*(z,B(0,R)) = \max(0,\log(\|z\|/R)))$. It follows that for each $1 \le k \le N$ the function $g_k(z,E)$ defined on $\mathbb{A}_k^N\backslash H_0$ by

$$g_k(z,E) = G^*(z,E) - \log(|z_k|)$$

extends uniquely to a plurisubharmonic function on $\mathbb{A}_k^N$. If $k = 0$, write $g_0(z,E) = G^*(z,E)$. Since $\log(|Z_i/Z_j|)$ is pluriharmonic on $\mathbb{A}_i^N \cap \mathbb{A}_j^N$, the currents $\omega_k := dd^c(2g_k(z,E))$ cohere to give a positive $(1,1)$-current on $\mathbb{P}^N(\mathbb{C})$ extending ω. We will denote it by ω as well.

Let $\omega^k = \omega \wedge \cdots \wedge \omega$ be the k-fold exterior product of ω with itself (for the existence of ω^k, see [3], Theorem 2.1), and put $\omega^0 = 1$. Write Y_k for the space $H_0 \cap \cdots \cap H_{k-1}$ and let $U_k = Y_k\backslash H_k$, so $U_k(\mathbb{C}) \cong \mathbb{C}^{N-k}$. Define the *iterated*

Robin constant by

$$
\begin{aligned}
\widetilde{V}(E) &= \frac{1}{N}\sum_{k=1}^{N}\int\limits_{U_k(\mathbb{C})} g_k(z,E)\,\omega^{N-k} \\
&= \frac{1}{N}\left(\int\limits_{Z_0=0,Z_1\neq 0} g_1(z,E)\,\omega^{N-1} \right. \\
&\qquad + \int\limits_{Z_0=Z_1=0,Z_2\neq 0} g_2(z,E)\,\omega^{N-2} \\
&\qquad\qquad \left. + \cdots + g_N((0:\cdots:0:1),E)\right).
\end{aligned} \tag{2}
$$

Note that when $N = 1$, the sum consists of a single term and reduces to the usual Robin constant, since $\infty = (0:1) \in \mathbb{P}^1(\mathbb{C})$, and $g_1(\infty, E) = \lim_{z\to\infty} G^*(z,E) - \log(|z|) = V(E)$.

Theorem 1 *If $E \subset \mathbb{C}^N$ is compact and not pluripolar, then*

$$d_\infty(E) = \mathrm{e}^{-\widetilde{V}(E)}. \tag{3}$$

Before giving the proof we will need some facts from arithmetic intersection theory. Let K be a number field, and let $\mathcal{O}_K$ be the ring of integers of K. Let X/K be a smooth, connected projective variety of dimension N; write $K(X)$ for the field of K-rational functions on X.

The sectional capacity $S_\gamma(\mathbb{E}, D)$ is a measure of size for an adelic set $\mathbb{E}$ on X relative to an ample divisor D. It was first proposed by Chinburg [4], and its existence was shown in [14], using methods from [16]. The name 'sectional capacity' refers to asymptotics of volumes of spaces of sections of $\mathcal{O}_X(nD)$ as $n \to \infty$.

Let D be an effective, ample K-rational Cartier divisor on X.

For each place v of K, let K_v be the completion of K at v, and let $\mathbb{C}_v$ be the minimal complete algebraically closed field containing K_v. Let $|x|_v$ be the absolute value on $\mathbb{C}_v$ extending the canonical absolute value on K_v given by the modulus of additive Haar measure. Without loss, we can assume that X is embedded in $\mathbb{P}^M$ for a suitable M. There is a natural distance function $d_v(x,y)$ on $\mathbb{P}^M(\mathbb{C}_v)$: the chordal metric associated to the Fubini-Study metric, if v is archimedean; the v-adic spherical metric, if v is nonarchimendean (see [12], Sect. 1.1).

For each v, let $E_v \subset X(\mathbb{C}_v)$ be a nonempty set, and write $\mathbb{E} = \prod_v E_v$; we will call $\mathbb{E}$ an *adelic set*. We will assume that the E_v and D satisfy the following 'Standard Hypotheses':

1. Each E_v is bounded away from $\mathrm{supp}(D)(\mathbb{C}_v)$ under $d_v(x,y)$ and is stable under the group of continuous automorphisms $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$.
2. For all but finitely many v, E_v and $\mathrm{supp}(D)(\mathbb{C}_v)$ specialize to disjoint sets (mod v); equivalently, for all but finitely many v, $d_v(x,y) = 1$ for all $x \in E_v$ and all $y \in \mathrm{supp}(D)(\mathbb{C}_v)$.

Note that if v is archimedean and $K_v \cong \mathbb{R}$, then $\mathrm{Gal}^c(\mathbb{C}_v/K_v) = \{1,\tau\}$ where τ is complex conjugation; if $K_v \cong \mathbb{C}$, then $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$ is trivial.

For each integer $n \geq 0$, put $\Gamma(nD) = H^0(X, \mathcal{O}_X(nD))$, a finite dimensional vector space over K. Dehomogenizing at D, identify $\Gamma(nD)$ with $\{f \in K(X) : \mathrm{div}(f) + nD \geq 0\}$. For each v, consider the space of K_v-rational functions on X with polar divisor at most nD, $\Gamma_v(nD) = K_v \otimes_K \Gamma(nD) = \{f \in K_v(X) : \mathrm{div}(f) + nD \geq 0\}$. For $f \in K_v(X)$, write $\|f\|_{E_v} = \sup_{x\in E_v} |f(x)|_v$ and let

$$\mathcal{F}(E_v, nD) \;=\; \{f \in \Gamma_v(nD) : \|f\|_{E_v} \leq 1\}.$$

Let $K_{\mathbb{A}} = \{(x_v) \in \prod_v K_v : |x_v|_v \leq 1 \text{ for all but finitely many } v\}$ be the adele ring of K. Identifying $K_{\mathbb{A}} \otimes_K \Gamma(nD)$ with a subset of $(\prod_v K_v) \otimes_K \Gamma(nD) \cong \prod_v (K_v \otimes_K \Gamma(nD))$, introduce the 'adelic unit ball'

$$\mathcal{F}(\mathbb{E}, nD) \;=\; (K_{\mathbb{A}} \otimes_K \Gamma(nD)) \bigcap \left(\prod_v \mathcal{F}(E_v, nD) \right).$$

Fix an additive Haar measure $\mathrm{vol}_{\mathbb{A}}$ on $K_{\mathbb{A}}$. By transport of structure using a K-basis for $\Gamma(nD)$, $\mathrm{vol}_{\mathbb{A}}$ induces a Haar measure on $K_{\mathbb{A}} \otimes_K \Gamma(nD)$. The Product Formula shows this measure is independent of the choice of basis; it too will be denoted $\mathrm{vol}_{\mathbb{A}}$. The sectional capacity $S_\gamma(\mathbb{E}, D)$ is defined by

$$-\log(S_\gamma(\mathbb{E}, D)) \;=\; \lim_{n\to\infty} \frac{(N+1)!}{n^{N+1}} \mathrm{vol}_{\mathbb{A}}(\mathcal{F}(\mathbb{E}, nD))$$

(see [14], pp. 23–24). The limit is independent of the choice of Haar measure on $K_{\mathbb{A}}$. Its existence is part of Rumely et al. ([14], Theorem C, p. 8).

We will now discuss Arakelov theory. Let $\mathfrak{X}$ be a model of X: an integral projective scheme of Krull dimension $N+1$, flat and proper over $\mathrm{Spec}(\mathcal{O}_K)$, whose generic fibre is K-isomorphic to X.

Put $X_{\mathbb{C}} = X \times_{\mathbb{Q}} \mathrm{Spec}(\mathbb{C})$; then $X_{\mathbb{C}}(\mathbb{C}) = \bigsqcup_\sigma X_\sigma(\mathbb{C})$ is a complex manifold with a component corresponding to each embedding $\sigma : K \hookrightarrow \mathbb{C}$. Write τ for complex conjugation on $\mathbb{C}/\mathbb{R}$; there is a natural operation of τ on $X_{\mathbb{C}}(\mathbb{C})$ coming from its action on the base. If σ is a real embedding, then τ acts as complex conjugation on $X_\sigma(\mathbb{C})$; if $\{\sigma, \overline{\sigma}\}$ is a pair of conjugate complex embeddings, then τ interchanges the components X_σ and $X_{\overline{\sigma}}$, mapping $P \in X_\sigma(\mathbb{C})$ to $\overline{P} \in X_{\overline{\sigma}}(\mathbb{C})$, where $\overline{P}$ is the point whose coordinates are the complex conjugates of those of P.

Let L be a line bundle on X. A *metrized line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, [\![\cdot]\!])$ on $\mathfrak{X}$, extending L, is a pair consisting of a locally free sheaf $\mathcal{L}$ of rank 1 on $\mathfrak{X}$ which induces L on X, and a smooth positive Hermitian metric $[\![\cdot]\!]$ on the fibres of L over $X_{\mathbb{C}}(\mathbb{C})$, which is invariant under the action of τ. Let $c_1(\overline{\mathcal{L}})$ be the first Chern class of $\overline{\mathcal{L}}$, the smooth $(1,1)$-form on $X_{\mathbb{C}}(\mathbb{C})$ defined in a neighborhood of any point $z_0 \in X_{\mathbb{C}}(\mathbb{C})$ by $dd^c(-2\log([\![s(z)]\!]))$, where s is a meromorphic section of L which is defined and does not vanish at z_0.

For each cycle $\mathcal{Y}$ on $\mathfrak{X}$, Bost, Gillet and Soulé [2] define the "height" $h_{\overline{\mathcal{L}}}(\mathcal{Y})$ of $\mathcal{Y}$ relative to $\overline{\mathcal{L}}$ to be the self-intersection number $\widehat{\deg}(\hat{c}_1(\overline{\mathcal{L}})^{\dim(\mathcal{Y})}|_{\mathcal{Y}})$ in the arithmetic intersection theory Gillet and Soulé developed in [6–8]. The height is additive in $\mathcal{Y}$. There is a recursive formula for $h_{\overline{\mathcal{L}}}(\mathcal{Y})$ (see [2], Proposition 3.2.1(iv), p. 949, and the remarks after it dealing with the non-regular case): when $\mathcal{Y}$ is irreducible, let $s \neq 0$ be a section of $\mathcal{L}|_{\mathcal{Y}}$, and let $\mathrm{div}_{\mathcal{Y}}(s)$ be its divisor. Write Y for the generic fibre $\mathcal{Y}_K$. Then

$$h_{\overline{\mathcal{L}}}(\mathcal{Y}) = \int_{Y_{\mathbb{C}}(\mathbb{C})} -\log([\![s(z)]\!])\, c_1(\overline{\mathcal{L}})^{\dim(Y)} + h_{\overline{\mathcal{L}}}(\mathrm{div}_{\mathcal{Y}}(s)). \tag{4}$$

Eventually the recursion abuts at a 0-cycle $\mathcal{Z} = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot \mathfrak{p}$, a finite sum of closed points, and then $h_{\overline{\mathcal{L}}}(\mathcal{Z}) = \sum n_{\mathfrak{p}} \log(N\mathfrak{p})$ where $N\mathfrak{p}$ is the order of the residue field at $\mathfrak{p}$ ([2], formula 3.1.4, p. 946).

It is customary to write $\overline{\mathcal{L}}^{N+1}$ for $h_{\overline{\mathcal{L}}}(\mathfrak{X})$. If $\overline{\mathcal{L}}$ is replaced by $\overline{\mathcal{L}}^{\otimes m}$, then $(\overline{\mathcal{L}}^{\otimes m})^{N+1} = m^{N+1} \cdot \overline{\mathcal{L}}^{N+1}$ ([2], Proposition 3.2.1(i)). This leads to the notion of a *fractional metrized line bundle*: if $n > 0$ is an integer, and $\overline{\mathcal{L}}$ is a metrized line bundle inducing $L^{\otimes n}$ on X, then we call the formal object $\frac{1}{n}\overline{\mathcal{L}}$ a fractional metrized line bundle, and define $(\frac{1}{n}\overline{\mathcal{L}})^{N+1} = n^{-(N+1)} \cdot \overline{\mathcal{L}}^{N+1}$.

Chinburg, Lau and Rumely [5] expressed the sectional capacity as a limit of self-intersection numbers of fractional metrized line bundles. Given an adelic set $\mathbb{E}$ and an effective ample divisor D on X satisfying the Standard Hypotheses, they constructed a sequence of models $\mathfrak{X}_n$, and metrized line bundles $\overline{\mathcal{L}}_n$ on $\mathfrak{X}_n$ extending $L^{\otimes n} = \mathcal{O}_X(nD)$ on X, such that

$$-\log(S_\gamma(\mathbb{E}, D)) = \lim_{n\to\infty} (\frac{1}{n}\overline{\mathcal{L}}_n)^{N+1} \ : \tag{5}$$

see ([5], Theorem 1.1, p. 233).

The models $\mathfrak{X}_n$ are defined using the nonarchimedean part of $\mathbb{E}$. For each n, put $S_n = \{f \in \Gamma(nD) : \|f\|_{E_v} \leq 1$ for all nonarchimedean $v\}$, and let $\mathcal{O}_K[S_n]$ be the graded $\mathcal{O}_K$-algebra generated in degree 1 by S_n. Then $\mathfrak{X}_n = \mathrm{Proj}(\mathcal{O}_K[S_n])$. Let $\mathcal{L}_n = \mathcal{O}_{\mathfrak{X}_n}(1)$; then $\mathcal{L}_n$ induces $L^{\otimes n}$ on X.

The metrics $[\![\cdot]\!]_n$ are constructed using the archimedean part of $\mathbb{E}$. For each archimedean place v of K, fix an isomorphism $\mathbb{C}_v \cong \mathbb{C}$, and choose a sequence of sets $E_{v,1} \supseteq E_{v,2} \supseteq \cdots$ containing E_v for which the extremal functions

$$G(z, E_{v,n}, D) := \lim_{m\to\infty} \sup_{\substack{f\in\Gamma(mD)\\ \|f\|_{E_{v,n}}\leq 1}} \frac{1}{m} \log(|f(z)|)$$

are continuous and increase monotonically to $G(z, E_v, D)$. (The existence of such sets $E_{v,n}$ follows from the proof of ([5], Lemma 1.2, p. 234).)

Since $G(z, E_{v,n}, D)$ is continuous, it is plurisubharmonic. By a theorem of Richburg ([11], Satz 4.7) there is a smooth plurisubharmonic function $G_{v,n}(z)$ such that

$$G(z, E_{v,n}, D) - \frac{1}{n} \le G_{v,n}(z) \le G(z, E_{v,n}, D) - \frac{1}{n+1}$$

for all $z \in X(\mathbb{C}_v)\backslash\mathrm{supp}(D)(\mathbb{C}_v)$. Hence

$$G(z, E_v, D) = \lim_{n\to\infty} G_{v,n}(z)$$

as an increasing limit.

By ([5], Theorem 2.13, p. 253), the smoothings $G_{v,n}(z)$ can be chosen so that for each $z_0 \in \mathrm{supp}(D)(\mathbb{C}_v)$, if s is a local equation for D at z_0, then $G_{v,n}(z) + \log(|s(z)|)$ extends to a smooth plurisubharmonic function in a neighborhood of z_0. If $K_v \cong \mathbb{R}$, then since E_v has been assumed to be stable under complex conjugation, the $G_{v,n}(z)$ can be chosen to be invariant under complex conjugation as well.

Each embedding $\sigma : K \hookrightarrow \mathbb{C}$ determines a place v of K. If σ is a real embedding, put $G_{\sigma,n}(z) = G_{v,n}(z)$. If σ is a complex embedding, then precisely one of σ and its complex conjugate $\overline{\sigma}$ induces the chosen isomorphism $\mathbb{C}_v \cong \mathbb{C}$; if it is σ, put $G_{\sigma,n}(z) = G_{v,n}(z)$, if not, put $G_{\sigma,n}(z) = G_{v,n}(\overline{z})$.

The metric $[\![\cdot]\!]_n$ is defined by requiring that for the tautological section 1 of $L^{\otimes n}$, if $z \in X_\sigma(\mathbb{C}) \subset X_{\mathbb{C}}(\mathbb{C})$ then

$$[\![1(z)]\!]_n = \exp(-nG_{\sigma,n}(z)).$$

By construction, $[\![\cdot]\!]_n$ is invariant under τ.

Proof of Theorem 1 Let $E \subset \mathbb{C}^N$ be a compact, non-pluripolar set. We will embed it as the archimedean component of an adelic set, and apply the machinery above.

Take $K = \mathbb{Q}(\sqrt{-1})$ as the ground field, so $\mathcal{O}_K$ is the ring of Gaussian integers. Write $\mathbb{P}^N_{\mathcal{O}_K}$ for $\mathbb{P}^N/\mathrm{Spec}(\mathcal{O}_K)$, and write $\mathbb{P}^N_K$ for its generic fibre. Let X/K be $\mathbb{P}^N_K$, and let D be defined by $\{Z_0 = 0\}$.

Put $\mathbb{A}^N = \mathbb{P}^N_K\backslash H_0$, and define $\mathbb{E} = \prod_v E_v \subset \prod_v \mathbb{A}^N(\mathbb{C}_v)$ as follows. For each place v, identify $\mathbb{A}^N(\mathbb{C}_v)$ with $\mathbb{C}_v^N$. There is one archimedean place v_∞ for K; fix an isomorphism $K_{v_\infty} \cong \mathbb{C}$ and put $E_{v_\infty} = E \subset \mathbb{C}^N$. Condition (1) of the Standard Hypotheses holds trivially. For each nonarchimedean v, put $E_v = B(0,1) = \{(z_1, \ldots, z_N) \in \mathbb{C}_v^N : \max(|z_i|_v) \le 1\}$. Since $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$ preserves $|x|_v$, again condition (1) of the Standard Hypotheses holds. The sets E_v and $H_0(\mathbb{C}_v)$ specialize to disjoint sets (mod v) for all nonarchimedean v, so condition (2) of the Standard Hypotheses holds as well.

By ([13], Theorem 3.1, p. 551) the sectional capacity $S_\gamma(\mathbb{E}, D)$ can be decomposed as a product of 'local sectional capacities'

$$S_\gamma(\mathbb{E}, D) = \prod_v S_\gamma(E_v, D).$$

Here the $S_\gamma(E_v, D)$ depend on the choice of an ordered basis for the graded ring $\oplus_{n=0}^{\infty}\Gamma(nD)$: we take this to be the monomial basis, equipped with the lexicographic order graded by the degree. In this situation, $S_\gamma(E_{v_\infty}, D) = d_\infty(E)^{2N}$. This follows from Rumely and Lau ([13], Theorems 2.3 and 2.6), combined with the discussion on ([13], p. 557). (In Rumely and Lau ([13], p. 557), the formula $S_\gamma(E_\infty, D) = d_\infty(E)^N$ is given when $\mathbb{R}$ is the ground field. Here, since $K_{v_\infty} \cong \mathbb{C}$, the normalized absolute value $|x|_{v_\infty} = |x|^2$ used in computing $S_\gamma(E_{v_\infty}, D)$ is the square of the usual absolute value; this is the source of the 2 in the exponent.) Furthermore, $S_\gamma(E_v, D) = 1$ for each nonarchimedean v by Rumely and Lau ([13], Example 4.1, p. 555). Hence

$$S_\gamma(\mathbb{E}, D) = d_\infty(E)^{2N}. \tag{6}$$

Let $\{\frac{1}{n}\overline{\mathcal{L}}_n\}$ be the sequence of fractional metrized line bundles constructed in Chinburg et al.([5], Theorem 1.1) as described above. Since E_v is the 'trivial set' $B(0,1)$ for each nonarchimedean v, the Maximum Modulus Principle of nonarchimedean analysis shows that

$$S_n = \oplus_{k_0+\cdots+k_N=n}\, \mathcal{O}_K \cdot Z_0^{k_0} \cdots Z_N^{k_N}.$$

It follows that $\mathfrak{X}_n \cong \mathbb{P}^N_{\mathcal{O}_K}$, and $\mathcal{L}_n \cong \mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(n)$. Let $\overline{\mathcal{L}}'_n$ be $\mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(1)$, equipped with the metric $[\![\cdot]\!]'_n$ defined by $[\![1(z)]\!]'_n = \exp(-G_n(z))$, where $G_n(z)$ is metric constructed as above by smoothing $G(z, E, D)$. Then $\overline{\mathcal{L}}_n \cong (\overline{\mathcal{L}}'_n)^{\otimes n}$, so $(\frac{1}{n}\overline{\mathcal{L}}_n)^{N+1} = (\overline{\mathcal{L}}'_n)^{N+1}$ and without loss we can replace $\frac{1}{n}\overline{\mathcal{L}}_n$ by $\overline{\mathcal{L}}'_n$.

We will now compute the intersection product $(\overline{\mathcal{L}}'_n)^{N+1}$ using Bost-Gillet-Soulé's recursive formula. There are two embeddings of K into $\mathbb{C}$, both of which correspond to the place v_∞, so $(\mathbb{P}^N_K)_{\mathbb{C}}(\mathbb{C})$ has two components which are interchanged by τ. Write $\mathbb{P}^N(\mathbb{C})$ for the one corresponding to our chosen isomorphism $K_{v_\infty} \cong \mathbb{C}$, and $(\mathbb{P}^N)_\tau(\mathbb{C})$ for the other. Since they are isomorphic, and the metrics $[\![\cdot]\!]_n$ are τ-invariant, the integrals over the two components in the archimedean part of the intersection product are the same. Therefore, in what follows, we will compute the integrals over $\mathbb{P}^N(\mathbb{C})$ and double the answer. For any K-rational subvariety $Y \subset \mathbb{P}^N_K$, write $Y(\mathbb{C})$ for the part of $Y_{\mathbb{C}}(\mathbb{C})$ in the chosen component $\mathbb{P}^N(\mathbb{C})$.

Write $\mathbb{A}^N_k$ for $\mathbb{P}^N_K \backslash H_k$, where $H_k = \{Z_k = 0\}$. Let $\overline{H}_k$ be the Zariski closure of H_k in $\mathbb{P}^N_{\mathcal{O}_K}$. Then $\overline{H}_0, \ldots, \overline{H}_N$ meet transversely on $\mathbb{P}^N_{\mathcal{O}_K}$. Write $g_{n,0}(x) = G_n(x)$ on $(\mathbb{A}^N_0)(\mathbb{C})$ and for each $k = 1, \ldots, N$ let $g_{n,k}(x)$ be the natural extension of

$G_n(x) - \log(|z_k(x)|)$ to a plurisubharmonic function on $\mathbb{A}_k^N(\mathbb{C})$; here $z_k = Z_k/Z_0$. As before, the $(1,1)$-forms $dd^c(2g_{n,k})$ glue to give a well-defined $(1,1)$-form ω_n on $\mathbb{P}^N(\mathbb{C})$.

Since $L = \mathcal{O}_{\mathbb{P}_K^N}(D)$ where $D = \{Z_0 = 0\}$, the canonical section '1' of L is Z_0. This means that for each $x \in \mathbb{A}_0^N(\mathbb{C})$

$$-\log([\![Z_0(x)]\!]_n') \ = \ G_n(x) \ = \ g_{n,0}(x).$$

Similarly, on $\mathbb{A}_k^N(\mathbb{C})$,

$$\begin{aligned} -\log([\![Z_k(x)]\!]_n') &= -\log([\![(Z_k/Z_0)(x) \cdot Z_0(x)]\!]_n') \\ &= -\log(|z_k(x)|) + G_n(x) \ = \ g_{n,k}(x). \end{aligned}$$

Put $\mathcal{Y}_0 = \mathfrak{X}_n = \mathbb{P}_{\mathcal{O}_K}^N$ and take $s = Z_0$. Let $Y_0 = \mathbb{P}_K^N$ be the generic fibre of $\mathcal{Y}_0$ and put $\mathcal{Y}_1 = \mathrm{div}(Z_0) = \overline{H}_0$. By Bost-Gillet-Soulé's formula (4) and the remarks above,

$$(\overline{\mathcal{L}}_n')^{N+1} \ = \ h_{\overline{\mathcal{L}}_n'}(\mathfrak{X}_n) \ = \ 2 \int\limits_{Y_0(\mathbb{C})} g_{n,0}(z)\, \omega^N \ + \ h_{\overline{\mathcal{L}}_n'}(\mathcal{Y}_1).$$

Inductively apply this formula to the sections $Z_1, \ldots, Z_N$, putting $\mathcal{Y}_{k+1} = \mathrm{div}_{\mathcal{Y}_k}(Z_k|\, \mathcal{Y}_k) = \overline{H}_0 \cdot \cdots \cdot \overline{H}_k$. Note that $\overline{H}_0 \cdot \cdots \cdot \overline{H}_N = 0$, so the final abutment term vanishes. It follows that

$$(\overline{\mathcal{L}}_n')^{N+1} \ = \ 2 \sum_{k=0}^{N} \int\limits_{Y_k(\mathbb{C})} g_{n,k}(z)\, \omega_n^{N-k}. \tag{7}$$

Since ω_n is smooth, the $(N-k-1)$-dimensional subspace $Y_{k+1}(\mathbb{C})$ of $Y_k(\mathbb{C})$ has measure 0 under ω_n^{N-k}. Put $U_k = Y_k \backslash Y_{k+1}$. Then for each k

$$\int\limits_{Y_k(\mathbb{C})} g_{n,k}(z)\, \omega_n^{N-k} \ = \ \int\limits_{U_k(\mathbb{C})} g_{n,k}(z)\, \omega_n^{N-k}. \tag{8}$$

Now let $n \to \infty$. For each k the $g_{n,k}$ increase monotonically on $U_k(\mathbb{C})$ to $G(z, E) - \log(|z_k|)$, whose upper semicontinuous regularization is $g_k(z, E)$. Noting that $U_k(\mathbb{C}) \cong \mathbb{C}^{N-k}$, it follows from Bedford and Taylor ([3], Theorem 7.4) that

$$\lim_{n \to \infty} \int\limits_{U_k(\mathbb{C})} g_{n,k}(z)\, \omega_n^{N-k} \ = \ \int\limits_{U_k(\mathbb{C})} g_k(z, E)\, \omega^{N-k}. \tag{9}$$

Combining (5), (7), (8), and (9) gives

$$-\log(S_\gamma(\mathbb{E}, D)) \;=\; 2\sum_{k=0}^{N} \int\limits_{U_k(\mathbb{C})} g_k(z,E)\,\omega^{N-k}. \tag{10}$$

When $k = 0$, $g_0(z,E)$ is the extremal plurisubharmonic function $G^*(z,E)$ on $U_0(\mathbb{C}) \cong \mathbb{C}^N$, and ω^N is the Bedford-Taylor measure, which is supported on E. The set $\{z \in E : G^*(z,E) > 0\}$ is a negligible set, so it has measure 0 under ω^N. Hence

$$\int\limits_{U_0(\mathbb{C})} g_0(z,E)\,\omega^N \;=\; 0. \tag{11}$$

Using (6), (10), and (11) we obtain

$$-2N \cdot \log(d_\infty(E)) \;=\; 2\sum_{k=1}^{N} \int\limits_{U_k(\mathbb{C})} g_k(z,E)\,\omega^{N-k}$$

which is equivalent to (3). □

Generalizations It is tempting to assert that the formula in Theorem 1 gives a new definition of the capacity. However, we refrain from doing so because (3) is only one of a class of formulas with the same property.

In particular, the order in which the hyperplanes H_k are intersected in the definition of $\widetilde{V}(E)$ is immaterial. More generally, if $A = (a_{kj}) \in \mathrm{GL}_N(\mathbb{C})$, and if $Z'_k = \sum a_{kj} Z_j$ for $k = 1, \ldots, N$ then the constant $\widetilde{V}(E)$ can equally well be defined by

$$\widetilde{V}(E) \;=\; \frac{1}{N}\sum_{k=1}^{N} \int\limits_{Z_0 = Z'_1 = \cdots = Z'_{k-1} = 0,\, Z'_k \neq 0} g'_k(z,E)\,\omega^{N-k} \;+\; \frac{1}{N}\log(|\det(A)|) \tag{12}$$

where $g'_k(z,E) = G^*(z,E) - \log(|\sum a_{kj} z_j|)$ on $\mathbb{C}^N$.

Indeed, if we put $z' = {}^t(z'_1, \ldots, z'_N) = A(z)$, where $z'_k = \sum a_{kj} z_j$ for $k = 1, \ldots, N$, this follows from Sheĭnov's formula (see [15], or [1], p. 287)

$$d_\infty(A(E)) \;=\; |\det(A)|^{1/N} \cdot d_\infty(E),$$

by applying Theorem 1 to the set $A(E)$, and noting that $G(z', A(E)) = G(z, E)$, as follows easily from the definitions.

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References

1. Bloom, T., Calvi, J.P.: On the multivariate transfinite diameter. Ann. Polonici Math. **72**, 285–305 (1999)
2. Bost, J.-B., Gillet, H., Soulé, C.: Heights of projective varieties and positive Green's forms. J. Am. Math. Soc. **7**, 903–1027 (1994)
3. Bedford, E., Taylor B.A.: A new capacity for plurisubharmonic functions. Acta Math **149**, 1–40 (1982)
4. Chinburg, T.: Capacity theory on varieties. Compositio Math. **80**, 71–84 (1991)
5. Chinburg, T.: Lau C.F., Rumely, R.: Capacity theory and arithmetic intersection theory. Duke Math. J. **117**, 229–285 (2003)
6. Gillet, H., Soulé, C.: Arithmetic intersection theory. IHES Publ Math. **72**, 94–174 (1990)
7. Gillet H., Soulé, C.: Characteristic classes for algebraic vector bundles with Hermitian metric, I, II. Ann. Math. **131** (2), 13–203, 205–238 (1990)
8. Gillet, H., Soulé, C.: An arithmetic Riemann-Roch Theorem. Invent. Math. **110**, 473–543 (1992)
9. Klimek M.: Pluripotential theory. Oxford (1991)
10. Levenberg, N., Taylor, B.A.: Comparison of capacities in $\mathbb{C}^N$. In: Complex Analysis (Toulouse, 1983), pp. 162–172, SLNM 1094. Springer, Berlin HeidelBerg New York (1984)
11. Richburg, R.: Stetige streng pseudokonvexe Functionen. Math. Ann. **175**, 257–286 (1968)
12. Rumely, R.: Capacity theory on algebraic curves. In: Lecture Notes in Mathematics, vol 1378. Springer, Berlin Heidelberg New York (1989)
13. Rumely, R., Lau, C.F.: Arithmetic capacities on $\mathbb{P}^n$. Math. Zeit. **215**, 533–560 (1994)
14. Rumely, R., Lau, C.F., Varley, R.: Existence of the Sectional Capacity. AMS Memoires 690 (vol. 145). American Mathematical Society, Providence (2000)
15. Sheĭnov, V.P.: Invariant form of Pólya's inequalities. Siberian Math. J. **14**, 138–145 (1973)
16. Zaharjuta, V.: Transfinite diameter, Chebyshev constants, and capacity for compacta in $\mathbb{C}^N$. Math. USSR Sbornik **25**, 350–364 (1975)
17. Zaharjuta, V.: Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein-Walsh theorem for analytic functions of several variables. Ann. Polonici Math. **33**, 137–148 (1976)