# ZEROS OF $p$-ADIC EXPONENTIAL POLYNOMIALS II

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## ABSTRACT

We give a bound for the total number of zeros of a $p$-adic exponential polynomial in its disc of convergence. For an exponential polynomial of order $n$ defined over $\mathbb{Q}_p$, with $p > n \geqslant 3$, the bound is $(n-2)p$. Similar bounds are obtained in general.

## 1. *Introduction*

Let $p$ be a fixed rational prime. Write $\mathbb{Q}_p$ for the field of $p$-adic numbers, $\mathbb{C}_p$ for the completion of the algebraic closure of $\mathbb{Q}_p$, $\hat{O}_p$ for the ring of integers in $\mathbb{C}_p$, and $m_p$ for its maximal ideal. Let $|x|$ be the absolute value on $\mathbb{C}_p$, and $\operatorname{ord}(x)$ the associated valuation, normalized so that $|p| = 1/p$ and $\operatorname{ord}(p) = 1$.

A $p$-adic exponential polynomial is a sum

$$b(z) = \sum_{i=1}^{d} B_i(z) \exp(\eta_i z),$$

where the $\eta_i$ are distinct elements of $\mathbb{C}_p$ and the $B_i(z)$ are polynomials in $\mathbb{C}_p[z]$. If the degrees of the $B_i(z)$ are $n(i)-1$ for $1 \leqslant i \leqslant d$, then

$$n = \sum_{i=1}^{d} n(i)$$

is called the order of $b(z)$. We say $b(z)$ is defined over a field $L$ if the $\eta_i$ and the coefficients of the $B_i(z)$ belong to $L$.

If $n \leqslant 2$, then $b(z)$ has at most one zero. The purpose of this paper is to bound the total numbers of zeros in the general case. When $L = \mathbb{Q}_p$ and $p > n \geqslant 3$, our bound is $(n-2)p$. In the course of the proof we shall recover an estimate of Robba [2] for the number of zeros of $b(z)$ in a subdisc of its disc of convergence.

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## 2. *Preliminaries*

The theory of exponential polynomials is governed by the interaction between three objects: generalized power sums, recurrence sequences, and rational functions.

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Let $\omega_i$ be distinct non-zero elements of $\mathbb{C}_p$, and let polynomials $A_i(z) \in \mathbb{C}_p[z]$ have degrees $n(i)-1$, for $1 \leqslant i \leqslant d$. Then

$$A(h) = \sum_{i=1}^{d} A_i(h)\,\omega_i^h, \qquad h = 0, 1, 2, \ldots,$$

is called a generalized power sum with roots $\omega_i$ and coefficients $A_i(z)$; $n = \sum n(i)$ is called its order. Write

$$\begin{aligned} D(x) &= \prod_{i=1}^{d} (1-\omega_i x)^{n(i)} \\ &= 1 - d_1 x - d_2 x^2 - \ldots - d_n x^n. \end{aligned}$$

The sequence $(a_h)$ with $a_h = a(h)$ satisfies the homogeneous linear recurrence relation

$$a_h = d_1 a_{h-1} + d_2 a_{h-2} + \ldots + d_n a_{h-n}$$

for $h \geqslant n$. To see this, let $E\colon f(h) \to f(h+1)$ be the translation operator and let $\Delta = E - 1$ be the difference operator. Then

$$(E-\omega_i)\,A_i(h)\,\omega_i^h = (\Delta A_i(h))\,\omega_i^{h+1},$$

and so since $\Delta A_i$ has lower degree than $A_i$, by linearity and induction it is plain that $\prod_{i=1}^{d}(E-\omega_i)^{n(i)}$ annihilates the sequence $(a_h)$. Moreover, $(a_h)$ is determined by its initial values $a_0, a_1, \ldots, a_{n-1}$ and the recurrence relation.

As a consequence, the power series with coefficients $a(h)$,

$$g(x) = \sum_{h=0}^{\infty} a(h)\,x^h = \frac{N(x)}{D(x)},$$

is a rational function. Here $D(x)$ is as before, and $N(x)$ is a polynomial of degree less than $n$. This follows easily by multiplying both sides by $D(x)$ and using the recurrence relation.

Conversely, if a rational function $g(x)$ as above is given, then its Taylor coefficients $a(h)$ can be expressed as a power sum by using the partial fraction decomposition of $g(x)$:

$$\begin{aligned} g(x) = \frac{N(x)}{D(x)} &= \sum_{i=1}^{d} \sum_{j=1}^{n(i)} \frac{a_{ij}}{(1-\omega_i x)^j} \\ &= \sum_{h=0}^{\infty} \left[ \sum_{i=1}^{d} \sum_{j=1}^{n(i)} a_{ij} \binom{h+j-1}{j-1} \omega_i^h \right] x^h. \end{aligned}$$

The connections between exponential polynomials and generalized power sums go two ways. If an exponential polynomial $b(z)$ is expanded as a power series,

$$\begin{aligned} b(z) &= \sum_{i=1}^{d} \mathbf{B}_i(z) \exp(\eta_i z) \\ &= \sum_{h=0}^{\infty} B(h) \frac{z^h}{h!}, \end{aligned}$$

then $B(h)$ is a generalized power sum of order $n$:

$$B(h) = \sum_{i=1}^{d} B_i(h)\,\eta_i^h,$$

where

$$B_i(h) = \sum_{j=0}^{n(i)} \mathbf{B}_{ij}\,\eta_i^{-j}\, h(h-1)\ldots(h-j+1).$$

On the other hand if the roots $\omega_i$ of the power sum $a(h)$ satisfy

$$\operatorname{ord}(\omega_i - 1) > \frac{1}{p-1},$$

then $a(h)$ can be interpolated by an exponential polynomial converging for $t \in \hat{O}_p$:

$$a(t) = \sum_{i-1}^{d} A_i(t) \exp(t \log \omega_i).$$

In particular $a(t)$ is a continuous $p$-adic function.

## 3. *Finiteness of the number of zeros*

The zeros of a power series $a(z) = \sum a_h z^h$ are governed by its Newton polygon. Recall that this is the lower convex hull of the set of points

$$(h, \operatorname{ord}(a_h)), \qquad h = 0, 1, 2, \ldots .$$

A finite segment of the Newton polygon with slope $-m$ and projection length $L$ on the horizontal axis corresponds to the existence of $L$ roots of $f(z)$ satisfying $\operatorname{ord}(\lambda_i) = m$. The Newton polygon either has infinitely many segments, or terminates in a ray. If it terminates in a ray, then $a(z)$ has only finitely many zeros. If in addition there are infinitely many points $(h, \operatorname{ord}(a_h))$ with bounded vertical distance to the ray, then the total number of zeros is the index $l$ of the final corner of the Newton polygon. Furthermore, the domain of convergence of $f(z)$ is the open disc

$$\operatorname{ord}(z) > w,$$

where $-w$ is the slope of the ray. See [**1**, pp. 91–99] for further discussion.

We now apply this to the exponential polynomial

$$b(z) = \sum_{i-1}^{d} \mathbf{B}_i(z) \exp(\eta_i z).$$

Put

$$V = \min_{1 \leqslant i \leqslant d} (\operatorname{ord}(\eta_i)).$$

THEOREM 1. *The exact disc of convergence of $b(z)$ is*

$$\left\{ z : \operatorname{ord}(z) > -\left( V - \frac{1}{p-1} \right) \right\}$$

*and the total number of zeros of $b(z)$ is finite.*

*Proof.* It is convenient to change variables. Put

$$\pi = p^{1/(p-1)}, \qquad \lambda = p^{V-(1/(p-1))}, \qquad \omega_i = \eta_i / p^V,$$

and set $a(z) = b(z/\lambda)$, $\mathbf{A}_i(z) = \mathbf{B}_i(z/\lambda)$. Then

$$\begin{aligned} a(z) &= \sum_{i-1}^{d} \mathbf{A}_i(z) \exp(\omega_i \pi z) \\ &= \sum_{h-0}^{\infty} \left( \sum_{i-1}^{d} A_i(h) \omega_i^h \right) \frac{\pi^h}{h!} z^h \\ &= \sum_{h-0}^{\infty} A(h) \frac{\pi^h}{h!} z^h, \end{aligned}$$

where $A(h)$ is a generalized power sum with roots $\omega_i$. Assertions about $b(z)$ on

$$\operatorname{ord}(z) > -\left(V-\frac{1}{p-1}\right)$$

are equivalent to ones for $a(z)$ on $\operatorname{ord}(z) > 0$.

By construction all the roots $\omega_i$ belong to $\hat{O}_p$, and at least one is a unit. Suppose first that all the $\omega_i$ are units. Then there is a number $Q = p^E(p^f-1)$ such that, for all $i$,

$$\operatorname{ord}(\omega_i^Q-1) > \frac{1}{p-1}.$$

The coefficients of $a(z)$ are given by $a_h = A(h)\pi^h/h!$. It is well known that

$$\operatorname{ord}(h!) = \frac{h-s(h)}{p-1},$$

where if $h$ is written in base $p$ as $h = h_0+h_1p+\ldots+h_mp^m$ then $s(h) = h_0+h_1+\ldots+h_m$. Consequently

$$\operatorname{ord}(a_h) = \operatorname{ord}(A(h))+\frac{s(h)}{p-1}.$$

Clearly $\operatorname{ord}(a_h)$ is bounded from below; this means that $a(z)$ converges in at least the open disc $\operatorname{ord}(z) > 0$. To prove that it converges in no larger disc, and that $a(z)$ has a finite number of zeros, we shall show that

(1) $\operatorname{ord}(a_h)$ achieves a minimum value;

(2) $\operatorname{ord}(a_h)$ is bounded above for infinitely many $h$.

To do so, consider $a_h$ on arithmetic progressions mod $Q$. For each $r$, $0 \leqslant r < Q$, write $h = r+tQ$ and put $a_r(t) = A(r+tQ)$. Then $a_r(t)$ can be interpolated by an exponential polynomial:

$$a_r(t) = \sum_{i=1}^{d} A_i(r+tQ)\,\omega_i^r \exp(t\log\omega_i^Q).$$

Here $\log(w)$ is the $p$-adic logarithm, which converges on $\operatorname{ord}(w) > 0$ and is inverse to $\exp(w)$ on $\operatorname{ord}(w) > 1/(p-1)$.

We claim that for any bound $M$, $\operatorname{ord}(a_r(t))$ takes on only a finite number of values less than $M$ as $t$ runs over $\mathbb{Z}$. In fact, this is true for $t \subset \mathbb{Z}_p$. Since $a_r(t)$ is continuous, for any $x$ such that $a_r(x) \neq 0$ there is a neighbourhood $\mathbf{D}_x$ on which $\operatorname{ord}(a_r(t))$ is constant. For any $x$ such that $a_r(x) = 0$ there is a neighbourhood $\mathbf{D}_x$ on which $\operatorname{ord}(a_r(t)) > M$. Since $\mathbb{Z}_p$ is compact, the claim is established.

Clearly $s(h)/(p-1)$ is greater than or equal to 0 and takes values that are integer multiples of $1/(p-1)$. Combining these facts for all $r$ mod $Q$, we see that $\operatorname{ord}(a_h)$ takes on only a finite number of values less than any bound $M$. Consequently there is $h_0$ for which $\operatorname{ord}(a_{h_0})$ is minimal; this establishes (1).

To prove (2) we consider $h$ of the form

$$h = h_0+p^mQ.$$

By the discussion above,

$$\operatorname{ord}(A(h_0+p^mQ)) = \operatorname{ord}(A(h_0))$$

for all sufficiently large $m$. On the other hand, as soon as $p^m > h_0$, we have

$$s(h_0+p^mQ) = s(h_0)+s(Q).$$

Thus there are infinitely many $h$ for which

$$\operatorname{ord}(a_h) = \operatorname{ord}(A(h_0))+\frac{s(h_0)+s(Q)}{p-1}.$$

This proves (2) and yields the theorem when all the $\omega_i$ are units.

In the general case, break $A(h)$ into two sums $A^0(h)+A^+(h)$, where

$$A^0(h) = \sum_{\operatorname{ord}(\omega_i)=0} A_i(h)\,\omega_i^h,$$

$$A^+(h) = \sum_{\operatorname{ord}(\omega_i)>0} A_i(h)\,\omega_i^h.$$

Then the argument above applies to $A^0(h)$, while $\lim_{h\to\infty} \operatorname{ord}(A^+(h)) = \infty$. By the ultrametric inequality, assertions (1) and (2) remain valid.

We now know the Newton polygon of $a(z)$ terminates in a ray of slope 0, and that the total number of zeros of $a(z)$ is the index of the final corner of the Newton polygon. This fact will be the basis for our future results.

## 4. *Some lemmas*

In this section we collect several lemmas about recurrence sequences, binomial coefficients, and $p$-adic digit sums that will be needed later.

LEMMA 1. *Let $(b_h)$ be a recurrence sequence of order $n$, satisfying the relation*

$$b_h = d_1 b_{h-1}+d_2 b_{h-2}+\ldots+d_n b_{h-n}$$

*for $h \geqslant n$. Suppose that the $d_i$ all belong to $\hat{O}_p$, and put $M = \inf_{h\geqslant 0}(\operatorname{ord}(b_h))$. Then there is $H < n$ such that $\operatorname{ord}(b_H) = M$.*

*Proof.* By the ultrametric inequality and induction

$$\operatorname{ord}(b_h) \geqslant \min_{1\leqslant i\leqslant n}(\operatorname{ord}(b_{h-i})) \geqslant \min_{0\leqslant H<n}(\operatorname{ord}(b_H)).$$

LEMMA 2. *Let $(b_h)$ be a recurrence sequence of order $n$, whose associated generalized power sum is $B(h)$. Suppose that the roots $\omega_i$ of $B(h)$ satisfy $\operatorname{ord}(\omega_i) \geqslant \varepsilon$ for some $\varepsilon > 0$. Put $M = \min_{h\geqslant 0}\operatorname{ord}(b_h)$. Then for all $h \geqslant n$,*

$$\operatorname{ord}(b_h) \geqslant M+(h-n+1)\,\varepsilon.$$

*Proof.* From

$$D(x) = \prod_{i=1}^{d}(1-\omega_i x)^{n(i)} = 1-d_1x^1-\ldots-d_nx^n$$

we see that $\operatorname{ord}(d_j) \geqslant j\varepsilon$. Hence for $h \geqslant n$,

$$\begin{aligned}\operatorname{ord}(b_h) &\geqslant \min_{1\leqslant j\leqslant n} (\operatorname{ord}(b_{h-j})+j\varepsilon)\\ &\geqslant \operatorname{ord}(b_{h-j_0})+j_0\varepsilon,\end{aligned}$$

where $j_0$ is chosen to achieve the minimum. By descending induction, there is $j$ for which $0 \leqslant h-j \leqslant n-1$ and such that

$$\begin{aligned}\operatorname{ord}(b_h) &\geqslant \operatorname{ord}_p(b_{h-j})+j\varepsilon\\ &\geqslant M+(h-n+1)\varepsilon.\end{aligned}$$

LEMMA 3. *Consider the binomial coefficient* $\binom{t}{l}$ *as a function of* $t$. *Set* $L = \lfloor \log_p l \rfloor$. *Then, for any natural numbers* $t$ *and* $F$,

$$\binom{t+p^F}{l} \equiv \binom{t}{l} \quad \operatorname{mod} p^{F-L}.$$

*Proof.* Here $\log_p l$ means the base $p$ logarithm, and $\lfloor x \rfloor$ is the greatest integer $k$ with $k \leqslant x$; similarly $\lceil x \rceil$ is the least integer $k$ with $k \geqslant x$.

If $F \leqslant L$ the result is trivial, so we can assume that $F > L$. This implies that $p^F > l$. Equating coefficients of $x^l$ in

$$(1+x)^{t+p^F} = (1+x)^t(1+x)^{p^F}$$

gives

$$\binom{t+p^F}{l} = \binom{t}{l} + \sum_{k=1}^{l} \binom{t}{l-k}\binom{p^F}{k}.$$

For each $j$, $1 \leqslant j < p^F$, the power of $p$ dividing $j$ is the same as that dividing $p^F-j$. Hence, by the product expansion of binomial coefficients,

$$\operatorname{ord}\binom{p^F}{k} = F-\operatorname{ord}(k) \geqslant F-L,$$

and this proves the lemma.

LEMMA 4. *For any* $h \in \mathbb{N}$, *the p-adic digit sum* $s(h)$ *satisfies*

$$s(h) \equiv h \quad \operatorname{mod}(p-1).$$

*Furthermore, if* $h = h_0+h_1p+\ldots+h_mp^m$ *in base* $p$, *where* $h_m > 0$, *and if* $\hat{h} = h-(p^m-p^k)$ *with* $0 \leqslant k \leqslant m$, *then* $s(\hat{h}) \leqslant s(h)$.

*Proof.* The first assertion holds since

$$h-s(h) = \sum_{i=1}^{m} h_i(p^i-1)$$

is obviously divisible by $p-1$. For the second assertion, if $m = k$ the result is trivial. If $m > k$ and $h_k \neq p-1$, then $s(\hat{h}) = s(h)$. However, if $h_k = p-1$, then writing $h = a+(p-1)p^k+p^{k+1}b$ we have

$$s(h) = s(a)+(p-1)+s(b), \qquad s(\hat{h}) = s(a)+s(\hat{b}),$$

where $\hat{b} = b-(p^{m-k-1}-1)$, and by induction $s(\hat{b}) \leqslant s(b)$.

## 5. *Bounds on the number of zeros*

We begin by giving a new proof for Robba's bound for the number of zeros of an exponential polynomial in a subdisc of its full disc of convergence. See also [3]. Although the result itself is not needed in what follows, it provides a context for our bound on the total number of zeros, and the proof introduces useful ideas. Notation will be the same as that established earlier.

Given $b(z) = \sum B_i(z) \exp(\eta_i z)$ of order $n$, let $R = -(V - 1/(p-1))$ be the (logarithmic) radius of its disc of convergence. Put

$$s_0 = \max_{0 \leqslant l < n} s(l).$$

If $n = n_0 + n_1 p + \ldots + n_m p^m$ in base $p$, then $s_0 = m(p-1) + n_m - 1$. Thus,

$$(p-1)\lfloor \log_p n \rfloor \leqslant s_0 \leqslant (p-1)\lceil \log_p n \rceil,$$

and if $p \geqslant n$ then $s_0 = n-1$.

THEOREM 2 (Robba [2]). *For any $\delta > 0$, the number of zeros of $b(z)$ in the disc* $\operatorname{ord}(z) \geqslant R + \delta$ *is at most*

$$n - 1 + \frac{1}{\delta}\frac{s_0 - 1}{p-1}.$$

*Proof.* By taking a limit, it suffices to consider the case where $\delta \in \mathbb{Q}$. Then, after a change of variables, if necessary, we can assume that $R = -\delta$ (note that $R$ is in any case rational, since $V = \min_i(\operatorname{ord}(\eta_i)) \in \mathbb{Q}$). Thus

$$b(z) = \sum_{h=0}^{\infty} b_h z^h = \sum_{h=0}^{\infty} B(h)(p^{\delta})^h \frac{\pi^h}{h!} z^h,$$

where the roots $\omega_i$ of $B(h)$ satisfy $\operatorname{ord}(\omega_i) \geqslant 0$.

By the theory of Newton polygons, the number of zeros of $b(z)$ in the disc $\operatorname{ord}(z) \geqslant 0$ is the largest index $h$ for which $\operatorname{ord}(b_h)$ is minimal. Here

$$\operatorname{ord}(b_h) = \operatorname{ord}(B(h)) + \frac{s(h)}{p-1} + h\delta$$

and $B(h)$ is a recurrence sequence of order $n$. By Lemma 1, the first $H$ for which $\operatorname{ord}(B(h))$ is minimal satisfies $H < n$. Now $\operatorname{ord}(b_H)$ is a strong candidate to be the minimum value of $\operatorname{ord}(b_h)$; note also that $s(H) \leqslant s_0$. Certainly $\operatorname{ord}(b_h)$ is not minimal for any $h$ satisfying

$$h > n - 1 + \frac{1}{\delta}\frac{s_0 - s(h)}{p-1},$$

for then

$$\operatorname{ord}(B(h)) + \frac{s(h)}{p-1} + h\delta > \operatorname{ord}(B(h)) + \frac{s_0}{p-1} + (n-1)\delta$$

$$\geqslant \operatorname{ord}(B(H)) + \frac{s(H)}{p-1} + H\delta.$$

Now $s(h) \geqslant 1$ for all $h \geqslant 1$. Hence if

$$h > n - 1 + \frac{1}{\delta}\frac{s_0 - 1}{p-1},$$

then both inequalities above are valid; this yields the bound.

Next we turn to bounds for the total number of zeros of $b(z)$. If $n = 1$, then $b(z) = b_1 \exp(\eta_1 z)$ has no zeros. If $n = 2$, then $b(z) = (b_1 + b_2 z)\exp(\eta_1 z)$ or $b(z) = b_1 \exp(\eta_1 z) + b_2 \exp(\eta_2 z)$. In the first case the only possible zero of $b(z)$ is at $-b_1/b_2$; in the second case it is at $(\eta_2 - \eta_1)^{-1} \log(-b_1/b_2)$. Thus $b(z)$ has at most one zero, and we can assume that $n \geqslant 3$ in what follows.

By making a change of variables $a(z) = b(z/\lambda)$ as in Theorem 1, it is enough to consider exponential polynomials of the form

$$a(z) = \sum_{i=1}^{d} A_i(z) \exp(\pi \omega_i z) = \sum_{h=0}^{\infty} A(h) \frac{\pi^h}{h!} z^h,$$

where $A(h) = \sum_{i=1}^{d} A_i(h) \omega_i^h$ has roots $\omega \in \hat{O}_p$, and at least one $\omega_i$ is a unit. The disc of convergence of $a(z)$ is $\operatorname{ord}(z) > 0$.

For each unit $\omega \in \hat{O}_p^\times$ there is a unique root of unity $\zeta$ of order prime to $p$ such that $\operatorname{ord}(\omega - \zeta) > 0$. This is called the Teichmuller representative of $\omega$, and written $\zeta = \operatorname{Teich}(\omega)$. For $\omega \in m_p$, put $\operatorname{Teich}(\omega) = 0$.

Our bound for the number of zeros depends on the following data intrinsic to $a(z)$: its order $n$, the proximity of the $\omega_i$ to their Teichmuller representatives, and the size of the residue field they generate. Let

$$\varepsilon = \min(1, \min_{1 \leqslant i \leqslant d} \operatorname{ord}(\omega_i - \operatorname{Teich}(\omega_i))).$$

Then $0 < \varepsilon \leqslant 1$. Also, let $R = \mathbb{Z}_p[\omega_1, \ldots, \omega_d]$ and put $m = R \cap m_p$. The residue class field $R/m$ has order a power of $p$; write this as $q = p^f$. Finally recall that

$$s_0 = \min_{0 \leqslant l < n} s(l),$$

where $n \geqslant 3$ and $s(l)$ is the sum of the base $p$ digits of $l$.

THEOREM 3. *The total number of zeros of $a(z)$ in its disc of convergence is at most $p^f(p^F - 1)$, where*

$$F = \left\lceil \frac{s_0 - 1}{p-1} \right\rceil + \left\lfloor \log_p \left( n - 1 + \frac{1}{\varepsilon} \frac{s_0 - 1}{p-1} \right) \right\rfloor.$$

*When $s_0 < p$ (that is, when $n < 2p$) this can be improved to $(s_0 - 1)p^{f+F-1}$.*

*Proof.* Our goal is to bound the location of the final corner of the Newton polygon of $a(z)$, or equivalently, to bound the occurrence of the first $h$ for which $\operatorname{ord}(a_h)$ is minimal. As before,

$$\operatorname{ord}(a_h) = \operatorname{ord}(A(h)) + \frac{s(h)}{p-1}.$$

By Lemma 1, the first $H$ for which $\operatorname{ord}(A(H))$ is minimal satisfies $H < n$. If $H$ is not the index of the final corner, then there is $h > H$ for which

$$\operatorname{ord}(A(h)) + \frac{s(h)}{p-1} < \operatorname{ord}(A(H)) + \frac{s(H)}{p-1}.$$

Without loss of generality we can assume that $a(z)$ has been normalized so that

$\min(\operatorname{ord}(A(l))) = 0$. Then $\operatorname{ord}(A(H)) = 0$, and by $s(H) \leqslant s_0$ and $s(h) \geqslant 1$, the above inequality yields

$$\operatorname{ord}(A(h)) + \frac{s(h)}{p-1} < \frac{s_0}{p-1}.$$

For such $h$, it will turn out that unless $h$ is less than the bound in the theorem, there is $\hat{h} < h$ such that $\operatorname{ord}(A(\hat{h})) = \operatorname{ord}(A(h))$ and $s(\hat{h}) \leqslant s(h)$.

Let us first prove the theorem under the assumption that all the $\omega_i$ are units. In order to find $h$ such that $\operatorname{ord}(A(h)) < (s_0-1)/(p-1)$ it suffices to consider $A(h) \bmod \pi^{s_0-1}$. We shall study $A(h)$ on progressions $h = r+t(q-1)$ because on such sequences, $A(h) \bmod \pi^{s_0-1}$ is congruent to a polynomial. This is due to the connection between recurrence sequences and Taylor expansions of rational functions. Fortunately the digit sums $s(h)$ also behave well on progressions $\bmod (q-1)$.

For each $r$, $0 \leqslant r < q-1$, put

$$\begin{aligned} A_r(t) &= A(r+t(q-1)) \\ &= \sum_{i=1}^{d} \omega_i^r A_i(r+t(q-1))(\omega_i^{q-1})^t. \end{aligned}$$

Thus, $A_r(t)$ is a generalized power sum of order $n$ with roots $\omega_i^{q-1}$. Consequently

$$g_r(z) := \sum_{t=0}^{\infty} A_r(t) z^t = \frac{N_r(z)}{D(z)}$$

is a rational function with denominator $D(z) = \prod_{i=1}^{d}(1-\omega_i^{q-1}z)^{n(i)}$ and numerator $N_r(z)$ of degree less than $n$, as was noted in Section 2.

For each $i$, write $\omega_i^{q-1} = 1+\delta_i$. Since $q-1$ is prime to $p$,

$$\operatorname{ord}(\delta_i) = \operatorname{ord}(\omega_i - \operatorname{Teich}(\omega_i)) \geqslant \varepsilon.$$

Choose $k \in \mathbb{Z}$ so that

$$ke < \frac{s_0-1}{p-1} \leqslant (k+1)\varepsilon.$$

Using the identity

$$1-\omega_i^{q-1}z = (1-z)(1+\delta_i)\left(1-\frac{\delta_i}{1+\delta_i}\frac{1}{1-z}\right)$$

we can expand

$$\begin{aligned} \frac{1}{D(z)} &= \frac{1}{(1-z)^n}\cdot\frac{1}{\prod(1+\delta_i)^{n(i)}}\cdot\frac{1}{\prod\left(1-\dfrac{\delta_i}{1+\delta_i}\dfrac{1}{1-z}\right)^{n(i)}} \\ &= \frac{1}{(1-z)^n}\sum_{j=0}^{\infty}\frac{d_j}{(1-z)^j}, \end{aligned}$$

where $\operatorname{ord}(d_j) \geqslant j\varepsilon$. Hence

$$\frac{1}{D(z)} \equiv \frac{1}{(1-z)^n}\sum_{j=0}^{k}\frac{d_j}{(1-z)^j} \quad \bmod \pi^{s_0-1}\hat{O}_p[[z]].$$

On the other hand, consider $N_r(z)$. Let $\|g(z)\|$ be the Gauss norm of a power series, that is, the maximum of the absolute values of its coefficients, and recall that the Gauss norm is multiplicative. By our normalization, $|A(h)| \leqslant 1$ for every

$h$ and hence certainly $\|g_r(z)\| \leqslant 1$. Clearly also $\|D(z)\| \leqslant 1$. Therefore $\|N_r(z)\| = \|g_r(z)\|\,\|D(z)\| \leqslant 1$, and we can write

$$N_r(z) = \sum_{i=1}^{n} c_i(1-z)^{n-i}$$

with $c_i \in \hat{\mathcal{O}}_p$, $1 \leqslant i \leqslant n$.

Combining these expressions yields

$$\sum_{t=0}^{\infty} A_r(t)\, z^t = g_r(z) \equiv \left(\sum_{i=1}^{n} \frac{c_i}{(1-z)^i}\right)\left(\sum_{j=0}^{k} \frac{d_j}{(1-z)^j}\right)$$

$$\equiv \sum_{l=1}^{n+k} \frac{\gamma_l}{(1-z)^l} \quad \operatorname{mod} \pi^{s_0-1}\hat{\mathcal{O}}_p[[z]],$$

with $\operatorname{ord}(\gamma_l) \geqslant \max(0, l-n)\,\varepsilon$. Using the binomial expansion for $1/(1-z)^l$, we obtain

$$A_r(t) \equiv \sum_{l=1}^{n+k} \gamma_l \binom{t+l-1}{l-1} \quad \operatorname{mod} \pi^{s_0-1}.$$

We now wish to estimate how large $M$ must be in order that

$$A_r(t) \equiv A_r(t-p^M) \quad \operatorname{mod} \pi^{s_0-1}$$

for all $t$. This will certainly hold if it is true for each term in the sum; consequently, by Lemma 3, it is enough if

$$\max(0, l-n)\,\varepsilon + M - \lfloor \log_p(l-1) \rfloor \geqslant \frac{s_0-1}{p-1}$$

for $l = 2, 3, \ldots, n+k$. Using

$$k < \frac{1}{\varepsilon}\frac{s_0-1}{p-1},$$

it suffices to have $M \geqslant F$, where

$$F = \left\lceil \frac{s_0-1}{p-1} \right\rceil + \left\lfloor \log_p\left(n-1+\frac{1}{\varepsilon}\frac{s_0-1}{p-1}\right) \right\rfloor.$$

To apply this to bound the number of zeros of $a(z)$, suppose that $h$ satisfies

$$\operatorname{ord}(A(h)) + \frac{s(h)}{p-1} < \frac{s_0}{p-1}.$$

Write $h = r + t(q-1)$. If $t \geqslant p^F$, let $M = \lfloor \log_p t \rfloor$ and put

$$\hat{t} = t - p^M,$$

$$\hat{h} = r + \hat{t}(q-1) = h - (p^{M+f} - p^M).$$

By the above discussion,

$$\operatorname{ord}(A(\hat{h})) = \operatorname{ord}(A(h)).$$

On the other hand, by Lemma 4, $s(\hat{h}) \leqslant s(h)$, and so

$$\operatorname{ord}(A(\hat{h})) + \frac{s(\hat{h})}{p-1} \leqslant \operatorname{ord}(A(h)) + \frac{s(h)}{p-1}.$$

Therefore, the first $h$ for which $\operatorname{ord}(a_h)$ is minimal must be such that $h \leqslant H \leqslant n-1$ or such that $h = r + t(q-1)$ with $r < q-1, t \leqslant p^F - 1$. In other words (since $n \geqslant 3$)

$$h \leqslant p^F(p^f - 1) < p^{F+f}.$$

In addition $s(h) \leqslant s_0 - 1$. To use this bit of information, note that if $s_0 < p$ then the largest permissible $h$ will have only one non-zero $p$-adic digit, namely

$$h \leqslant (s_0 - 1) p^{f+F-1}.$$

(Note that this is the final assertion in the theorem.) However, if $s_0 \geqslant p$ then in any case since $F \geqslant (s_0 - 1)/(p-1)$ we have $s(h) \leqslant (p-1)F$. Thus

$$h \leqslant (p-1)p^f + \ldots + (p-1)p^{f+F-1},$$

or

$$h \leqslant p^f(p^F - 1).$$

This concludes the proof when all the $\omega_i$ are units.

In the general case it is possible to proceed in several ways. If there is a coset of $R/m$ which does not contain any $\omega_i$, then let $\zeta$ be the corresponding Teichmuller representative. Multiplying $a(z) = \Sigma A_i(z) \exp(\pi \omega_i z)$ by $\exp(-\pi \zeta z)$ replaces each $\omega_i$ by $\omega_i - \zeta$, and does not change the disc of convergence or the number of zeros. Furthermore, each $\omega_i - \zeta$ is a unit, hence we are reduced to the previous case. (Note that if $\zeta_1$, $\zeta_2$ and $\zeta_3$ are Teichmuller elements such that $\zeta_1 + \zeta_2 \equiv \zeta_3 \bmod m_p$, then $\operatorname{ord}(\zeta_1 + \zeta_2 - \zeta_3) \geqslant 1$ because $\mathbb{Q}_p(\zeta_1, \zeta_2, \zeta_3)/\mathbb{Q}_p$ is unramified. This shows that $\varepsilon$ in the theorem remains unchanged.)

This completes the proof when $n < p^f$. If $n \geqslant p^f$, one can always extend the residue field and find an acceptable $\zeta$; however this would degrade $p^f$ to $p^{2f}$ in the estimate for $h$ so we proceed differently.

As in Theorem 1, break $A(h)$ into a sum $A^0(h) + A^+(h)$, where

$$A^0(h) = \sum_{\operatorname{ord}(\omega_i) = 0} A_i(h)\omega_i^h, \qquad A^+(h) = \sum_{\operatorname{ord}(\omega_i) > 0} A_i(h)\omega_i^h.$$

We shall show that $\operatorname{ord}(A^+(h))$ grows too rapidly to affect the conclusion of the theorem.

First we claim that $\operatorname{ord}(A^+(h)) \geqslant 0$ for all $h$. Because $A^+(h) = A(h) - A^0(h)$ and $\operatorname{ord}(A(h)) \geqslant 0$, it suffices to show that $\operatorname{ord}(A^0(h)) \geqslant 0$ for all $h$. The roots of $A^0(h)$ are units, and so

$$\inf_h \operatorname{ord}(A^0(h)) = \liminf_{h \to \infty} \operatorname{ord}(A^0(h)).$$

(This is because on suitable arithmetic progressions $A^0(h)$ can be interpolated by a continuous $p$-adic function; see the proof of Theorem 1.) On the other hand

$$\liminf_{h \to \infty} \operatorname{ord}(A^0(h)) = \liminf_{h \to \infty} \operatorname{ord}(A(h)) \geqslant 0$$

because $\lim_{h \to \infty} \operatorname{ord}(A^+(h)) = \infty$.

It now follows from Lemma 2 that for $h \geqslant n, \operatorname{ord}(A^+(h)) \geqslant (h - n + 1)\varepsilon$. Consequently if

$$h \geqslant n + k$$

then $\operatorname{ord}(A^+(h)) \geqslant (k+1)\varepsilon \geqslant (s_0 - 1)/(p-1)$, and so

$$A(h) \equiv A^0(h) \mod \pi^{s_0 - 1}.$$

Note that $n+k \leqslant p^F$; if to the contrary we had $n+k-1 \geqslant p^F$ then from

$$F \geqslant \left\lceil \frac{s_0-1}{p-1} \right\rceil + \lfloor \log_p(n+k-1) \rfloor$$

we would derive $s_0-1 \leqslant 0$, which holds only if $n \leqslant 2$.

The first part of the proof shows that if $h$ satisfies

$$\operatorname{ord}(A^0(h)) + \frac{s(h)}{p-1} < \frac{s_0}{p-1},$$

then there is $\acute{h} \equiv h \bmod p^F(q-1)$ with $\acute{h} < p^F(q-1)$ such that

$$A^0(\acute{h}) \equiv A^0(h) \mod \pi^{s_0-1}$$

and $s(\acute{h}) \leqslant s(h)$. Obviously $\acute{h}$ is the least positive residue of $h \bmod p^F(q-1)$. Hence if

$$\operatorname{ord}(A(h)) + \frac{s(h)}{p-1} < \frac{s_0}{p-1} \tag{$*$}$$

and $\acute{h} \geqslant n+k$, then

$$\operatorname{ord}(A(\acute{h})) + \frac{s(\acute{h})}{p-1} \leqslant \operatorname{ord}(A(h)) + \frac{s(h)}{p-1} \tag{$**$}$$

and the proof remains valid. However if $\acute{h} < n+k$ then it is conceivable that $A^0(\acute{h})$ and $A^+(\acute{h})$ could cancel, leaving $\operatorname{ord}(A(h))$ quite large. There are now two cases.

If $s_0 < p$, we shall show that $\acute{h} < n+k$ together with ($*$) implies that $\acute{h} = h$, hence ($**$) holds by assumption. Suppose to the contrary that $\acute{h} < h$. Then since $\acute{h} \equiv h \bmod p^F$ and $\acute{h} < n+k \leqslant p^F$, the digits of $\acute{h}$ are embedded in the low-order $p$-adic digits of $h$ and consequently $s(h) > s(\acute{h})$. However, we also have $h \equiv \acute{h} \bmod (p-1)$ and hence $s(h) \equiv s(\acute{h}) \bmod (p-1)$ by Lemma 4. Thus $s(h) \geqslant s(\acute{h})+p-1$ which contradicts $s(h) < s_0 < p$.

When $s_0 \geqslant p$, the argument is more complicated. First of all $n < p^F$: we have shown above that $n+k \leqslant p^F$; if $k = 0$ and $n = p^F$, then

$$F \geqslant \left\lceil \frac{s_0-1}{p-1} \right\rceil + \lfloor \log_p(n+k-1) \rfloor$$

forces $n \leqslant p$. However, this means that $s_0 \leqslant p-1$, a contradiction.

If $f \geqslant F$ then the trick of multiplying $a(z)$ by $\exp(-\pi\zeta z)$ is applicable; thus we can assume that $f < F$.

It is no longer true that $\acute{h} < n+k$ and ($*$) imply that $h = \acute{h}$. However, when $f < F$ and $s(h) < s_0$ there is a second index $\tilde{h} < p^f(p^F-1)$ such that $\tilde{h} \equiv h \bmod p^F(p^f-1)$. We shall show that either $\acute{h}$ or $\tilde{h}$ meets the needs of the theorem.

If $h = \acute{h}$ there is nothing to prove. However, if $h > \acute{h}$ then since $\acute{h} < n+k \leqslant p^F$ and $\acute{h} \equiv h \bmod p^F(q-1)$, it follows as before that $\acute{h}$ is embedded in $h$. The proof of the first part of the theorem actually produced a chain of values

$$h > \acute{h}_1 > \ldots > \acute{h}_t = \acute{h}$$

in which $\acute{h}_{i+1} = \acute{h}_i - p^{M_i}(p^f-1)$ with $M_i \geqslant F$, and with

$$s(h) \geqslant s(\acute{h}_1) \geqslant \ldots \geqslant s(\acute{h}).$$

Examining the last step of the chain, it follows that

$$s(\acute{h}+p^M(p^f-1)) \leqslant s(h)$$

for some $M \geqslant F$. Now put $\tilde{h} = \hat{h} + p^F(p^f - 1)$. Since $\hat{h} < p^F$,

$$s(\tilde{h}) = s(\hat{h} + p^M(p^f - 1)) \leqslant s(h) \leqslant s_0 - 1.$$

Also $\tilde{h} < p^{F+f}$; these two facts imply that $\tilde{h} \leqslant p^f(p^F - 1)$ as in the proof of the first part.

Since obviously $\tilde{h} \geqslant n + k$ and $\tilde{h} \equiv h \bmod p^F(q-1)$, we conclude that

$$\operatorname{ord}(A(\tilde{h})) = \operatorname{ord}(A(h)).$$

Thus, the index of the final corner of the Newton polygon is still no larger than $p^f(p^F - 1)$.

Let $L$ be a finite extension of $\mathbb{Q}_p$. For exponential polynomials defined over $L$, there is an a priori bound on the number of zeros. Write $e$ for the ramification index of $L/\mathbb{Q}_p$, and $f$ for the residue degree.

Corollary 1. *Let $b(z)$ be an exponential polynomial of order $n \geqslant 3$ whose roots and coefficients belong to $L$. The total number $N$ of zeros of $b(z)$ in its disc of convergence satisfies $N \leqslant p^f(p^F - 1)$, where*

$$F = \left\lceil \frac{s_0 - 1}{p - 1} \right\rceil + \left\lfloor \log_p \left( n - 1 + e\frac{s_0 - 1}{p - 1} \right) \right\rfloor \leqslant \lceil \log_p n \rceil + \lfloor \log_p (n + e\lceil \log_p n \rceil) \rfloor.$$

*If $p \geqslant n$, this can be improved as follows.*

(a) *If $L = \mathbb{Q}_p$, then $N \leqslant (n-2)p$.*

(b) *If $L/\mathbb{Q}_p$ is unramified, then $N \leqslant (n-2)p^f$.*

(c) *If $L/\mathbb{Q}_p$ is ramified, then $N \leqslant (n-2)p^{f+E}$, where*

$$E = \left\lfloor \log_p \left( n - 1 + e\left( \frac{n-2}{p-1} \right) \right) \right\rfloor \leqslant \lceil \log_p e \rceil.$$

*Proof.* This is an immediate consequence of the theorem, using the trivial bounds $\varepsilon \geqslant 1/e$ and $\lceil \log_p n \rceil \geqslant (s_0 - 1)/(p-1)$, with $s_0 - 1 = n - 2$ if $p \geqslant n$.

## 6. *An example*

Suppose that $p > n$. In this section we shall construct an exponential polynomial $b(z)$ of order $n$ defined over $\mathbb{Q}_p$ which has $(\lfloor \sqrt{n} \rfloor - 1)p$ zeros in its disc of convergence. This does not approach our bound $(n-2)p$, but it at least shows the true bound has joint dependence on $n$ and $p$.

To carry out the construction, we need some preliminary lemmas.

Lemma 5. *Fix $m < p$, and put $\lambda_j = 1 + jp$ for $j = 0, 1, \ldots, m$. Then for any polynomial $Q(t) \in \mathbb{Z}_p[t]$ of degree less than or equal to $m$, there are coefficients $D_j \in \mathbb{Q}_p$ such that the power sum $d(t) = \sum_{j=0}^{m} D_j \lambda_j^t$ satisfies*

$$d(t) \equiv Q(t) \mod p$$

*for all $t \in \mathbb{N}$.*

*Proof.* Note that

$$\lambda_j^t = (1 + jp)^t = \sum_{l=0}^{t} \binom{t}{l} j^l p^l.$$

For each $k < p$ the Vandermonde determinant $V(0, 1, \ldots, k)$ is a unit in $\mathbb{Z}_p$, and so there exist constants $D_{j,k} \in (1/p^k)\mathbb{Z}_p$, $0 \leqslant j \leqslant k$, such that for $0 \leqslant l \leqslant k$

$$\sum_{j=0}^{k} D_{j,k} j^l = \begin{cases} 0 & \text{if } l \neq k, \\ \dfrac{k!}{p^k} & \text{if } l = k. \end{cases}$$

Thus $\sum_{j=0}^{k} D_{j,k} \lambda_j^t \equiv t(t-1)\ldots(t-k+1) \bmod p$. Taking linear combinations gives the lemma.

Lemma 6. *Suppose that $m < p-1$, and let $\alpha_i \in \mathbb{Z}_p^\times$, $0 \leqslant i \leqslant m$, be units such that $\alpha_i \not\equiv \alpha_k \bmod p$ for $i \neq k$. Then, for any constants $H_r \in \mathbb{Z}_p$, $0 \leqslant r \leqslant m$, there exist coefficients $c_i \in \mathbb{Z}_p$ such that*

$$H_r = \sum_{i=0}^{m} c_i \alpha_i^r, \qquad 0 \leqslant r \leqslant m.$$

*Proof.* This is because $V(\alpha_0, \ldots, \alpha_m) \not\equiv 0 \bmod p$.

We can now exhibit $b(z)$. Take $m = \lfloor \sqrt{n} \rfloor - 1$, put $\alpha_i = 1 + i$ for $0 \leqslant i \leqslant m$, and $\lambda_j = 1 + jp$ for $0 \leqslant j \leqslant m$. Then $b(z)$ will have the form

$$b(z) = \sum_{i,j=0}^{m} B_{ij} \exp(\alpha_i \lambda_j z).$$

To construct the coefficients $B_{ij}$, first define polynomials $Q_r(x) \in \mathbb{Z}_p[x]$ for $0 \leqslant r \leqslant m$ by

$$Q_r(x) = \begin{cases} x(x-1)\ldots(x-r) & \text{if } 0 \leqslant r \leqslant m-1, \\ (x-1)\ldots(x-m) & \text{if } r = m, \end{cases}$$

and write $Q_r(x) = \sum_{k=0}^{m} Q_{rk} x^k$. Next find $c_{ik} \in \mathbb{Z}_p$ such that for each $k = 0, \ldots, m$,

$$Q_{rk} = \sum_{i=0}^{m} c_{ik} \alpha_i^r, \qquad 0 \leqslant r \leqslant m.$$

This can be done by Lemma 6. Finally, using Lemma 5, choose $B_{ij} \in \mathbb{Q}_p$ such that for each $i = 0, \ldots, m$

$$\sum_{j=0}^{m} B_{ij} \lambda_j^t \equiv \sum_{k=0}^{m} c_{ik} t^k \mod p \qquad \text{for all } t \in \mathbb{Z}.$$

As defined, $b(z)$ has order $(m+1)^2 \leqslant n$. If an exponential polynomial of exact order $n$ is desired, add on additional terms of the form $\exp(p\omega_k z)$.

The disc of convergence of $b(z)$ is $\operatorname{ord}(z) > 1/(p-1)$. Expanding $b(z)$ as a power series,

$$b(z) = \sum_{h=0}^{\infty} b_h \left(\frac{z}{\pi}\right)^h = \sum_{h=0}^{\infty} B(h) \frac{\pi^h}{h!} \left(\frac{z}{\pi}\right)^h,$$

where $B(h) = \sum_{i,j} B_{ij} (\alpha_i \lambda_j)^h$. As usual,

$$\operatorname{ord}(b_h) = \operatorname{ord}(B(h)) + \frac{s(h)}{p-1}.$$

To show that $b(z)$ has $mp$ zeros, we must show that $mp$ is the first index for which $\operatorname{ord}(b_h)$ is minimal. First, note that $\operatorname{ord}(B(h)) \geqslant 0$ for all $h$. Indeed

$$B(h) = \sum_i \alpha_i^h \Big(\sum_j B_{ij} \lambda_j^h\Big) \equiv \sum_i \alpha_i^h \Big(\sum_k c_{ik} h^k\Big) \mod p,$$

and the $c_{ik}$ belong to $\mathbb{Z}_p$.

Write $h = r+t(p-1)$, and for each $r$, $0 \leqslant r < p-1$, put $B_r(t) = B(r+t(p-1))$. Then

$$h \equiv \begin{cases} r & \operatorname{mod}(p-1), \\ r-t & \operatorname{mod} p. \end{cases}$$

Therefore

$$\begin{aligned} B_r(t) &\equiv \sum_i \alpha_i^r \Big(\sum_k c_{ik}(r-t)^k\Big) \\ &\equiv \sum_k \Big(\sum_i c_{ik}\alpha_i^r\Big)(r-t)^k \\ &\equiv \sum_k Q_{rk}(r-t)^k \\ &\equiv Q_r(r-t) \operatorname{mod} p. \end{aligned}$$

By our choice of $Q_m(x)$, $\operatorname{ord}(B(mp)) = 0$. Note that $s(mp) = m$. We now consider which values of $h$ satisfy $s(h) \leqslant m$. Since $s(h) \equiv h \operatorname{mod}(p-1)$, it must be that $h \equiv r \operatorname{mod}(p-1)$ for some $r \leqslant m$. In addition, $t$ is constrained. Writing $h = r+t(p-1) = h_0+h_1p+\ldots+h_kp^k$, we find that

$$\begin{aligned} t = \frac{h-r}{p-1} &= \sum_{i=1}^{k} h_i\left(\frac{p^i-1}{p-1}\right) \\ &= \sum_{i=1}^{k} h_i(1+p+\ldots+p^{i-1}) \\ &\equiv \sum_{i=1}^{k} h_i \equiv r-h_0 \quad \operatorname{mod} p. \end{aligned}$$

Now if $s(h) = r$ then obviously $0 \leqslant h_0 \leqslant r$. Thus, the only possible values of $h$ with $s(h) < m$ are such that $h = r+t(p-1)$ with $0 \leqslant r < m$ and $t \equiv 0, 1, \ldots, r \operatorname{mod} p$. By our choice of the $Q_r(x)$ and the congruence above, $\operatorname{ord}(B(h)) \geqslant 1$ for all these $h$. Furthermore, the only values of $h < mp$ such that $s(h) = m$ are $h = m+t(p-1)$ for $t = 0, 1, \ldots, m-1$. By the choice of $Q_m(x)$, $\operatorname{ord}(B(h)) \geqslant 1$ for these $h$ as well.

This proves the result.

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